

# $\mathfrak{o}$ -BOUNDED GROUPS AND OTHER TOPOLOGICAL GROUPS WITH STRONG COMBINATORIAL PROPERTIES

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**ABSTRACT.** We construct several topological groups with very strong combinatorial properties. In particular, we give simple examples of subgroups of  $\mathbb{R}$  (thus strictly  $\mathfrak{o}$ -bounded) which have the Hurewicz property but are not  $\sigma$ -compact, and show that the product of two  $\mathfrak{o}$ -bounded subgroups of  ${}^{\mathbb{N}}\mathbb{R}$  may fail to be  $\mathfrak{o}$ -bounded, even when they satisfy the stronger property  $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ . This solves a problem of Tkačenko and Hernandez, and extends independent solutions of Krawczyk and Michałewski and of Banach, Nickolas, and Sanchis. We also construct separable metrizable groups  $G$  of size continuum such that every countable Borel  $\omega$ -cover of  $G$  contains a  $\gamma$ -cover of  $G$ .

## 1. INTRODUCTION

In [15, 11], a unified framework for topological diagonalizations is established, which turns out closely related to several notions which appear in a recently flourishing study of topological groups in terms of their covering properties (see, e.g., [18, 9, 10, 12, 13] and references therein). A comprehensive study of these interrelations is currently being carried by Babinkostova, Kočinac, and Scheepers [1]. The purpose of this paper is to adopt several recent construction techniques from the general theory of topological diagonalizations to the theory of topological groups.

**1.1. Topological diagonalizations.** We briefly describe the general framework. Let  $X$  be a topological space. An open cover  $\mathcal{U}$  of  $X$  is an  $\omega$ -cover of  $X$  if  $X$  is not in  $\mathcal{U}$  and for each finite subset  $F$  of  $X$ , there is a set  $U \in \mathcal{U}$  such that  $F \subseteq U$ .  $\mathcal{U}$  is a  $\gamma$ -cover of  $X$  if it is infinite and for each  $x$  in  $X$ ,  $x \in U$  for all but finitely many  $U \in \mathcal{U}$ . Let  $\mathcal{O}$ ,  $\Omega$ , and  $\Gamma$  denote the collections of all countable open covers,  $\omega$ -covers, and  $\gamma$ -covers of  $X$ , respectively. Let  $\mathfrak{U}$  and  $\mathfrak{V}$  be collections of covers of a space  $X$ . Following are selection hypotheses which  $X$  might satisfy or not satisfy.

- $S_1(\mathfrak{U}, \mathfrak{V})$ : For each sequence  $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$  of members of  $\mathfrak{U}$ , there is a sequence  $\{U_n\}_{n \in \mathbb{N}}$  such that for each  $n$   $U_n \in \mathcal{U}_n$ , and  $\{U_n\}_{n \in \mathbb{N}} \in \mathfrak{V}$ .
- $S_{fin}(\mathfrak{U}, \mathfrak{V})$ : For each sequence  $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$  of members of  $\mathfrak{U}$ , there is a sequence  $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$  such that each  $\mathcal{F}_n$  is a finite (possibly empty) subset of  $\mathcal{U}_n$ , and  $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n \in \mathfrak{V}$ .
- $\mathfrak{U}_{fin}(\mathfrak{U}, \mathfrak{V})$ : For each sequence  $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$  of members of  $\mathfrak{U}$  which do not contain a finite subcover, there exists a sequence  $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$  such that for each  $n$   $\mathcal{F}_n$  is a finite (possibly empty) subset of  $\mathcal{U}_n$ , and  $\{\bigcup \mathcal{F}_n\}_{n \in \mathbb{N}} \in \mathfrak{V}$ .

Some of the properties defined in this manner were studied earlier by Hurewicz ( $\mathfrak{U}_{fin}(\mathcal{O}, \Gamma)$ ), Menger ( $S_{fin}(\mathcal{O}, \mathcal{O})$ ), Rothberger ( $S_1(\mathcal{O}, \mathcal{O})$ ), traditionally known as

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the  $C'''$  property), Gerlits and Nagy ( $S_1(\Omega, \Gamma)$ , traditionally known as the  $\gamma$ -property), and others. Many equivalences hold among these properties, and the surviving ones appear in Figure 1 (where an arrow denotes implication), to which no arrow can be added except perhaps from  $U_{fin}(\mathcal{O}, \Gamma)$  or  $U_{fin}(\mathcal{O}, \Omega)$  to  $S_{fin}(\Gamma, \Omega)$  [11].

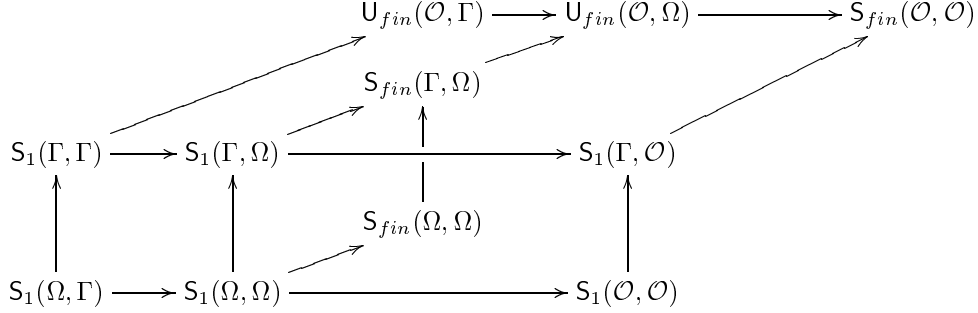


FIGURE 1. The Scheepers Diagram

Each selection principle has a naturally associated game, but we will restrict attention to the game  $G_{fin}(\mathfrak{U}, \mathfrak{V})$ , which is played as follows: In the  $n$ th inning, ONE chooses an element  $U_n$  of  $\mathfrak{U}$  and then TWO responds by choosing a finite subset  $\mathcal{F}_n$  of  $U_n$ . They play an inning per natural number. A play  $(U_0, \mathcal{F}_0, U_1, \mathcal{F}_1, \dots)$  is won by TWO if  $\bigcup_n \mathcal{F}_n \in \mathfrak{V}$ ; otherwise ONE wins. We will write  $\text{ONE} \uparrow G_{fin}(\mathfrak{U}, \mathfrak{V})$  (respectively,  $\text{TWO} \uparrow G_{fin}(\mathfrak{U}, \mathfrak{V})$ ) for “ONE (respectively, TWO) has a winning strategy in  $G_{fin}(\mathfrak{U}, \mathfrak{V})$ ”.

**1.2.  $\sigma$ -bounded groups.** Okunev introduced the following notion as an approximation of  $\sigma$ -compact groups: A topological group  $G$  is  *$\sigma$ -bounded* if for each sequence  $\{U_n\}_{n \in \mathbb{N}}$  of neighborhoods of the unit element of  $G$ , there exists a sequence  $\{F_n\}_{n \in \mathbb{N}}$  of finite subsets of  $G$  such that  $G = \bigcup_n F_n \cdot U_n$ . It is possible to state this definition in the language of selection principles. Let  $\mathcal{O}_{\text{nb}}$  denote the covers of  $G$  of the form  $\{g \cdot U : g \in G\}$  where  $U$  is a neighborhood of the unit element of  $G$ . Then  $G$  is  $\sigma$ -bounded if, and only if,  $G$  satisfies  $S_{fin}(\mathcal{O}_{\text{nb}}, \mathcal{O})$ .

According to Tkačenko, a topological group  $G$  is *strictly  $\sigma$ -bounded* if TWO has a winning strategy in the game  $G_{fin}(\mathcal{O}_{\text{nb}}, \mathcal{O})$ . Clearly, every subgroup of a  $\sigma$ -compact group is strictly  $\sigma$ -bounded, but the converse does not hold [9].

*Notational convention.* For sets  $X, Y$ ,  ${}^X Y$  denotes the collection of all functions from  $X$  to  $Y$ . If  $Y$  is a topological space, then the topology on  ${}^X Y$  is the Tychonoff product topology.

## 2. TWO ALMOST $\sigma$ -COMPACT SUBGROUPS OF $\mathbb{R}$

The *Baire space*  ${}^{\mathbb{N}}\mathbb{N}$  is (quasi)ordered by eventual dominance:  $f \leq^* g$  if  $f(n) \leq g(n)$  for all but finitely many  $n$ . A subset of  ${}^{\mathbb{N}}\mathbb{N}$  is *dominating* if it is cofinal in  ${}^{\mathbb{N}}\mathbb{N}$  with respect to  $\leq^*$ . If a subset of  ${}^{\mathbb{N}}\mathbb{N}$  is unbounded with respect to  $\leq^*$  then we simply say that it is *unbounded*. Let  $\mathfrak{b}$  (respectively,  $\mathfrak{d}$ ) denote the minimal cardinality of an unbounded (respectively, dominating) subset of  ${}^{\mathbb{N}}\mathbb{N}$ .

We use the following setting from [5]. Let  $\mathbb{N} \cup \{\infty\}$  be the one point compactification of  $\mathbb{N}$ . (A subset  $A \subseteq \mathbb{N} \cup \{\infty\}$  is open if:  $A \subseteq \mathbb{N}$ , or  $\infty \in A$  and  $A$  is cofinite.)

Let  $\mathcal{Z} \subseteq {}^{\mathbb{N}}(\mathbb{N} \cup \{\infty\})$  consist of the *nondecreasing* functions  $f$  such that for all  $n$ , if  $f(n) < \infty$ , then  $f(n) < f(n+1)$ . ( $\mathcal{Z}$  is homeomorphic to the Cantor set of reals.) For each increasing finite sequence  $s$  of natural numbers, let  $q_s \in \mathcal{Z}$  be defined as

$$q_s(k) = \begin{cases} s(k) & \text{if } k < |s| \\ \infty & \text{otherwise} \end{cases}$$

for each  $k \in \mathbb{N}$ . Note that the set

$$Q = \{q_s : s \text{ is an increasing finite sequence in } \mathbb{N}\}$$

is dense in  $\mathcal{Z}$ .

**Theorem 1.** *There exists a non  $\sigma$ -compact subgroup  $G_H$  of  $\mathbb{R}$  of cardinality  $\mathfrak{b}$  such that all finite powers of  $G_H$  satisfy  $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$  (in particular, they satisfy  $\mathsf{S}_{fin}(\Omega, \Omega)$ ).*

*Proof.* Let  $B = \{f_\alpha : \alpha < \mathfrak{b}\} \subseteq {}^{\mathbb{N}}\mathbb{N}$  be a  $\leq^*$ -unbounded set of strictly increasing elements of  ${}^{\mathbb{N}}\mathbb{N}$  which forms a  $\mathfrak{b}$ -scale (that is, for each  $\alpha < \beta$ ,  $f_\alpha \leq^* f_\beta$ ), and set  $H = B \cup Q$ . In [5] it is proved that all finite powers of  $H$  satisfy  $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$ .

Think of  $H$  as a set of real numbers. For each  $n$ , the set

$$G_n = \{\alpha_1 g_1 + \cdots + \alpha_n g_n : \alpha_1, \dots, \alpha_n \in \mathbb{Z}, g_1, \dots, g_n \in H\}$$

is a union of countably many continuous images of  $H^n$ , thus for each  $k$ ,  $(G_n)^k$  is a union of countably many continuous images of  $H^{nk}$ . As the property  $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$  is preserved under taking continuous images and countable unions [11, 4], we have that each set  $(G_n)^k$  satisfies  $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$ .

Take  $G_H = \langle H \rangle$ . Then  $G_H = \bigcup_{n \in \mathbb{N}} G_n$  and for each  $n$ ,  $G_n \subseteq G_{n+1}$ . Thus,  $(G_H)^k = \bigcup_{n \in \mathbb{N}} (G_n)^k$  for each  $k$ ; therefore  $(G_H)^k$  satisfies  $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$  for each  $k$ . Now, satisfying  $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$  in all finite powers implies  $\mathsf{S}_{fin}(\Omega, \Omega)$  [11].

Since  $|G_H| = \mathfrak{b}$  is consistently smaller than the continuum,  $G_H$  is not  $\sigma$ -compact.  $\square$

**Problem 2.** *Does  $G_H$  satisfy  $\mathsf{S}_1(\Gamma, \Gamma)$ ?*

**Corollary 3.**  *$\text{TWO} \uparrow \mathsf{G}_{fin}(\mathcal{O}, \mathcal{O})$  is strictly stronger than  $\text{TWO} \uparrow \mathsf{G}_{fin}(\mathcal{O}_{\text{nbd}}, \mathcal{O})$  (strict  $o$ -boundedness).*

*Proof.* By a well known theorem of Telgársky,  $\text{TWO} \uparrow \mathsf{G}_{fin}(\mathcal{O}, \mathcal{O})$  if, and only if, the space  $G$  is  $\sigma$ -compact. Contrast this with Theorem 1.  $\square$

Theorem 1 has a group-theoretic consequence.

**Corollary 4.** *Assume that for each  $m$ ,  $\{g_{m,n} + U_m\}_{n \in \mathbb{N}}$  is a cover of  $G_H$ . Then there exists a sequence  $\{m_n\}_{n \in \mathbb{N}}$  such that*

$$G_H = \bigcup_k \bigcap_{n > k} (\{g_{n,1}, \dots, g_{n,m_n}\} + U_n).$$

*In fact, this property is satisfied by all finite powers of  $G_H$ .*

*Proof.* This property is an immediate consequence of  $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$ .  $\square$

Let  $D = \{g_\alpha : \alpha < \mathfrak{d}\}$  be a dominating subset of  ${}^{\mathbb{N}}\mathbb{N}$  where each  $g_\alpha$  is increasing, and for each  $f \in {}^{\mathbb{N}}\mathbb{N}$  there exists  $\alpha_0 < \mathfrak{d}$  such that for any finite set  $F \subseteq \mathfrak{d} \setminus \alpha_0$ ,  $f(n) < \min\{g_\beta(n) : \beta \in F\}$  for infinitely many  $n$ . Such a set was constructed in [5].

A subset  $F$  of  ${}^{\mathbb{N}}\mathbb{N}$  is *finitely-dominating* if for each  $g \in {}^{\mathbb{N}}\mathbb{N}$  there exist  $k$  and  $f_1, \dots, f_k \in {}^{\mathbb{N}}\mathbb{N}$  such that  $g(n) \leq^* \max\{f_1(n), \dots, f_k(n)\}$ . For conciseness, we use the following shortened notation.

**Ax:** Either a union of less than  $\mathfrak{d}$  many not dominating sets is not dominating (in other words,  $\mathfrak{b} = \mathfrak{d}$ ), or else a union of less than  $\mathfrak{d}$  many not finitely-dominating sets of increasing functions is not finitely-dominating.

In [5] it is shown that **Ax** implies that  $M = D \cup Q$  satisfies  $\mathsf{S}_{fin}(\Omega, \Omega)$ . (Observe that  $\mathsf{S}_{fin}(\Omega, \Omega)$  is preserved under taking finite powers [11].) Assuming **Ax**, one shows as in Theorem 1 that all finite powers of  $G_M = \langle M \rangle$  satisfy  $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$  and gets the following.

**Theorem 5 (Ax).**

- (1) *There exists a non  $\sigma$ -compact subgroup  $G_M$  of  $\mathbb{R}$  of cardinality  $\mathfrak{d}$  such that  $G_M$  satisfies  $\mathsf{S}_{fin}(\Omega, \Omega)$ .*
- (2) *Assume that for each  $m$ ,  $\{g_{m,n} + U_m\}_{n \in \mathbb{N}}$  is a cover of  $G_M$ . Then there exists a sequence  $\{m_n\}_{n \in \mathbb{N}}$  such that for each finite subset  $F$  of  $G_M$ , there exists  $n$  such that  $F \subseteq \{g_{n,1}, \dots, g_{n,m_n}\} + U_n$ . Moreover, this holds for all finite powers of  $G_M$ .*

It follows from the next section that the hypothesis **Ax** is not necessary to prove Theorem 5 (namely, it also follows from the incomparable assumption  $\text{cov}(\mathcal{M}) = \mathfrak{c}$ ).

**Problem 6.** *Is Theorem 5 provable in ZFC?*

### 3. PRODUCTS OF $\mathfrak{o}$ -BOUNDED GROUPS

In Problem 3.2 of [18] and Problem 5.2 of [9] it is asked whether the (Tychonoff) product of two  $\mathfrak{o}$ -bounded groups is  $\mathfrak{o}$ -bounded. We give a negative answer. A negative answer was independently given by Krawczyk and Michalewski [13], but our result is stronger in the following sense: Let  $G$  be a topological group such that all finite powers of  $G$  are Lindelöf. Then each open  $\omega$ -cover of  $G$  contains a *countable*  $\omega$ -cover of  $G$ . Let  $\mathcal{B}_\Omega$  denote the collection of all *countable Borel*  $\omega$ -covers of  $G$ . In this case,

$$\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega) \rightarrow \mathsf{S}_1(\Omega, \Omega) \rightarrow \mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$$

where no implication can be reversed [11, 17], and the last property (the Menger property) implies  $\mathsf{S}_{fin}(\mathcal{O}_{\text{nbd}}, \mathcal{O})$  ( $\mathfrak{o}$ -boundedness). In [13] it is proved that, assuming  $\text{cov}(\mathcal{M}) = \mathfrak{c}$  (this is a small portion of the Continuum Hypothesis), there exist groups  $G_1$  and  $G_2$  satisfying the Menger property  $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ , such that  $G_1 \times G_2$  is not  $\mathfrak{o}$ -bounded. We use the same hypothesis to show that there exist such groups satisfying  $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ .

**Fact 7.** *Assume that  $G$  is a  $\leq^*$ -dominating subgroup of  ${}^{\mathbb{N}}\mathbb{R}$ . Then  $G$  is not  $\mathfrak{o}$ -bounded.*

*Proof.* As  $\mathbb{N}$  can be partitioned into infinitely many infinite sets, the following holds.

**Lemma 8.** *For each  $\mathfrak{o}$ -bounded group  $G$  and sequence  $\{U_n\}_{n \in \mathbb{N}}$  of neighborhoods of the identity of  $G$ , there exists a sequence  $\{F_n\}_{n \in \mathbb{N}}$  of finite subsets of  $G$  with  $G = \bigcap_m \bigcup_{n > m} F_n \cdot G$ .*

Consider the open sets  $U_n = \{f \in {}^{\mathbb{N}}\mathbb{R} : |f(n)| < 1\} \subseteq {}^{\mathbb{N}}\mathbb{R}$ . For each sequence  $\{F_n\}_{n \in \mathbb{N}}$  of finite subsets of  ${}^{\mathbb{N}}\mathbb{R}$ ,  $\bigcup_n F_n + U_n$  is  $\leq^*$ -bounded in  ${}^{\mathbb{N}}\mathbb{R}$ . Let  $h \in {}^{\mathbb{N}}\mathbb{R}$  be a witness for that. As  $G$  is dominating, there exists  $g \in G$  such that  $h \leq^* g$ . Then  $g \notin \bigcap_m \bigcup_{n > m} F_n \cdot G$ .  $\square$

Let  $\mathfrak{c} = |\mathbb{R}|$ . The assertion  $\text{cov}(\mathcal{M}) = \mathfrak{c}$  means that  $\mathbb{R}$  (or any complete, separable, metric space) is not the union of less than continuum many meager (=first category) sets. We say that  $L$  is a  $\kappa$ -Luzin group if  $|L| \geq \kappa$ , and for each meager set  $M$  in  $L$ ,  $|L \cap M| < \kappa$ .

**Theorem 9.** *Assume that  $\text{cov}(\mathcal{M}) = \mathfrak{c}$ . Then there exist  $\mathfrak{c}$ -Luzin subgroups  $L_1, L_2$  of  ${}^{\mathbb{N}}\mathbb{R}$  satisfying  $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ , such that the group  $L_1 \times L_2$  is not  $o$ -bounded. (These Luzin groups are, in fact, linear vector spaces over  $\mathbb{Q}$ .)*

*Proof.* We extend the technique of [17, 4]. We stress that there exists a much easier proof if we only require that  $L_1$  and  $L_2$  satisfy  $S_1(\Omega, \Omega)$ ; however we do not supply this easier proof to avoid repetitions.

A cover  $\mathcal{U}$  of  $X$  is  $\omega$ -fat if for each finite  $F \subseteq X$  and each finite family  $\mathcal{F}$  of nonempty open sets, there exists  $U \in \mathcal{U}$  such that  $F \subseteq U$  and for each  $O \in \mathcal{F}$ ,  $U \cap O$  is not meager. Let  $\mathcal{M}$  denote the meager subsets of  ${}^{\mathbb{N}}\mathbb{R}$ .

**Lemma 10** ([4]). *Assume that  $\mathcal{U}$  is an  $\omega$ -fat cover of a set  $X \subseteq {}^{\mathbb{N}}\mathbb{R}$ . Then:*

- (1)  $\bigcup \mathcal{U}$  is comeager,
- (2) For each finite  $F \subseteq G$  and finite family  $\mathcal{F}$  of nonempty open sets,

$$\mathcal{U}_{F, \mathcal{F}} := \{U \in \mathcal{U} : F \subseteq U \text{ and for each } O \in \mathcal{F}, U \cap O \notin \mathcal{M}\}$$

*is an  $\omega$ -fat cover of  $G$ . Consequently,  $\bigcup \mathcal{U}_{F, \mathcal{F}}$  is comeager.*

For shortness, we will say that a cover  $\mathcal{U}$  of  $X$  is *good* if: For each finite  $A \subseteq \mathbb{Q} \setminus \{0\}$  and finite  $B \subseteq G$ , the family

$$\mathcal{U}^{A, B} := \left\{ \bigcap_{q \in A, g \in B} q(U - g) : U \in \mathcal{U} \right\}$$

is an  $\omega$ -fat cover of  $X$ . These covers allow the inductive construction hinted in the following lemma.

**Lemma 11.** *Assume that  $\mathcal{U}$  is a good cover of a group  $G \subseteq {}^{\mathbb{N}}\mathbb{R}$ . Then for each element  $x$  in the intersection of all sets of the form  $\bigcup (\mathcal{U}^{A, B})_{F, \mathcal{F}}$  where the members of  $\mathcal{F}$  are basic open sets,  $\mathcal{U}$  is a good cover of the group  $G + \mathbb{Q}x$ .*

*Proof.* Fix finite sets  $A \subseteq \mathbb{Q} \setminus \{0\}$  and  $B \subseteq G$ . We may assume that  $1 \in A$ . We must show that  $\mathcal{U}^{A, B}$  is an  $\omega$ -fat cover of  $G + \mathbb{Q}x$ . Let  $F$  be a finite subset of  $G + \mathbb{Q}x$ , and  $\mathcal{F}$  be a finite family of nonempty open sets. By moving to subsets we may assume that all members of  $\mathcal{F}$  are basic open sets.

Choose finite sets  $\tilde{A} \subseteq \mathbb{Q} \setminus \{0\}$ ,  $\tilde{B} \subseteq G$ , and  $\tilde{F} \subseteq G$ , such that  $F \subseteq (\tilde{B} + \tilde{A}x) \cup \tilde{F}$ ,  $1 \in \tilde{A}$ , and  $0 \in \tilde{B}$ . As  $x \in \bigcup (\mathcal{U}^{\tilde{A}^{-1}A, B + A^{-1}\tilde{B}})_{\tilde{F}, \mathcal{F}}$ , there exists  $U \in \mathcal{U}$  such that  $x \in \tilde{V} := \bigcap_{q \in \tilde{A}^{-1}A, g \in B + A^{-1}\tilde{B}} q(U - g)$ ,  $\tilde{F} \subseteq \tilde{V}$ , and for each  $O \in \mathcal{F}$ ,  $\tilde{V} \cap O$  is not meager. Take  $V = \bigcap_{q \in A, g \in B} q(U - g)$ . Then  $\tilde{V} \subseteq V$ , thus  $\tilde{F} \subseteq V$  and for each  $O \in \mathcal{F}$ ,  $V \cap O$  is not meager. Now,  $x \in \tilde{V}$ , thus for each  $\tilde{a} \in \tilde{A}$  and  $\tilde{b} \in \tilde{B}$ ,  $x \in \bigcap_{q \in \tilde{a}^{-1}A, g \in B + A^{-1}\tilde{b}} q(U - g)$ , thus  $x \in \bigcap_{q \in A, g \in B} \tilde{a}^{-1}q(U - (g + q^{-1}\tilde{b}))$ ,

therefore  $x \in \bigcap_{q \in A, g \in B} \tilde{a}^{-1}(q(U - g) - \tilde{b})$ , thus  $\tilde{a}x + \tilde{b} \in V$ . This shows that  $F \subseteq (\tilde{A}x + \tilde{B}) \cup \tilde{F} \subseteq V \in \mathcal{U}^{A,B}$ , and we are done.  $\square$

Since we are going to construct Luzin groups, the following lemma tells that we need not consider covers which are not good.

**Lemma 12.** *Assume that  $L$  is a subgroup of  ${}^{\mathbb{N}}\mathbb{R}$  such that  $\mathbb{Q} \cdot L \subseteq L$  and for each nonempty basic open set  $O$ ,  $L \cap O$  is not meager. Then every countable Borel  $\omega$ -cover  $\mathcal{U}$  of  $L$  is a good cover of  $L$ .*

*Proof.* Assume that  $\mathcal{U}$  is a countable collection of Borel sets which is not a good cover of  $L$ . Then there exist finite sets  $A \subseteq \mathbb{Q} \setminus \{0\}$ ,  $B \subseteq L$ ,  $F \subseteq L$ , and  $\mathcal{F}$  of nonempty open sets such that for each  $V \in \mathcal{U}^{A,B}$  containing  $F$ ,  $V \cap O$  is meager for some  $O \in \mathcal{F}$ . For each  $O \in \mathcal{F}$  let

$$M_O = \bigcup \{V \in \mathcal{U}^{A,B} : F \subseteq V \text{ and } V \cap O \in \mathcal{M}\}.$$

Then  $M_O \cap O$  is meager, thus there exists  $x_O \in (L \cap O) \setminus M_O$ . Then  $F \cup \{x_O : O \in \mathcal{F}\}$  is not covered by any  $V \in \mathcal{U}^{A,B}$ . We will show that this cannot be the case.

Put  $F' = A^{-1}F + B$ . As  $\mathbb{Q} \cdot L \subseteq L$ ,  $F'$  is a finite subset of  $L$ . Thus, there exists  $U \in \mathcal{U}$  such that  $F' \subseteq U$ . Consequently, for each  $q \in A$  and  $g \in B$ ,  $x \in q(U - g)$  for each  $x \in F$ , that is,  $F \subseteq \bigcap_{q \in A, g \in B} q(U - g) \in \mathcal{U}^{A,B}$ .  $\square$

We need one more lemma. Denote the collection of countable Borel good covers of a set  $X$  by  $\mathcal{B}_{\text{good}}$ .

**Lemma 13.** *If  $|X| < \text{cov}(\mathcal{M})$ , then  $X$  satisfies  $S_1(\mathcal{B}_{\text{good}}, \mathcal{B}_{\text{good}})$ .*

*Proof.* Assume that  $|X| < \text{cov}(\mathcal{M})$ , and let  $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$  be a sequence of countable Borel good covers of  $X$ . Enumerate each cover  $\mathcal{U}_n$  by  $\{U_k^n\}_{k \in \mathbb{N}}$ . Let  $\{Y_n\}_{n \in \mathbb{N}}$  be a partition of  $\mathbb{N}$  into infinitely many infinite sets. For each  $m$ , let  $y_m \in {}^{\mathbb{N}}\mathbb{N}$  be an increasing enumeration of  $Y_m$ . Let  $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$  be an enumeration of all finite families of nonempty basic open sets.

For finite sets  $F, B \subseteq X$  and  $A \subseteq \mathbb{Q} \setminus \{0\}$ , and each  $m$  define a function  $\Psi_{F,m}^{A,B} \in {}^{\mathbb{N}}\mathbb{N}$  by

$$\Psi_{F,m}^{A,B}(n) = \min\{k : F \subseteq V := \bigcap_{q \in A, g \in B} q(U_k^{y_m(n)} - g) \text{ and } (\forall O \in \mathcal{F}_m) V \cap O \notin \mathcal{M}\}.$$

Since there are less than  $\text{cov}(\mathcal{M})$  many functions  $\Psi_{F,m}^{A,B}$ , there exists by [3] a function  $f \in {}^{\mathbb{N}}\mathbb{N}$  such that for each  $m, F, A$ , and  $B$ ,  $\Psi_{F,m}^{A,B}(n) = f(n)$  for infinitely many  $n$ . Consequently,  $\mathcal{V} = \{U_{f(n)}^{y_m(n)} : m, n \in \mathbb{N}\}$  is a good cover of  $X$ .  $\square$

We are now ready to carry out the construction. Let  ${}^{\mathbb{N}}\mathbb{R} = \{y_\alpha : \alpha < \mathfrak{c}\}$ ,  $\{M_\alpha : \alpha < \mathfrak{c}\}$  be all  $F_\sigma$  meager subsets of  ${}^{\mathbb{N}}\mathbb{R}$ , and  $\{\{\mathcal{U}_n^\alpha\}_{n \in \mathbb{N}} : \alpha < \mathfrak{c}\}$  be all sequences of countable families of Borel sets. Let  $\{O_k : k \in \mathbb{N}\}$  and  $\{\mathcal{F}_m : m \in \mathbb{N}\}$  be all nonempty basic open sets and all finite families of nonempty basic open sets, respectively, in  ${}^{\mathbb{N}}\mathbb{R}$ .

We construct  $L_1$  and  $L_2$  by induction on  $\alpha < \mathfrak{c}$  as follows. At stage  $\alpha \geq 0$  set  $L_\alpha^i = \bigcup_{\beta < \alpha} L_\beta^i$  and consider the sequence  $\{\mathcal{U}_n^\alpha\}_{n \in \mathbb{N}}$ . Say that  $\alpha$  is  $i$ -good if for each  $n$   $\mathcal{U}_n^\alpha$  is a good cover of  $L_\alpha^i$ . In this case, by Lemma 13 there exist elements  $U_n^{\alpha,i} \in \mathcal{U}_n^\alpha$  such that  $\mathcal{U}^{\alpha,i} = \{U_n^{\alpha,i}\}_{n \in \mathbb{N}}$  is a good cover of  $L_\alpha^i$ . We make the inductive

hypothesis that for each  $i$ -good  $\beta < \alpha$ ,  $\mathcal{U}^{\beta,i}$  is a good cover of  $L_\alpha^i$ . For finite sets  $F, B \subseteq L_\alpha^i$  and  $A \subseteq \mathbb{Q} \setminus \{0\}$ , each  $i$ -good  $\beta \leq \alpha$ , and each  $m$  define

$$G_{A,B,F,m}^{\beta,i} = \cup((\mathcal{U}^{\beta,i})^{A,B})_{F,\mathcal{F}_m}.$$

As  $\mathcal{U}^{\beta,i}$  is a good cover of  $X_\alpha^i$ ,  $(\mathcal{U}^{\beta,i})^{A,B}$  is  $\omega$ -fat cover of  $L_\alpha^i$ , and by Lemma 10,  $G_{A,B,F,m}^{\beta,i}$  is comeager in  ${}^{\mathbb{N}}\mathbb{R}$ . Set

$$Y_\alpha = \bigcup_{\beta < \alpha} M_\beta \cup \bigcup \left\{ {}^{\mathbb{N}}\mathbb{R} \setminus G_{A,B,F,m}^{\beta,i} : \begin{array}{l} i < 2, \text{ } i\text{-good } \beta \leq \alpha, m \in \mathbb{N}, \\ \text{finite sets } F, B \subseteq L_\alpha^i, A \subseteq \mathbb{Q} \setminus \{0\} \end{array} \right\},$$

and  $Y_\alpha^* = \{x \in {}^{\mathbb{N}}\mathbb{R} : (\exists y \in Y_\alpha) x =^* y\}$  (where  $x =^* y$  means that  $x(n) = y(n)$  for all but finitely many  $n$ .) Then  $Y_\alpha^*$  is a union of less than  $\text{cov}(\mathcal{M})$  many meager sets.

**Lemma 14** ([4]). *If  $X$  is a union of less than  $\text{cov}(\mathcal{M})$  many meager sets in  ${}^{\mathbb{N}}\mathbb{R}$ , then for each  $x \in {}^{\mathbb{N}}\mathbb{R}$  there exist  $y, z \in {}^{\mathbb{N}}\mathbb{R} \setminus X$  such that  $y + z = x$ .*

Use Lemma 14 to pick  $x_\alpha^0, x_\alpha^1 \in {}^{\mathbb{N}}\mathbb{R} \setminus Y_\alpha^*$  such that  $x_\alpha^0 + x_\alpha^1 = y_\alpha$ . Let  $k = \alpha \bmod \omega$ , and change a finite initial segment of  $x_\alpha^0$  and  $x_\alpha^1$  so that they both become members of  $O_k$ . Then  $x_\alpha^0, x_\alpha^1 \in O_k \setminus Y_\alpha$ , and  $x_\alpha^0 + x_\alpha^1 =^* y_\alpha$ . Finally, define  $X_{\alpha+1}^i = L_\alpha^i + \mathbb{Q} \cdot x_\alpha^i$ . By Lemma 11, the inductive hypothesis is preserved. This completes the construction.

Take  $L_i = \bigcup_{\alpha < \mathfrak{c}} L_\alpha^i$ ,  $i = 1, 2$ . By Lemma 13, each  $L_i$  satisfies  $S_1(\mathcal{B}_{\text{good}}, \mathcal{B}_{\text{good}})$ , and by the construction, its intersection with each nonempty basic open set has size  $\mathfrak{c}$ . By Lemma 12,  $\mathcal{B}_{\text{good}} = \mathcal{B}_\Omega$  for  $L_i$ . Finally,  $L_0 + L_1$  (a homomorphic image of  $L_0 \times L_1$ ) is a dominating subset of  ${}^{\mathbb{N}}\mathbb{R}$ , thus  $L_0 \times L_1$  is not  $\mathfrak{o}$ -bounded.  $\square$

We do not know whether there exists a simple proof for the following consequence of Theorem 9. Say that a subset  $S$  of a topological group  $G$  is  $\aleph_0$ -bounding if there exists a countable set  $F \subseteq G$  such that  $F \cdot S = G$ . For example,  $G$  is  $\aleph_0$ -bounded if each nonempty open set in  $G$  is  $\aleph_0$ -bounding. The first property in the following corollary may be called *Borel  $\mathfrak{o}$ -boundedness*. This property is more interesting when the group in question is  $\aleph_0$ -bounded, in which case it is stronger than  $\mathfrak{o}$ -boundedness.

**Corollary 15.** *Assume that  $\text{cov}(\mathcal{M}) = \mathfrak{c}$ . Then there exists an  $\aleph_0$ -bounded topological group  $G$  of size continuum such that:*

- (1) *For each sequence  $\{B_n\}_{n \in \mathbb{N}}$  of  $\aleph_0$ -bounding Borel sets in  $G$ , there exists a sequence  $\{F_n\}_{n \in \mathbb{N}}$  of finite subsets of  $G$  such that  $G = \bigcup_n F_n \cdot B_n$ .*
- (2) *Moreover, the sequence in (1) will have the property that for each finite  $F \subseteq G$  there exists  $n$  such that  $F \subseteq F_n \cdot U_n$ .*

*Remark 16.* Banach, Nickolas, and Sanchis have also, independently, proved the consistency of  $\mathfrak{o}$ -bounded groups not being closed under taking finite products (however, they do not consider stronger combinatorial properties as done in [13] and here). Their construction uses ultrafilters which are not nearly coherent – see [2].

#### 4. GROUPS SATISFYING $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ OR $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$

One may wonder whether Theorem 9 can be strengthened further so that  $L_0$  and  $L_1$  will satisfy a stronger property. By inspection of the Scheepers Diagram 1, the only candidate for a stronger property (among the ones considered in this

paper) is  $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ . This is far from possible: A result of [1] (see [16, Theorem 32]) implies that whenever  $G_1$  is a topological group satisfying  $S_1(\Omega, \Gamma)$ , and  $G_2$  is  $\mathfrak{o}$ -bounded, the group  $G_1 \times G_2$  is  $\mathfrak{o}$ -bounded. However strong, though, the notion of a topological group satisfying  $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$  is not trivial.

**Theorem 17.** *For each cardinal  $\kappa$  with  $\text{cf}(\kappa) > \aleph_0$ , it is consistent that  $\mathfrak{c} = \kappa$  and there exists a topological subgroup of  $\mathbb{R}$  of size  $\mathfrak{c}$ , which satisfies  $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ .*

*Proof.* This is really a theorem of Miller: Let  $M$  be a countable standard model of ZFC satisfying  $\mathfrak{c} = \kappa$ . In [14] it is proved that there exists a ccc poset  $\mathbb{P}$  in  $M$  of size continuum (so that forcing with  $\mathbb{P}$  does not change the size of the continuum) such that the old reals  $M \cap \mathbb{R}$  satisfy  $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$  in  $V^\mathbb{P}$ .

But observe that, as the operations of addition and subtraction in  $\mathbb{R}$  are absolute,  $G = M \cap \mathbb{R}$  is a group in  $V^\mathbb{P}$ .  $\square$

We do not know to translate this result into a standard mathematical theorem.

**Problem 18.** *Does the Continuum Hypothesis imply the existence of a separable metrizable group of size  $\mathfrak{c}$  which satisfies  $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ ?*

We have some approximate results. For a sequence  $\{X_n\}_{n \in \mathbb{N}}$  of subsets of  $X$ , define  $\liminf X_n = \bigcup_m \bigcap_{n \geq m} X_n$ . For a family  $\mathcal{F}$  of subsets of  $X$ ,  $L(\mathcal{F})$  denotes its closure under the operation  $\liminf$ . According to [8],  $X$  is a  $\delta$ -set if for each  $\omega$ -cover  $\mathcal{U}$  of  $X$ ,  $X \in L(\mathcal{U})$ . It is easy to see that the  $\delta$ -property is preserved under taking countable *increasing* unions. Clearly  $S_1(\Omega, \Gamma)$  implies the  $\delta$ -property. The converse is still an open problem [19]. If a  $\delta$ -set is a group, we will call it a  $\delta$ -group.

Let  $\mathbb{Z}_2$  denote the usual group  $\{0, 1\}$  with modulo 2 addition.

**Theorem 19.** *Assume that  $\mathfrak{p} = \mathfrak{c}$ . Then there exists a subgroup  $G$  of  ${}^\mathbb{N}\mathbb{Z}_2$  such that for each  $k$   $G^k$  is a countable increasing union of sets satisfying  $S_1(\Omega, \Gamma)$ . In particular, all finite powers of  $G$  are  $\delta$ -groups.*

*Proof.* As  $\mathfrak{p} = \mathfrak{c}$ , there exists a subset  $X$  of  ${}^\mathbb{N}\mathbb{Z}_2$  of size continuum which satisfies  $S_1(\Omega, \Gamma)$  [7]. We may assume that  $0 \in X$ . Consequently,

$$G := \langle X \rangle = \bigcup_{n \in \mathbb{N}} \{x_1 + \cdots + x_n : x_1, \dots, x_n \in X\}$$

is a countable increasing union of continuous images of powers of  $X$ . But the property  $S_1(\Omega, \Gamma)$  is closed under taking finite powers and continuous images [11]. Observe that each finite power of  $G$  is the countable increasing union of the same power of the original sets.  $\square$

Assuming the Continuum Hypothesis, there exists a set of reals  $X$  of size  $\mathfrak{c}$  satisfying  $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$  (e.g., [14]). It is an open problem whether  $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$  is provably preserved under taking finite powers [19]. If it is, then we have a positive answer to Problem 18.

**Theorem 20.** *Assume that  $X \subseteq {}^\mathbb{N}\mathbb{Z}_2$  satisfies  $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$  in all finite powers. Then  $G = \langle X \rangle$  satisfies  $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$  in all finite powers.*

*Proof.* By the above arguments, it suffices to prove the following.

**Lemma 21.**  *$S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$  is preserved under taking countable increasing unions.*



To prove the lemma, assume that  $X = \bigcup_n X_n$  is an increasing union, and observe that  $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$  implies  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ , which in turn implies that all Borel images of each  $X_k$  in  ${}^\mathbb{N}\mathbb{N}$  are bounded [17].

Assume that  $\mathcal{U}_n = \{U_m^n : m \in \mathbb{N}\}$ ,  $n \in \mathbb{N}$ , are countable Borel  $\omega$ -covers of  $X$ . By  $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ , we may assume that each  $\mathcal{U}_n$  is a  $\gamma$ -cover of  $X_n$ . For each  $k$ , define a function  $\Psi_k$  from  $X_k$  to  ${}^\mathbb{N}\mathbb{N}$  so that for each  $x$  and  $n$ :

$$\Psi_k(x)(n) = \min\{i : (\forall m \geq i) \ x \in U_m^n\}.$$

Then  $\Psi_k$  is a Borel map, and so each  $\Psi_k[X]$  is bounded. Consequently,  $\bigcup_k \Psi_k[X]$  is bounded, say by the sequence  $m_n$ . Then  $\{U_{m_n}^n\}_{n \in \mathbb{N}}$  is a  $\gamma$ -cover of  $X$ , as required.  $\square$

We conclude the paper with a group satisfying  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$  in all finite powers. Let  $\mathcal{N}$  denote the collection of null (Lebesgue measure zero) sets of reals,  $\text{cov}(\mathcal{N})$  is the minimal size of a cover of  $\mathbb{R}$  by null sets, and  $\text{cof}(\mathcal{N})$  is the minimal size of a cofinal family in  $\mathcal{N}$  with respect to inclusion. Let  $\kappa$  be an uncountable cardinal.  $S \subseteq \mathbb{R}$  is a  $\kappa$ -Sierpiński set if  $|S| \geq \kappa$  and for each null set  $N$ ,  $|S \cap N| < \kappa$ .  $\mathfrak{b}$ -Sierpiński sets satisfy  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ , but powers of  $\kappa$ -Sierpiński sets are never  $\kappa$ -Sierpiński sets.

**Theorem 22.** *Assume that  $\text{cov}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \mathfrak{b}$ . Then there exists a (non  $\sigma$ -compact) group  $G \subseteq \mathbb{R}$  of size  $\mathfrak{b}$ , such that all finite powers of  $G$  satisfy  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ . Consequently, all finite powers of  $G$  also satisfy  $S_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ .*

*Proof.* The proof is similar to that of Theorem 7.2 of [6].

**Lemma 23.** *Assume that  $\text{cov}(\mathcal{N}) = \text{cof}(\mathcal{N})$ . Then there exists a  $\text{cov}(\mathcal{N})$ -Sierpiński set  $S$  such that for each  $k$  and each null set  $N$  in  $\mathbb{R}^k$ ,  $S^k \cap N$  is contained in a union of less than  $\text{cov}(\mathcal{N})$  many continuous images of  $S^{k-1}$ .*

*Proof.* Let  $\kappa = \text{cov}(\mathcal{N}) = \text{cof}(\mathcal{N})$ . Let  $\{N_\alpha^{(k)} : \alpha < \kappa\}$  be a cofinal family of null sets in  $\mathbb{R}^k$ ,  $k \in \mathbb{N}$ .

For  $J \subseteq \mathbb{R}^k$ ,  $x \in \mathbb{R}$ , and  $i < k$ , define  $J_{(x,i)} = \{v_1 \frown v_2 : \ell(v_1) = i, v_1 \frown x \frown v_2 \in J\}$ . By the Fubini Theorem, for each null set  $N \subseteq \mathbb{R}^k$  and  $i < k$ ,

$$\tilde{N} = \{x : (\exists i < k) \ N_{(x,i)} \text{ is not null in } \mathbb{R}^{k-1}\} \in \mathcal{N}.$$

We make an inductive construction on  $\alpha < \kappa$  of elements  $x_\alpha \in \mathbb{R}$  with auxiliary collections  $\mathcal{F}_\alpha$  of null sets, as follows. For  $\alpha < \kappa$  let  $\mathcal{P}_\alpha = \{N_\alpha^{(k)} : k \in \mathbb{N}\}$ . At step  $\alpha$  do the following:

- (1) Choose  $x_\alpha \notin \bigcup_{\beta < \alpha} \left( \bigcup_{N \in (\mathcal{P}_\beta \cup \mathcal{F}_\beta) \setminus P(\mathbb{R})} \tilde{N} \cup \bigcup_{N \in (\mathcal{P}_\beta \cup \mathcal{F}_\beta) \cap P(\mathbb{R})} N \right)$ .
- (2) Set  $\mathcal{F}_\alpha = \{N_{(x_\alpha, i)} : \beta < \alpha, N \in (\mathcal{P}_\beta \cup \mathcal{F}_\beta) \setminus P(\mathbb{R}), i \in \mathbb{N}\}$ .

This is possible because  $x_\alpha$  is required to avoid membership in a union of less than  $\text{cov}(\mathcal{N})$  many null sets.

Take  $S = \{x_\alpha : \alpha < \kappa\}$ . Then  $S$  is a  $\kappa$ -Sierpiński set. Fix  $k$ . For each null  $N \subseteq \mathbb{R}^k$ , there exists  $\beta < \kappa$  with  $N \subseteq N_\beta^{(k)}$ . Whenever  $\beta < \alpha_0 < \dots < \alpha_{k-1}$ , and  $\pi$  is a permutation on  $\{0, \dots, k-1\}$ ,  $(N_\beta^{(k)})_{(x_{\alpha_0}, \pi^{-1}(0))} \in \mathcal{F}_{\alpha_0}$ , thus  $(N_\beta^{(k)})_{(x_{\alpha_0}, \pi^{-1}(0)), (x_{\alpha_1}, \pi^{-1}(1))} \in \mathcal{F}_{\alpha_1}, \dots, (N_\beta^{(k)})_{(x_{\alpha_0}, \pi^{-1}(0)), \dots, (x_{\alpha_{k-2}}, \pi^{-1}(k-2))} \in \mathcal{F}_{\alpha_{k-2}}$ , thus  $x_{\alpha_{k-1}} \notin (N_\beta^{(k)})_{(x_{\alpha_0}, \pi^{-1}(0)), \dots, (x_{\alpha_{k-2}}, \pi^{-1}(k-2))}$ , that is,  $(x_{\alpha_{\pi(0)}}, \dots, x_{\alpha_{\pi(k-1)}}) \notin N_\beta^{(k)}$ .

Consequently,  $S^k \cap N$  is contained in the union of all sets of the form  $S^i \times \{x_\xi : \xi \leq \beta\} \times S^{k-i-1}$  ( $i < k$ )—a union of  $|\beta| < \kappa$  copies of  $S^{k-1}$ —and  $\{v \in S^k : v_i = v_j\}$  ( $i < j < k$ ), which are continuous images of  $S^{k-1}$ .  $\square$

Now assume that  $\text{cov}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \mathfrak{b}$ , and let  $S$  be a  $\mathfrak{b}$ -Sierpiński set as in Lemma 23. We will show by induction that for each  $k$ ,  $S^k$  satisfies  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ . By [17] it is enough to show that for each null  $N \subseteq \mathbb{R}^k$ ,  $S^k \cap N$  satisfies  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ . By Lemma 23 and the induction hypothesis,  $S^k \cap N$  is contained in a union of less than  $\mathfrak{b}$  many sets satisfying  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ . In [4, full version] it is shown that  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$  is preserved under taking unions of size less than  $\mathfrak{b}$ , and in [5] it is shown that  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$  is preserved under taking subsets. This proves the assertion.

So all finite powers of  $S$  satisfy  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ , and  $G = \langle S \rangle$  works, since as we mentioned before,  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$  is preserved under taking countable unions.

Finally, by [17]  $S_1(\mathcal{B}_\Gamma, \mathcal{B})$  in all finite powers implies  $S_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ .  $\square$

It seems that the following was not known before.

**Corollary 24.** *Assume that  $\text{cov}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \mathfrak{b}$ . Then there exists a set of reals satisfying  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$  and  $S_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ , but not  $S_1(\mathcal{O}, \mathcal{O})$ .*

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