EXISTENCE OF TRAVELLING WAVES IN DISCRETE SINE-GORDON RINGS

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Abstract. We prove existence results for travelling waves in discrete, damped, dc-driven sine-Gordon equations with periodic boundary conditions.

1. Introduction

The damped, dc-driven discrete sine-Gordon equation, known also as the driven Frenkel-Kontorova model, with periodic boundary conditions, arises as a model of many physical systems, including circular arrays of Josephson junctions, the motions of disclocations in a crystal, the adsorbate layer on on the surface of a crystal, ionic conductors, glassy materials, charge-density wave transport, sliding friction, as well as the mechanical interpretation as a model for a ring of pendula coupled by torsional springs (we refer to [7, 8, 9] and references therein). This model has thus become a fundamental one for nonlinear physics, and been the subject of many theoretical, numerical and experimental studies. The system of equations is

$$\phi_j'' + \Gamma\phi_j' + \sin(\phi_j) = F + K[\phi_{j+1} - 2\phi_j + \phi_{j-1}], \quad \forall j \in \mathbb{Z} \tag{1}$$

with the parameters $\Gamma > 0, K > 0, F > 0$, with the periodic boundary-condition

$$\phi_{j+n}(t) = \phi_j(t) + 2\pi m \quad \forall j \in \mathbb{Z} \tag{2}$$

where $m \geq 1$ (we note that in view of the boundary conditions we are really dealing with an n-dimensional system of ODE's rather than an infinite-dimensional one). In numerical simulations, as well as in experimental work on systems modelled by (1),(2), it is observed that solutions often converge to a travelling wave: a solution satisfying

$$\phi_j(t) = f\Big(t + j\frac{m}{n}T\Big), \tag{3}$$

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where the waveform $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying

$$f(t+T) = f(t) + 2\pi \quad \forall t \in \mathbb{R}. \tag{4}$$

However, as has been pointed out in [8], even the *existence* of such a solution, has not been proven, except for the case of small K in which existence of a travelling wave for some values of F had been proven in [3].

In the 'super-damped' case, in which the second-derivative term in (1) is removed, there are very satisfying results about existence and also global stability of travelling waves ([1], theorem 2). Such results rely strongly on monotonicity arguments. Recently Baesens and Mackay [2] have managed to extend these arguments to the 'overdamped' of (1): their result applies when

$$\Gamma > 2\sqrt{2K+1}$$

and says that whenever (1),(2) does not have stationary solutions, there exists a travelling-wave solution which is globally stable. We do not know whether in general the non-existence of stationary solutions implies the existence of a travelling wave.

We note that a function f is a waveform if and only if it satisfies (4) and

$$f''(t) + \Gamma f'(t) + \sin(f(t)) = F + K\Big[f\Big(t + \frac{m}{n}T\Big) - 2f(t) + f\Big(t - \frac{m}{n}T\Big)\Big]. \tag{5}$$

Here we obtain two existence results for travelling-waves under conditions not covered by the existing work, described above.

Theorem 1. *For any $F > 1$ there exists a travelling-wave solution of (1), (2).*

Theorem 2. *Assume that n does not divide m. Fixing any $\tilde{F} > 0$, for K sufficiently large there exists a travelling-wave solution of (1), (2) for any $F \geq \tilde{F}$.*

We note that the assumption that n does not divide m cannot be removed from theorem 2, since if n divides m the coupling term vanishes and (5),(4) reduce to the equation of a running solution of a dc-forced pendulum, which is known to have a solution only when F exceeds a positive critical value [4].

Uniqueness of the travelling wave cannot be expected for all the parameter values for which existence holds. Indeed the numerical and experimental evidence ([7, 8]) shows that for some parameter values there is more than one travelling wave.

Along the way we will prove that

Proposition 3. *A lower bound for the period T of any travelling wave is given by*

$$T > \frac{2\pi\Gamma}{F}. \tag{6}$$

and an upper bound, in the case $F > 1$, is given by

$$T < \frac{2\pi\Gamma}{F-1}. \tag{7}$$

Let us note some questions that arise from our results and remain open:
(i) Let us define, for fixed $\Gamma > 0$,

$$F_0(\Gamma, K) = \inf\{\tilde{F} \geq 0 \mid \forall F \geq \tilde{F} \text{ a travelling wave of (1), (2) exists}\}.$$

Theorem 1 shows that $F_0(\Gamma, K) \leq 1$ for all K. Theorem 2 shows that, when n does not divide m, $\lim_{K\to\infty} F_0(\Gamma, K) = 0$. Is it true, though, that for each fixed $\Gamma, K > 0$ we have $F_0(\Gamma, K) > 0$ (we would expect this due to the 'pinning' phenomenon)?
(ii) Clarify the connection, if any, between the travelling waves obtained by Levi [3] for small values of $K > 0$ and those obtained by us for large values of K in theorem 2.
(iii) Investigate the issues of uniqueness vs. multiplicity and of stability of the travelling waves, especially in the large-K regime for which existence has been established by theorem 2.

2. Proofs of the results

Our method of proof involves re-formulating the problem as a fixed point problem in a Banach space, and applying results of nonlinear functional analysis. Our approach is thus close in spirit to [5], which deals with travelling waves in globally coupled Josephson junctions.

We transform the problem (4),(5) by setting

$$\omega = \frac{2\pi}{T}$$
$$f(t) = u(\omega t) + \omega t,$$

where u satisfies:

$$u(t + 2\pi) = u(t) \quad \forall t \in \mathbb{R}. \tag{8}$$

(5) can then be written as

$$\begin{aligned} \omega^2 u''(t) \quad &+ \quad \Gamma\omega u'(t) + \sin(t + u(t)) \\ &= \quad F - \Gamma\omega + K\Big[u\Big(t + 2\pi\frac{m}{n}\Big) - 2u(t) + u\Big(t - 2\pi\frac{m}{n}\Big)\Big] \end{aligned} \tag{9}$$

Dividing by ω^2 and setting

$$\lambda = \frac{1}{\omega}$$

we re-write (9) in the form

$$\begin{aligned} u''(t) \quad &+ \quad \lambda\Gamma u'(t) + \lambda^2 \sin(t + u(t)) \\ &= \quad \lambda^2 F - \lambda\Gamma + \lambda^2 K\Big[u\Big(t + 2\pi\frac{m}{n}\Big) - 2u(t) + u\Big(t - 2\pi\frac{m}{n}\Big)\Big]. \end{aligned} \tag{10}$$

We note that if $u(t)$ satisfies (8),(10) then so does $\tilde{u}(t) = u(t+c)+c$, for any $c \in \mathbb{R}$. Thus by adjusting c we may assume that u satisfies

$$\int_0^{2\pi} u(t)dt = 0. \tag{11}$$

We note now that if u satisfies (8),(10), then by integrating both sides of (10) over $[0,2\pi]$ we obtain

$$F = \frac{\Gamma}{\lambda} + \frac{1}{2\pi}\int_0^{2\pi} \sin(s+u(s))ds. \tag{12}$$

We can thus re-write (10) as

$$\begin{aligned} u''(t) + \lambda\Gamma u'(t) + \lambda^2 \sin(t+u(t)) \quad &= \quad \lambda^2 \frac{1}{2\pi}\int_0^{2\pi} \sin(s+u(s))ds \\ &+ \quad \lambda^2 K\Big[u\Big(t+2\pi\frac{m}{n}\Big) - 2u(t) + u\Big(t-2\pi\frac{m}{n}\Big)\Big]. \end{aligned} \tag{13}$$

Conversely, if u satisfies (12) and (13) then it satisfies (10). We have thus reformulated our problem as: find solutions (λ, u) of (8),(11),(12),(13). The idea now is to consider λ as a *parameter* in (13) and try to find solutions u satisfying (8),(11),(13) and then substitute λ and u into (12) to obtain the corresponding value of F. This is the same idea as used in the numerical method presented in [7], but here it is used as part of existence proofs. We claim that

Proposition 4. *For any value λ, there exists a solution u of (13) satisfying (8),(11).*

To prove this we will use the Schauder fixed-point theorem. We denote by X, Y the Banach spaces of real-valued functions

$$X = \{u \in H^2[0,2\pi] \mid u(0) = u(2\pi),\ \ u'(0) = u'(2\pi),\ \ \int_0^{2\pi} u(s)ds = 0\},$$

$$Y = \{u \in L^2[0,2\pi] \mid \int_0^{2\pi} u(s)ds = 0\},$$

with the norm

$$\|u\|_Y = \Big(\frac{1}{2\pi}\int_0^{2\pi} (u(s))^2 ds\Big)^{\frac{1}{2}},$$

and by $L_\lambda : X \to Y$ the linear mapping

$$L_\lambda(u) = u'' + \lambda\Gamma u' - \lambda^2 K\Big[u\Big(t+2\pi\frac{m}{n}\Big) - 2u(t) + u\Big(t-2\pi\frac{m}{n}\Big)\Big].$$

We want to show that this mapping is invertible and derive an upper bound for the norm of its inverse. Noting that any $u \in X$ can be decomposed in a Fourier series $u(t) = \sum_{l\neq 0} a_l e^{ilt}$ (with $a_{-l} = \overline{a_l}$), we apply L_λ to the Fourier elements, obtaining,

$$L_\lambda(e^{ilt}) = \mu_l e^{ilt},$$

where

$$\mu_l = -l^2 - 2K\lambda^2\Big(\cos\Big(\frac{2\pi ml}{n}\Big) - 1\Big) + \lambda l\Gamma i,$$

so that

$$|\mu_l| = \Big[\Big(l^2 + 2K\lambda^2\Big(\cos\Big(\frac{2\pi ml}{n}\Big) - 1\Big)\Big)^2 + \lambda^2 l^2\Gamma^2\Big]^{\frac{1}{2}}, \tag{14}$$

which does not vanish if $\Gamma > 0$. Thus the mapping L_λ has an inverse satisfying $L_\lambda^{-1}(e^{ilt}) = \frac{1}{\mu_l}e^{ilt}$. Since L_λ^{-1} takes Y onto X, and since X is compactly embedded in Y, we may consider L_λ^{-1} as a mapping from Y to itself, in which case it is a *compact* mapping. We also note using (14) that

$$\|L_\lambda^{-1}\|_{Y,Y} \leq \max_{l\geq 1} \frac{1}{|\mu_l|} \leq \frac{1}{\lambda\Gamma}. \tag{15}$$

We also define the nonlinear operator $N : Y \to Y$ by

$$N(u)(t) = -\sin(t + u(t)) + \frac{1}{2\pi}\int_0^{2\pi} \sin(s + u(s))ds.$$

It is easy to see that N is continuous, and that the range of N is contained in a bounded ball in Y, indeed we have

$$\|\sin(t + u(t))\|_{L_2} = \Big(\frac{1}{2\pi}\int_0^{2\pi} (\sin(s + u(s)))^2 ds\Big)^{\frac{1}{2}} \leq 1,$$

and since $N(u)$ is the orthogonal projection of $-\sin(t+u(t))$ into Y, we have

$$\|N(u)\|_Y \leq 1 \quad \forall u \in Y. \tag{16}$$

We can now rewrite the problem (8),(11),(13) as the fixed-point problem:

$$u = \lambda^2 L_\lambda^{-1} \circ N(u) \tag{17}$$

The operator on the right-hand side is compact by the compactness of L_λ^{-1}, and has a bounded range by (15),(16), so that Schauder's fixed-point theorem implies that (17) has a solution, proving proposition 4 (we note that by a simple bootstrap argument a solution in Y is in fact smooth). Moreover, defining

$$\Sigma = \{(\lambda, u) \in [0,\infty) \times Y \mid u = \lambda^2 L_\lambda^{-1} \circ N(u)\},$$

Rabinowitz's continuation theorem [6] implies that the connected component of Σ containing $(\lambda, u) = (0, 0)$, which we denote by C, is unbounded in $[0,\infty)\times Y$. Since for any $\lambda_0 > 0$ we have, from (16), (17), the bound $\|u\|_Y \leq \frac{\lambda_0}{\Gamma}$ for solutions (λ, u) of (17) with $\lambda \in [0, \lambda_0]$, the unboundedness of the set C must be in the λ-direction, that is, there exist $(\lambda, u) \in C$ with arbitrarily large values of λ. We can now consider the right-hand side of (12) as a functional on $[0,\infty) \times Y$:

$$\Phi(\lambda, u) = \frac{\Gamma}{\lambda} + \frac{1}{2\pi}\int_0^{2\pi} \sin(s + u(s))ds, \tag{18}$$

and our aim is to prove solvability of the equation

$$\Phi(\lambda, u) = F, \quad (\lambda, u) \in C.$$

We note that by the boundedness of the sine function we have

$$\lim_{\lambda \to 0+,\ (\lambda,u)\in C} \Phi(\lambda, u) = +\infty, \tag{19}$$

$$\limsup_{\lambda \to +\infty,\ (\lambda,u)\in C} \Phi(\lambda, u) \le 1. \tag{20}$$

Since C is a connected set and Φ is continuous, (19) implies that

Proposition 5. *For any F satisfying*

$$F > \underline{F} \equiv \inf_{(\lambda,u)\in C} \Phi(\lambda, u), \tag{21}$$

there exists a travelling wave.

Since (20) implies that $\underline{F} \le 1$, this proves theorem 1.

We now prove the lower and upper bounds for the period T of travelling-waves given in proposition 3. These follow from (12) and from

Proposition 6. *For any $(\lambda, u) \in \Sigma$ with $\lambda > 0$ we have*

$$0 < \frac{\Gamma}{\lambda} < \Phi(\lambda, u) < \frac{\Gamma}{\lambda} + 1.$$

The upper bound follows immediately from the definition (18) of $\Phi(\lambda, u)$ since $\frac{1}{2\pi}\int_0^{2\pi} sin(s + u(s))ds < 1$. The lower bound follows from the claim that

$$(\lambda, u) \in \Sigma \ \Rightarrow \ \int_0^{2\pi} sin(s + u(s))ds > 0.$$

To prove this claim we multiply (13) by $1 + u'(t)$ and integrate over $[0, 2\pi]$, noting that

$$\int_0^{2\pi} u\Big(s + 2\pi\frac{m}{n}\Big)u'(s)ds = \int_0^{2\pi} u(s)u'\Big(s - 2\pi\frac{m}{n}\Big)ds = -\int_0^{2\pi} u'(s)u\Big(s - 2\pi\frac{m}{n}\Big)ds,$$

so that we obtain

$$\Gamma\frac{1}{2\pi}\int_0^{2\pi} (u'(s))^2 ds = \lambda\frac{1}{2\pi}\int_0^{2\pi} sin(s + u(s))ds.$$

This proves the claim since the right-hand side is non-negative and cannot vanish unless $u \equiv 0$, but $(\lambda, 0) \notin \Sigma$ for $\lambda > 0$.

We now turn to the proof of theorem 2.

Proposition 7. *Assume n does not divide l and $\Gamma > 0$. Given any $\lambda_0 > 0$ and $\epsilon > 0$, there exists K_0 such that for $K \geq K_0$ we have that*

$$\Big|\frac{1}{2\pi}\int_0^{2\pi} \sin(s+u(s))ds\Big| < \epsilon \quad if\ (\lambda_0, u) \in \Sigma. \tag{22}$$

To see that proposition 7 implies theorem 2, we fix some $\tilde{F} > 0$ and we choose $\lambda_0 > \frac{\Gamma}{\tilde{F}}$. We then set $\epsilon = \tilde{F} - \frac{\Gamma}{\lambda_0}$ and choose K_0 according to proposition 7, so that (22) holds, which implies that when $K \geq K_0$ we have $\Phi(\lambda_0, u) < \tilde{F}$ for any u with $(\lambda_0, u) \in C$. Thus $\underline{F} < \tilde{F}$, where $\underline{F}$ is defined by (21), so proposition 5 implies the existence of a travelling wave for any $F \geq \tilde{F}$.

We now prove proposition 7. Let $\lambda_0 > 0$ and $\epsilon > 0$ be given. Assume $(\lambda_0, u) \in \Sigma$, so that (17) holds with $\lambda = \lambda_0$. Let (m, n) denote the greatest common divisor of m, n and let

$$p = \frac{m}{(m,n)}, \quad q = \frac{n}{(m,n)}.$$

Since we assume n does not divide m we have $q \geq 2$. Let Y_0 be the subspace of Y consisting of $\frac{2\pi}{q}$-periodic functions, and let Y_1 be its orthogonal complement in Y. We denote by P the orthogonal projection of Y to Y_0. Setting

$$u_0 = P(u), \quad u_1 = (I-P)(u),$$

we have $u = u_0 + u_1$ with $u_0 \in Y_0$, $u_1 \in Y_1$. Applying P and $I - P$ to (17), and noting that L_λ commutes with P, we have

$$u_0 = \lambda_0^2 L_{\lambda_0}^{-1} \circ P \circ N(u_0 + u_1) \tag{23}$$

$$u_1 = \lambda_0^2 L_{\lambda_0}^{-1} \circ (I-P) \circ N(u_0 + u_1) \tag{24}$$

We will now use (14) to derive a bound for $\|L_{\lambda_0}^{-1}|_{Y_1}\|_{Y_1,Y_1}$ which goes to 0 as $K \to \infty$. We note that

$$\|L_{\lambda_0}^{-1}|_{Y_1}\|_{Y_1,Y_1} \leq \max_{l\geq 1,\ q \not| \, l} \frac{1}{|\mu_l|}, \tag{25}$$

so we need to find lower bounds for the $|\mu_l|$'s for which q does not divide l. We define

$$\rho = \max_{l \geq 1,\ q\not|\, l} \cos\Big(\frac{2\pi p l}{q}\Big)$$

and note that since p, q are coprime we have $\rho < 1$.

We define

$$\alpha = 2K\lambda_0^2(1-\rho) - \sqrt{K},$$

and we shall henceforth assume that K is sufficiently large so that $\alpha > 0$. For each $l \geq 1$ we have either $l^2 < \alpha$ or $l^2 \geq \alpha$, and we treat each of these cases seperately.

(1) In case $l^2 < \alpha$, we have

$$l^2 + 2K\lambda_0^2(\rho - 1) < -\sqrt{K},$$

and by the definition of ρ

$$\cos\Big(\frac{2\pi ml}{n}\Big) = \cos\Big(\frac{2\pi pl}{q}\Big) \leq \rho,$$

so that

$$l^2 + 2K\lambda_0^2\Big(\cos\Big(\frac{2\pi ml}{n}\Big) - 1\Big) < -\sqrt{K},$$

which by (14) implies

$$|\mu_l| > \sqrt{K}. \tag{26}$$

(2) In case $l^2 \geq \alpha$, we have, since (14) implies $|\mu_l| > \lambda_0 \Gamma l$:

$$|\mu_l| \geq \lambda_0\Gamma\sqrt{\alpha} = \lambda_0\Gamma\Big[2K\lambda_0^2(1-\rho) - \sqrt{K}\Big]^{\frac{1}{2}}. \tag{27}$$

From (26),(27) we obtain that $\lim_{K\to\infty}|\mu_l| = +\infty$ uniformly with respect to $l \geq 1$ which are not multiples of q, hence by (25)

$$\lim_{K\to\infty} \|L_{\lambda_0}^{-1}|_{Y_1}\|_{Y_1,Y_1} = 0.$$

In particular we may choose K_0 such that for $K \geq K_0$ we will have

$$\|L_{\lambda_0}^{-1}|_{Y_1}\|_{Y_1,Y_1} < \frac{\epsilon}{\lambda_0^2}.$$

By (24) and (16) this implies

$$\|u_1\|_Y \leq \epsilon. \tag{28}$$

Thus

$$\begin{aligned}
\Big|\frac{1}{2\pi}\int_0^{2\pi}\sin(s+u(s))ds\Big| &\leq \Big|\frac{1}{2\pi}\int_0^{2\pi}\sin(s+u_0(s))ds\Big| \\
&+ \Big|\frac{1}{2\pi}\int_0^{2\pi}[\sin(s+u(s)) - \sin(s+u_0(s))]ds\Big| \\
&\leq \Big|\frac{1}{2\pi}\int_0^{2\pi}\sin(s+u_0(s))ds\Big| + \frac{1}{2\pi}\int_0^{2\pi}|u_1(s)|ds \qquad (29)
\end{aligned}$$

From (28) and the Cauchy-Schwartz inequality we have

$$\frac{1}{2\pi}\int_0^{2\pi}|u_1(s)|ds \leq \frac{1}{\sqrt{2\pi}}\Big(\int_0^{2\pi}(u_1(s))^2ds\Big)^{\frac{1}{2}} \leq \epsilon. \tag{30}$$

From trigonometry we have

$$\int_0^{2\pi}\sin(s+u_0(s))ds = \int_0^{2\pi}\sin(s)\cos(u_0(s))ds + \int_0^{2\pi}\cos(s)\sin(u_0(s))ds,$$

but the functions $\cos(u_0(s))$, $\sin(u_0(s))$ are $\frac{2\pi}{q}$-periodic with $q \geq 2$, which implies that they are orthogonal to $\cos(s)$, $\sin(s)$, so that we have

$$\int_0^{2\pi} \sin(s + u_0(s))ds = 0,$$

which together with (29) and (30) implies (22), concluding the proof of proposition 7.

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