

TOPOLOGICAL DIAGONALIZATIONS AND HAUSDORFF DIMENSION

TOMASZ WEISS AND BOAZ TSABAN

ABSTRACT. The Hausdorff dimension of a product $X \times Y$ can be strictly greater than that of Y , even when the Hausdorff dimension of X is zero. But when X is countable, the Hausdorff dimensions of Y and $X \times Y$ are the same. Diagonalizations of covers define a natural hierarchy of properties which are weaker than “being countable” and stronger than “having Hausdorff dimension zero”. Fremlin asked whether it is enough for X to have the strongest property in this hierarchy (namely, being a γ -set) in order to assure that the Hausdorff dimensions of Y and $X \times Y$ are the same.

We give a negative answer: Assuming the Continuum Hypothesis, there exists a γ -set $X \subseteq \mathbb{R}$ and a set $Y \subseteq \mathbb{R}$ with Hausdorff dimension zero, such that the Hausdorff dimension of $X+Y$ (a Lipschitz image of $X \times Y$) is maximal, that is, 1. However, we show that for the notion of a *strong* γ -set the answer is positive. Some related problems remain open.

1. INTRODUCTION

The Hausdorff dimension of a subset of \mathbb{R}^k is a derivative of the notion of Hausdorff *measures* [4]. However, for our purposes it will be more convenient to use the following equivalent definition. Denote the diameter of a subset A of \mathbb{R}^k by $\text{diam}(A)$. The *Hausdorff dimension* of a set $X \subseteq \mathbb{R}^k$, $\dim(X)$, is the infimum of all positive δ such that for each positive ϵ there exists a cover $\{I_n\}_{n \in \mathbb{N}}$ of X with

$$\sum_{n \in \mathbb{N}} \text{diam}(I_n)^\delta < \epsilon.$$

From the many properties of Hausdorff dimension, we will need the following easy ones.

Lemma 1.

- (1) If $X \subseteq Y \subseteq \mathbb{R}^k$, then $\dim(X) \leq \dim(Y)$.
- (2) Assume that X_1, X_2, \dots are subsets of \mathbb{R}^k such that $\dim(X_n) = \delta$ for each n . Then $\dim(\bigcup_n X_n) = \delta$.
- (3) Assume that $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^m$ is such that there exists a Lipschitz surjection $\phi : X \rightarrow Y$. Then $\dim(X) \geq \dim(Y)$.
- (4) For each $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^m$, $\dim(X \times Y) \geq \dim(X) + \dim(Y)$.

Equality need not hold in item (4) of the last lemma. In particular, one can construct a set X with Hausdorff dimension zero and a set Y such that $\dim(X \times$

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$Y) > \dim(Y)$. On the other hand, when X is countable, $X \times Y$ is a union of countably many copies of Y , and therefore

$$(1) \quad \dim(X \times Y) = \dim(Y).$$

Having Hausdorff dimension zero can be thought of as a notion of smallness. Being countable is another notion of smallness, and we know that the first notion is not enough restrictive in order to have Equation 1 hold, but the second is.

Notions of smallness for sets of real numbers have a long history and many applications – see, e.g., [11]. We will consider some notions which are weaker than being countable and stronger than having Hausdorff dimension zero.

According to Borel [3], a set $X \subseteq \mathbb{R}^k$ has *strong measure zero* if for each sequence of positive reals $\{\epsilon_n\}_{n \in \mathbb{N}}$, there exists a cover $\{I_n\}_{n \in \mathbb{N}}$ of X such that $\text{diam}(I_n) < \epsilon_n$ for all n . Clearly strong measure zero implies Hausdorff dimension zero. It does not require any special assumptions in order to see that the converse is false. A perfect set can be mapped onto the unit interval by a uniformly continuous function and therefore cannot have strong measure zero.

Proposition 2 (folklore). *There exists a perfect set of reals X with Hausdorff dimension zero.*

Proof. For $0 < \lambda < 1$, denote by $C(\lambda)$ the Cantor set obtained by starting with the unit interval, and at each step removing from the middle of each interval a subinterval of size λ times the size of the interval (So that $C(1/3)$ is the canonical middle-third Cantor set, which has Hausdorff dimension $\log 2 / \log 3$.) It is easy to see that if $\lambda_n \nearrow 1$, then $\dim(C(\lambda_n)) \searrow 0$.

Thus, define a special Cantor set $C(\{\lambda_n\}_{n \in \mathbb{N}})$ by starting with the unit interval, and at step n removing from the middle of each interval a subinterval of size λ_n times the size of the interval. For each n , $C(\{\lambda_n\}_{n \in \mathbb{N}})$ is contained in a union of 2^n (shrunk) copies of $C(\lambda_n)$, and therefore $\dim(C(\{\lambda_n\}_{n \in \mathbb{N}})) \leq \dim(C(\lambda_n))$. \square

As every countable set has strong measure zero, the latter notion can be thought of an “approximation” of countability. In fact, Borel conjectured in [3] that every strong measure zero set is countable, and it turns out that the usual axioms of mathematics (ZFC) are not strong enough to prove or disprove this conjecture: Assuming the Continuum Hypothesis there exists an uncountable strong measure zero set (namely, a Luzin set), but Laver [10] proved that one cannot prove the existence of such an object from the usual axioms of mathematics.

The property of strong measure zero (which depends on the metric) has a natural topological counterpart. A topological space X has *Rothberger’s property C''* [13] if for each sequence $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ of covers of X there is a sequence $\{U_n\}_{n \in \mathbb{N}}$ such that for each n $U_n \in \mathcal{U}_n$, and $\{U_n\}_{n \in \mathbb{N}}$ is a cover of X . Using Scheepers’ notation [15], this property is a particular instance of the following selection hypothesis (where \mathfrak{U} and \mathfrak{V} are any collections of covers of X):

$S_1(\mathfrak{U}, \mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ of members of \mathfrak{U} , there is a sequence $\{U_n\}_{n \in \mathbb{N}}$ such that $U_n \in \mathcal{U}_n$ for each n , and $\{U_n\}_{n \in \mathbb{N}} \in \mathfrak{V}$.

Let \mathcal{O} denote the collection of all open covers of X . Then the property considered by Rothberger is $S_1(\mathcal{O}, \mathcal{O})$. Fremlin and Miller [5] proved that a set $X \subseteq \mathbb{R}^k$ satisfies $S_1(\mathcal{O}, \mathcal{O})$ if, and only if, X has strong measure zero with respect to each metric which generates the standard topology on \mathbb{R}^k .

But even Rothberger's property for X is not strong enough to have Equation 1 hold: It is well-known that every Luzin set satisfies Rothberger's property (and, in particular, has Hausdorff dimension zero).

Lemma 3. *The mapping $(x, y) \mapsto x + y$ from \mathbb{R}^2 to \mathbb{R} is Lipschitz.*

Proof. Observe that for nonnegative reals a and b , $(a - b)^2 \geq 0$ and therefore $a^2 + b^2 \geq 2ab$. Consequently,

$$a + b = \sqrt{a^2 + 2ab + b^2} \leq \sqrt{2(a^2 + b^2)} = \sqrt{2}\sqrt{a^2 + b^2}.$$

Thus, $|(x_1 + y_1) - (x_2 + y_2)| \leq \sqrt{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. \square

Assuming the Continuum Hypothesis, there exists a Luzin set $L \subseteq \mathbb{R}$ such that $L + L$, a Lipschitz image of $L \times L$, is equal to \mathbb{R} [9].

We therefore consider some stronger properties. An open cover \mathcal{U} of X is an ω -cover of X if each finite subset of X is contained in some member of the cover, but X is not contained in any member of \mathcal{U} . \mathcal{U} is a γ -cover of X if it is infinite, and each element of X belongs to all but finitely many members of \mathcal{U} . Let Ω and Γ denote the collections of open ω -covers and γ -covers of X , respectively. Then $\Gamma \subseteq \Omega \subseteq \mathcal{O}$, and these three classes of covers introduce 9 properties of the form $S_1(\mathfrak{U}, \mathfrak{V})$. If we remove the trivial ones and check for equivalences [9, 19], then it turns out that only six of these properties are really distinct, and only three of them imply Hausdorff dimension zero:

$$S_1(\Omega, \Gamma) \rightarrow S_1(\Omega, \Omega) \rightarrow S_1(\mathcal{O}, \mathcal{O}).$$

The properties $S_1(\Omega, \Gamma)$ and $S_1(\Omega, \Omega)$ were also studied before. $S_1(\Omega, \Omega)$ was studied by Sakai [14], and $S_1(\Omega, \Gamma)$ was studied by Gerlits and Nagy in [8]: A topological space X is a γ -set if each ω -cover of X contains a γ -cover of X . Gerlits and Nagy proved that X is a γ -set if, and only if, X satisfies $S_1(\Omega, \Gamma)$. It is not difficult to see that every countable space is a γ -set. But this property is not trivial: Assuming the Continuum Hypothesis, there exist uncountable γ -sets [7].

$S_1(\Omega, \Omega)$ is closed under taking finite powers [9], thus the Luzin set we used to see that Equation 1 need not hold when X satisfies $S_1(\mathcal{O}, \mathcal{O})$ does not rule out that possibility that this Equation holds when X satisfies $S_1(\Omega, \Omega)$. However, in [2] it is shown that assuming the Continuum Hypothesis, there exist Luzin sets L_0 and L_1 satisfying $S_1(\Omega, \Omega)$, such that $L_0 + L_1 = \mathbb{R}$. Thus, the only remaining candidate for a nontrivial property of X where Equation 1 holds is $S_1(\Omega, \Gamma)$ (γ -sets). Fremlin (personal communication) asked whether Equation 1 is indeed provable in this case. We give a negative answer, but show that for a yet stricter (but nontrivial) property which was considered in the literature, the answer is positive.

The notion of a strong γ -set was introduced in [7]. However, we will adopt the following simple characterization from [19] as our formal definition. Assume that $\{\mathfrak{U}_n\}_{n \in \mathbb{N}}$ is a sequence of collections of covers of a space X , and that \mathfrak{V} is a collection of covers of X . Define the following selection hypothesis.

$S_1(\{\mathfrak{U}_n\}_{n \in \mathbb{N}}, \mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ where $\mathcal{U}_n \in \mathfrak{U}_n$ for each n , there is a sequence $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ such that $\mathcal{U}_n \in \mathfrak{U}_n$ for each n , and $\{\mathcal{U}_n\}_{n \in \mathbb{N}} \in \mathfrak{V}$.

A cover \mathcal{U} of a space X is an n -cover if each n -element subset of X is contained in some member of \mathcal{U} . For each n denote by \mathcal{O}_n the collection of all open n -covers of a space X . Then X is a *strong γ -set* if X satisfies $S_1(\{\mathcal{O}_n\}_{n \in \mathbb{N}}, \Gamma)$.

In most cases $S_1(\{\mathcal{O}_n\}_{n \in \mathbb{N}}, \mathfrak{V})$ is equivalent to $S_1(\Omega, \mathfrak{V})$ [19], but not in the case $\mathfrak{V} = \Gamma$: It is known that for a strong γ -set $G \subseteq \{0, 1\}^{\mathbb{N}}$ and each $A \subseteq \{0, 1\}^{\mathbb{N}}$ of measure zero, $G \oplus A$ has measure zero too [7]; this can be contrasted with Theorem 5 below. In Section 3 we show that Equation 1 is provable in the case that X is a strong γ -set, establishing another difference between the notions of γ -sets and strong γ -sets, and giving a positive answer to Fremlin's question under a stronger assumption on X .

2. THE PRODUCT OF A γ -SET AND A SET OF HAUSDORFF DIMENSION ZERO

Theorem 4. *Assuming the Continuum Hypothesis (or just $\mathfrak{p} = \mathfrak{c}$), there exist a γ -set $X \subseteq \mathbb{R}$ and a set $Y \subseteq \mathbb{R}$ with Hausdorff dimension zero such that the Hausdorff dimension of the algebraic sum*

$$X + Y = \{x + y : x \in X, y \in Y\}$$

(a Lipschitz image of $X \times Y$ in \mathbb{R}) is 1. In particular, $\dim(X \times Y) \geq 1$.

Our theorem will follow from the following related theorem. This theorem involves the *Cantor space* $\{0, 1\}^{\mathbb{N}}$ of infinite binary sequences. The Cantor space is equipped with the product topology and with the product measure.

Theorem 5 (Bartoszyński and Reclaw [1]). *Assume the Continuum Hypothesis (or just $\mathfrak{p} = \mathfrak{c}$). Fix an increasing sequence $\{k_n\}_{n \in \mathbb{N}}$ of natural numbers, and for each n define*

$$A_n = \{f \in \{0, 1\}^{\mathbb{N}} : f \upharpoonright [k_n, k_{n+1}) \equiv 0\}.$$

If the set

$$A = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} A_n$$

has measure zero, then there exists a γ -set $G \subseteq \{0, 1\}^{\mathbb{N}}$ such that the algebraic sum $G \oplus A$ is equal to $\{0, 1\}^{\mathbb{N}}$ (where \oplus denotes the modulo 2 coordinatewise addition).

Observe that the assumption in Theorem 5 holds whenever $\sum_n 2^{-(k_{n+1} - k_n)}$ converges.

Lemma 6. *There exists an increasing sequence of natural numbers $\{k_n\}_{n \in \mathbb{N}}$ such that $\sum_n 2^{-(k_{n+1} - k_n)}$ converges, and such that for the sequence $\{B_n\}_{n \in \mathbb{N}}$ defined by*

$$B_n = \left\{ \sum_{i \in \mathbb{N}} \frac{f(i)}{2^{i+1}} : f \in \{-1, 0, 1\}^{\mathbb{N}} \text{ and } f \upharpoonright [k_n, k_{n+1}) \equiv 0 \right\}$$

for each n , the set

$$Y = \bigcap_{m \in \omega} \bigcup_{n \geq m} B_n$$

has Hausdorff dimension zero.

Proof. Fix a sequence p_n of positive reals which converges to 0. Let $k_0 = 0$. Given k_n find k_{n+1} satisfying

$$3^{k_n} \cdot \frac{1}{2^{p_n(k_{n+1}-2)}} \leq \frac{1}{2^n}.$$

Clearly, every B_n is contained in a union of 3^{k_n} intervals such that each of the intervals has diameter $1/2^{k_{n+1}-2}$. For each positive δ and ϵ , choose m such that

$\sum_{n \geq m} 1/2^n < \epsilon$ and such that $p_n < \delta$ for all $n \geq m$. Now, Y is a subset of $\bigcup_{n \geq m} B_n$, and

$$\sum_{n \geq m} 3^{k_n} \left(\frac{1}{2^{k_{n+1}-2}} \right)^\delta < \sum_{n \geq m} 3^{k_n} \left(\frac{1}{2^{k_{n+1}-2}} \right)^{p_n} < \sum_{n \geq m} \frac{1}{2^n} < \epsilon.$$

Thus, the Hausdorff dimension of Y is zero. \square

The following lemma concludes the proof of Theorem 4.

Lemma 7. *There exists a γ -set $X \subseteq \mathbb{R}$ and set $Y \subseteq \mathbb{R}$ with Hausdorff dimension zero such that $X + Y = \mathbb{R}$. In particular, $\dim(X + Y) = 1$.*

Proof. Choose a sequence $\{k_n\}_{n \in \mathbb{N}}$ and a set Y as in Lemma 6. Then $\sum_n 2^{-(k_{n+1}-k_n)}$ converges, and the corresponding set A defined in Theorem 5 has measure zero. Thus, there exists a γ -set G such that $G \oplus A = \{0, 1\}^\mathbb{N}$.

Define $\Phi : \{0, 1\}^\mathbb{N} \rightarrow \mathbb{R}$ by

$$\Phi(f) = \sum_{i \in \mathbb{N}} \frac{f(i)}{2^{i+1}}.$$

As Φ is continuous, $X = \Phi[G]$ is a γ -set of reals. Assume that z is a member of the interval $[0, 1]$, let $f \in \{0, 1\}^\mathbb{N}$ be such that $z = \sum_i f(i)/2^{i+1}$. Then $f = g \oplus a$ for appropriate $g \in G$ and $a \in A$. Define $h \in \{0, 1\}^\mathbb{N}$ by $h(i) = f(i) - g(i)$. For infinitely many n , $a \upharpoonright [k_n, k_{n+1}) \equiv 0$ and therefore $f \upharpoonright [k_n, k_{n+1}) \equiv g \upharpoonright [k_n, k_{n+1})$, that is, $h \upharpoonright [k_n, k_{n+1}) \equiv 0$ for infinitely many n . Thus, $y = \sum_i h(i)/2^{i+1} \in Y$, and for $x = \Phi(g)$,

$$x + y = \sum_{i \in \mathbb{N}} \frac{g(i)}{2^{i+1}} + \sum_{i \in \mathbb{N}} \frac{h(i)}{2^{i+1}} = \sum_{i \in \mathbb{N}} \frac{g(i) + h(i)}{2^{i+1}} = \sum_{i \in \mathbb{N}} \frac{f(i)}{2^{i+1}} = z.$$

This shows that $[0, 1] \subseteq X + Y$. Consequently, $X + (Y + \mathbb{Q}) = (X + Y) + \mathbb{Q} = \mathbb{R}$. Now, observe that $Y + \mathbb{Q}$ has Hausdorff dimension zero since Y has. \square

3. THE PRODUCT OF A STRONG γ -SET AND A SET OF HAUSDORFF DIMENSION ZERO

Theorem 8. *Assume that $X \subseteq \mathbb{R}^k$ is a strong γ -set. Then for each $Y \subseteq \mathbb{R}^l$, $\dim(X \times Y) = \dim(Y)$.*

Proof. The proof for this is similar to that of Theorem 7 in [7].

It is enough to show that $\dim(X \times Y) \leq \dim(Y)$.

Lemma 9. *Assume that $Y \subseteq \mathbb{R}^l$ is such that $\dim(Y) < \delta$. Then for each positive ϵ there exists a large cover $\{I_n\}_{n \in \mathbb{N}}$ of Y (i.e., such that each $y \in Y$ is a member of infinitely many sets I_n) such that $\sum_n \text{diam}(I_n)^\delta < \epsilon$.*

Proof. For each m choose a cover $\{I_n^m\}_{n \in \mathbb{N}}$ of Y such that $\sum_n \text{diam}(I_n^m)^\delta < \epsilon/2^m$. Then $\{I_n^m : m, n \in \mathbb{N}\}$ is a large cover of Y , and $\sum_{m,n} \text{diam}(I_n^m)^\delta < \sum_n \epsilon/2^m = \epsilon$. \square

Lemma 10. *Assume that $Y \subseteq \mathbb{R}^l$ is such that $\dim(Y) < \delta$. Then for each sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ of positive reals there exists a large cover $\{A_n\}_{n \in \mathbb{N}}$ of Y such that for each n A_n is a union of finitely many sets, $I_1^n, \dots, I_{m_n}^n$, such that $\sum_j \text{diam}(I_j^n)^\delta < \epsilon_n$.*

Proof. Assume that $\{\epsilon_n\}_{n \in \mathbb{N}}$ is a sequence of positive reals. By Lemma 9, there exists a large cover $\{I_n\}_{n \in \mathbb{N}}$ of Y such that $\sum_n \text{diam}(I_n)^\delta < \epsilon_1$. For each n let $k_n = \min\{m : \sum_{j \geq m} \text{diam}(I_j)^\delta < \epsilon_n\}$. Take

$$A_n = \bigcup_{j=k_n}^{k_{n+1}-1} I_j.$$

□

Fix $\delta > \dim(Y)$ and $\epsilon > 0$. Choose a sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ of positive reals such that $\sum_n 2n\epsilon_n < \epsilon$, and use Lemma 10 to get the corresponding large cover $\{A_n\}_{n \in \mathbb{N}}$.

For each n we define an n -cover \mathcal{U}_n of X as follows. Let F be an n -element subset of X . For each $x \in F$, find an open interval I_x such that $x \in I_x$ and

$$\sum_{j=1}^{m_n} \text{diam}(I_x \times I_j^n)^\delta < 2\epsilon_n.$$

Let $U_F = \bigcup_{x \in F} I_x$. Set

$$\mathcal{U}_n = \{U_F : F \text{ is an } n\text{-element subset of } X\}.$$

As X is a strong γ -set, there exist elements $U_{F_n} \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_{F_n}\}_{n \in \mathbb{N}}$ is a γ -cover of X . Consequently,

$$X \times Y \subseteq \bigcup_{n \in \mathbb{N}} (U_{F_n} \times A_n) \subseteq \bigcup_{n \in \mathbb{N}} \bigcup_{x \in F_n} \bigcup_{j=1}^{m_n} I_x \times I_j^n$$

and

$$\sum_{n \in \mathbb{N}} \sum_{x \in F_n} \sum_{j=1}^{m_n} \text{diam}(I_x \times I_j^n)^\delta < \sum_n n \cdot 2\epsilon_n < \epsilon.$$

□

4. OPEN PROBLEMS

There are ways to strengthen the notion of γ -sets other than moving to strong γ -sets. Let \mathcal{B}_Ω and \mathcal{B}_Γ denote the collections of *countable Borel* ω -covers and γ -covers of X , respectively. As every open ω -cover of a set of reals contains a countable ω -subcover [9], we have that $\Omega \subseteq \mathcal{B}_\Omega$ and therefore $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ implies $S_1(\Omega, \Gamma)$. The converse is not true [16].

Problem 11. Assume that $X \subseteq \mathbb{R}$ satisfies $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$. Is it true that for each $Y \subseteq \mathbb{R}$, $\dim(X \times Y) = \dim(Y)$?

We conjecture that assuming the Continuum Hypothesis, the answer to this problem is negative. We therefore introduce the following problem. For infinite sets of natural numbers A, B , we write $A \subseteq^* B$ if $A \setminus B$ is finite. Assume that \mathcal{F} is a family of infinite sets of natural numbers. A set P is a *pseudointersection* of \mathcal{F} if it is infinite, and for each $B \in \mathcal{F}$, $A \subseteq^* B$. \mathcal{F} is *centered* if each finite subcollection of \mathcal{F} has a pseudointersection. Let \mathfrak{p} denote the minimal cardinality of a centered family which does not have a pseudointersection. In [16] it is proved that \mathfrak{p} is also the minimal cardinality of a set of reals which does not satisfy $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$.

Problem 12. Assume that the cardinality of X is smaller than \mathfrak{p} . Is it true that for each $Y \subseteq \mathbb{R}$, $\dim(X \times Y) = \dim(Y)$?

Another interesting open problem involves the following notion [17, 18]. A cover \mathcal{U} of X is a τ -cover of X if it is a large cover, and for each $x, y \in X$, one of the sets $\{U \in \mathcal{U} : x \in U \text{ and } y \notin U\}$ or $\{U \in \mathcal{U} : y \in U \text{ and } x \notin U\}$ is finite. Let \mathbf{T} denote the collection of open τ -covers of X . Then $\Gamma \subseteq \mathbf{T} \subseteq \Omega$, therefore $S_1(\{\mathcal{O}_n\}_{n \in \mathbb{N}}, \Gamma)$ implies $S_1(\{\mathcal{O}_n\}_{n \in \mathbb{N}}, \mathbf{T})$.

Problem 13. *Assume that $X \subseteq \mathbb{R}$ satisfies $S_1(\{\mathcal{O}_n\}_{n \in \mathbb{N}}, \mathbf{T})$. Is it true that for each $Y \subseteq \mathbb{R}$, $\dim(X \times Y) = \dim(Y)$?*

It is conjectured that $S_1(\{\mathcal{O}_n\}_{n \in \mathbb{N}}, \mathbf{T})$ is strictly stronger than $S_1(\Omega, \mathbf{T})$ [19]. If this conjecture is false, then the results in this paper imply a negative answer to Problem 13.

Another type of problems is the following: We have seen that the assumption that X is a γ -set and Y has Hausdorff dimension zero is not enough in order to prove that $X \times Y$ has Hausdorff dimension zero. We also saw that if X satisfies a stronger property (strong γ -set), then $\dim(X \times Y) = \dim(Y)$ for all Y . Another approach to get a positive answer would be to strengthen the assumption on Y rather than X .

Assume that X, Y are γ -sets. Does it necessarily follow that $\dim(X \times Y) = 0$? What about the case that (X is a γ -set, and) Y satisfies the Sakai property $S_1(\Omega, \Omega)$, or Rothberger's property $S_1(\mathcal{O}, \mathcal{O})$, or just strong measure zero? An argument similar to that of Theorem 8 shows that if X is a strong γ -set we get a positive answer.

Theorem 14. *Assume that $X \subseteq \mathbb{R}^k$ is a strong γ -set. Then for each strong measure zero set $Y \subseteq \mathbb{R}^l$, $X \times Y$ has strong measure zero. In particular, the Hausdorff dimension of $X \times Y$ is zero.*

Proof. Assume that $\{\epsilon_n\}_{n \in \mathbb{N}}$ is a sequence of positive reals. For each n let $k_n = 1 + \dots + (n - 1)$, and define $\tilde{\epsilon}_n = \min\{\epsilon_i : i < k_{n+1}\}/2$. As Y has strong measure zero, there exists a large cover $\{I_n\}_{n \in \mathbb{N}}$ of Y such that for each n , $\text{diam}(I_n) < \tilde{\epsilon}_n$.

For each n define an n -cover \mathcal{U}_n of X as follows. Let F be an n -element subset of X . For each $x \in F$, find an open interval I_x such that $x \in I_x$ and $\text{diam}(I_x \times I_n) < 2\tilde{\epsilon}_n$. Let $U_F = \bigcup_{x \in F} I_x$. Set

$$\mathcal{U}_n = \{U_F : F \text{ is an } n\text{-element subset of } X\}$$

and choose elements $U_{F_n} \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_{F_n}\}_{n \in \mathbb{N}}$ is a γ -cover of X . Then

$$X \times Y \subseteq \bigcup_{n \in \mathbb{N}} (U_{F_n} \times I_n) \subseteq \bigcup_{n \in \mathbb{N}} \bigcup_{x \in F_n} I_x \times I_n$$

Enumerate each F_n as $\{x_0^n, \dots, x_{n-1}^n\}$, and define a cover $\{J_n\}_{n \in \mathbb{N}}$ of $X \times Y$ by $J_{k_n+i} = I_{x_i^n} \times I_n$ for each n and $i < n$. For each natural number N , let n be such that $N = k_n + i$ for some $i < n$. Then

$$\text{diam}(J_N) = \text{diam}(I_{x_i^n} \times I_n) < 2\tilde{\epsilon}_n \leq \epsilon_{k_n+i} = \epsilon_N.$$

□

In fact, the corresponding result for γ -sets also holds; we plan to treat this problem in [20].

REFERENCES

- [1] T. Bartoszyński and I. Reclaw, *Not every γ -set is strongly meager*, Contemporary Mathematics **192** (1996), 25–29.
- [2] T. Bartoszyński, S. Shelah, and B. Tsaban, *Additivity properties of topological diagonalizations*, The Journal of Symbolic Logic **68** (2003), 1254–1260. (Full version: <http://arxiv.org/abs/math.LO/0112262>)
- [3] É. Borel, *Sur la classification des ensembles de mesure nulle*, Bulletin de la Société Mathématique de France **47** (1919), 97–125.
- [4] K. Falconer, *The geometry of fractal sets*, Cambridge University Press, 1990.
- [5] D.H. Fremlin and A.W. Miller, *On some properties of Hurewicz, Menger and Rothberger*, Fundamenta Mathematica **129** (1988), 17–33.
- [6] F. Galvin, J. Mycielski, and R. Solovay, *Strong measure zero sets*, Notices of the American Mathematical Society (1973), A–280.
- [7] F. Galvin and A.W. Miller, *γ -sets and other singular sets of real numbers*, Topology and its Applications **17** (1984), 145–155.
- [8] J. Gerlits and Zs. Nagy, *Some properties of $C(X)$, I*, Topology and its applications **14** (1982), 151–161.
- [9] W. Just, A. W. Miller, M. Scheepers, and P. Szeptycki, *Combinatorics of open covers II*, Topology and Its Applications, **73** (1996), 241–266.
- [10] R. Laver, *On the consistency of Borel’s conjecture*, Acta Mathematica **137** (1976), 151–169.
- [11] A.W. Miller, *Special subsets of the real line*, in: **Handbook of Set Theoretic Topology** (eds. K. Kunen and J.E. Vaughan), 201–233, North Holland, Amsterdam: 1984.
- [12] A. Nowik, M. Scheepers, and T. Weiss, *The algebraic sum of sets of real numbers with strong measure zero sets*, The Journal of Symbolic Logic **63** (1998), 301–324.
- [13] F. Rothberger, *Sur des familles indénombrables de suites de nombres naturels, et les problèmes concernant la propriété C* , Proceedings of the Cambridge Philosophical Society **37** (1941), 109–126.
- [14] M. Sakai, *Property C'' and function spaces*, Proceedings of the American Mathematical Society **104** (1988), 917–919.
- [15] M. Scheepers, *Combinatorics of open covers I: Ramsey Theory*, Topology and its Applications **69** (1996), 31–62.
- [16] M. Scheepers and B. Tsaban, *The combinatorics of Borel covers*, Topology and its Applications **121** (2002), 357–382.
<http://arxiv.org/abs/math.GN/0302322>
- [17] B. Tsaban, *A topological interpretation of \mathfrak{t}* , Real Analysis Exchange **25** (1999/2000), 391–404.
<http://arxiv.org/abs/math.LO/9705209>
- [18] B. Tsaban, *Selection principles and the minimal tower problem*, Note di Matematica, to appear.
<http://arxiv.org/abs/math.LO/0105045>
- [19] B. Tsaban, *Strong γ -sets and other singular spaces*, submitted.
<http://arxiv.org/abs/math.LO/0208057>
- [20] B. Tsaban and T. Weiss, *Products of special sets of real numbers*, submitted.

INSTITUTE OF MATHEMATICS, AKADEMIA PODLASKA 08-119 SIEDLCE, POLAND
E-mail address: tomaszweiss@go2.pl

EINSTEIN INSTITUTE OF MATHEMATICS, HEBREW UNIVERSITY OF JERUSALEM, GIVAT RAM,
 JERUSALEM 91904, ISRAEL
E-mail address: tsaban@math.huji.ac.il, <http://www.cs.biu.ac.il/~tsaban>