# The Radon transform of functions of matrix argument

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ABSTRACT. The monograph contains a systematic treatment of a circle of problems in analysis and integral geometry related to inversion of the Radon transform on the space of real rectangular matrices. This transform assigns to a function $f$ on the matrix space the integrals of $f$ over the so-called matrix planes, the linear manifolds determined by the corresponding matrix equations. Different inversion methods are discussed. They rely on close connection between the Radon transform, the Fourier transform, the Gårding-Gindikin fractional integrals, and matrix modifications of the Riesz potentials. A special emphasis is made on new higher rank phenomena, in particular, on possibly minimal conditions under which the Radon transform is well defined and can be explicitly inverted. Apart of the space of Schwartz functions, we also consider $L^p$- spaces and the space of continuous functions. Many classical results for the Radon transform on $\mathbb{R}^n$ are generalized to the higher rank case.

# Contents

# Introduction

## 0.1. Preface

One of the basic problems of integral geometry reads as follows [**Gel**]. Let $\mathfrak{X}$ be some space of points $x$, and let $\mathfrak{T}$ be a certain family of manifolds $\tau \subset \mathfrak{X}$. Consider the mapping $f(x) \to \hat{f}(\tau) = \int_\tau f$ which assigns to $f$ its integrals over all $\tau \in \mathfrak{T}$. This mapping is usually called the Radon transform of $f$. How do we recover $f(x)$ from $\hat{f}(\tau)$? This rather general problem goes back to Lorentz, Minkowski, Blaschke, Funk, Radon etc., and has numerous applications; see [**Eh**], [**GGG**], [**GGV**], [**H**], [**Na**], [**Ru3**], [**RK**], [**SSW**], and references therein.

A typical example is the so-called $k$-plane transform on $\mathbb{R}^n$ which assigns to $f(x),\ x \in \mathbb{R}^n$, a collection of integrals of $f$ over all $k$-dimensional planes in $\mathbb{R}^n$. This well known transformation gets a new flavor if we formally replace the dimension $n$ by the product $nm$, $n \geq m$, and regard $\mathbb{R}^{nm}$ as the space $\mathfrak{M}_{n,m}$ of $n \times m$ matrices $x = (x_{i,j})$. Once we have accepted this point of view, then $f(x)$ becomes a function of matrix argument and new "higher rank" phenomena come into play.

Let $V_{n,n-k}$ be the Stiefel manifold of orthonormal $(n-k)$-frames in $\mathbb{R}^n$, i.e., $V_{n,n-k} = \{\xi : \xi \in \mathfrak{M}_{n,n-k},\ \xi'\xi = I_{n-k}\}$ where $\xi'$ denotes the transpose of $\xi$, $I_{n-k}$ is the identity matrix, and $\xi'\xi$ is understood in the sense of matrix multiplication. Given $\xi \in V_{n,n-k}$ and $t \in \mathfrak{M}_{n-k,m}$, we define *the matrix $k$-plane*

$$\tau \equiv \tau(\xi, t) = \{x : x \in \mathfrak{M}_{n,m},\ \xi' x = t\}. \tag{0.1}$$

Let $\mathfrak{T}$ be the manifold of all matrix $k$-planes in $\mathfrak{M}_{n,m}$. Each $\tau \in \mathfrak{T}$ is topologically an ordinary $km$-dimensional plane in $\mathbb{R}^{nm}$, but the set $\mathfrak{T}$ has measure zero in the manifold of all such planes. The relevant Radon transform has the form

$$\hat{f}(\tau) = \int_{x \in \tau} f(x), \qquad \tau \in \mathfrak{T}, \tag{0.2}$$

the integration being performed against the corresponding canonical measure. If $m = 1$, then $\mathfrak{T}$ is the manifold of $k$-dimensional affine planes in $\mathbb{R}^n$, and $\hat{f}(\tau)$ is the usual $k$-plane transform. The inversion problem for (0.2) reads as follows: How do we reconstruct a function $f$ from the integrals $\hat{f}(\tau)$, $\tau \in \mathfrak{T}$? Our aim is to obtain explicit inversion formulas for the Radon transform (0.2) for continuous and, more generally, locally integrable functions $f$ subject to possibly minimal assumptions at infinity.

A systematic study of Radon transforms on matrix spaces was initiated by E.E. Petrov [**P1**] in 1967 and continued in [**Č**], [**Gr1**], [**P2**]–[**P4**], [**Sh1**], [**Sh2**]. Petrov [**P1**], [**P2**] considered the case $k = n - m$ and obtained inversion formulas for functions belonging to the Schwartz space. His method is based on the classical idea of decomposition of the delta function in plane waves; cf. [**GSh1**]. Results of Petrov were extended in part by L.P. Shibasov [**Sh2**] to all $1 \leq k \leq n - m$.

The following four inversion methods are customarily used in the theory of Radon transforms. These are (a) *the Fourier transform method* (based on the projection-slice theorem), (b) *the method of mean value operators*, (c) *the method of Riesz potentials*, and (d) *decomposition in plane waves*. Of course, this classification is vague and other names are also attributed in the literature. One could also mention implementation of wavelet transforms, but this can be viewed as a certain regularization or computational procedure for divergent integrals arising in (b)–(d). In Section 0.2, we briefly recall some known facts for the $k$-plane transform on $\mathbb{R}^n$ which corresponds to the case $m = 1$ in (0.2). Our task is to extend this theory to the higher rank case $m > 1$. The main results are exhibited in Section 0.3. A considerable part of the monograph (Chapters 1–3) is devoted to developing a necessary background and analytic tools which one usually gets almost "for free" in the rank-one case. We are not concerned with such important topics as the range characterization, the relevant Paley-Wiener theorems [**P3**], [**P4**], the support theorems, the convolution-backprojection method, and numerical implementation. One should also mention a series of papers devoted to the Radon transform on Grassmann manifolds; see, e.g., [**Go**], [**GK1**], [**GK2**], [**Gr2**], [**GR**], [**Ka**], [**Ru5**], [**Str2**], and references therein. This topic is conceptually close to ours.

## 0.2. The $k$-plane Radon transform on $\mathbb{R}^n$

Many authors contributed to this theory; see, e.g., [**GGG**], [**H**], [**Ke**], [**Ru4**] where one can find further references. We recall some main results which will be extended in the sequel to the higher rank case.

**0.2.1. Definitions and the framework of the inversion problem.** Let $\mathcal{G}_{n,k}$ be the manifold of affine $k$-dimensional planes in $\mathbb{R}^n$, $0 < k < n$. The integral (0.2) with $m = 1$ can be interpreted in different ways. Namely, if we define a $k$-plane $\tau \in \mathcal{G}_{n,k}$ by (0.1) where $\xi \in V_{n,n-k}$ and $t \in \mathbb{R}^{n-k}$ is a column vector, then *the $k$-plane Radon transform* is represented as

$$\hat{f}(\tau) \equiv \hat{f}(\xi, t) = \int\limits_{\{y:\, y\in\mathbb{R}^n;\, \xi' y=0\}} f(y + \xi t)\, d_\xi y, \tag{0.3}$$

where $d_\xi y$ is the induced Lebesgue measure on the plane $\{y : \xi' y = 0\}$. For $k = n - 1$, this is the classical hyperplane Radon transform [**H**], [**GGV**]. Clearly, the parameterization $\tau \equiv \tau(\xi, t)$ is not one-to-one. An alternative parameterization is as follows. Let $\eta$ belong to the ordinary Grassmann manifold $G_{n,k}$ of $k$-dimensional linear subspaces of $\mathbb{R}^n$, and $\eta^\perp$ be the orthogonal complement of $\eta$. The parameterization

$$\tau \equiv \tau(\eta, \lambda), \qquad \eta \in G_{n,k}, \qquad \lambda \in \eta^\perp,$$

is one-to-one and gives

$$\hat{f}(\tau) \equiv \hat{f}(\eta, \lambda) = \int\limits_{\eta} f(y + \lambda) dy. \tag{0.4}$$

The corresponding *dual Radon transform* of a function $\varphi(\tau)$ on $\mathcal{G}_{n,k}$ is defined as the mean value of $\varphi(\tau)$ over all $k$-dimensional planes $\tau$ through $x \in \mathbb{R}^n$:

$$\check{\varphi}(x) = \int\limits_{SO(n)} \varphi(\gamma\eta_0 + x)d\gamma = \int\limits_{G_{n,k}} \varphi(\eta, \mathrm{Pr}_{\eta^\perp} x)d\eta. \tag{0.5}$$

Here, $\eta_0$ is an arbitrary fixed $k$-plane through the origin, and $\mathrm{Pr}_{\eta^\perp} x$ denotes the orthogonal projection of $x$ onto $\eta^\perp$. A duality relation

$$\int\limits_{\mathbb{R}^n} f(x)\check{\varphi}(x)dx = \int\limits_{\mathcal{G}_{n,k}} \hat{f}(\tau)\varphi(\tau)d\tau, \tag{0.6}$$

holds provided that either side of this equality is finite for $f$ and $\varphi$ replaced by $|f|$ and $|\varphi|$, respectively [**H**], [**So**]. Here and on, $d\tau$ stands for the canonical measure on $\mathcal{G}_{n,k}$; see [**Ru4**].

The $k$-plane transform is injective for all $0 < k < n$ on "standard" function spaces where it is well defined (see Theorem 0.1 below). If $k < n-1$, then the inversion problem for $\hat{f}$ is overdetermined because

$$\dim \mathcal{G}_{n,k} = (n-k)(k+1) > n = \dim \mathbb{R}^n,$$

and the dual Radon transform is non-injective. If $k = n-1$, then the dimension of the affine Grassmanian $\mathcal{G}_{n,n-1}$ is just $n$. In this case, the Radon transform and its dual are injective simultaneously; see also [**Ru5**] where this fact is extended to Radon transforms on affine Grassmann manifolds.

The following theorem specifies standard classes of functions for which the $k$-plane transform is well defined.

THEOREM 0.1.
(i) *Let $f(x)$ be a continuous function on $\mathbb{R}^n$ satisfying $f(x) = O(|x|^{-a})$. If $a > k$ then the Radon transform $\hat{f}(\tau)$ is finite for all $\tau \in \mathcal{G}_{n,k}$.*
(ii) *If $(1+|x|)^\alpha f(x) \in L^1$ for some $\alpha \ge k-n$, in particular, if $f \in L^p$, $1 \le p < n/k$, then $\hat{f}(\tau)$ is finite for almost all $\tau \in \mathcal{G}_{n,k}$.*

Here, (i) is obvious, and (ii) is due to Solmon [**So**]. The conditions $a > k$, $\alpha \ge k-n$, and $p < n/k$ cannot be improved. For instance, if $p \ge n/k$ and $f(x) = (2+|x|)^{-n/p}(\log(2+|x|))^{-1}$ $(\in L^p)$, then $\hat{f}(\tau) \equiv \infty$.

More informative statements can be found in [**Ru4**]. For instance, if $|\tau|$ denotes the euclidean distance from the $k$-plane $\tau$ to the origin, then

$$\int\limits_{\mathcal{G}_{n,k}} \hat{f}(\tau)\frac{|\tau|^{\lambda-n}}{(1+|\tau|^2)^{\lambda/2}}d\tau = c\int\limits_{\mathbb{R}^n} f(x)\frac{|x|^{\lambda-n}}{(1+|x|^2)^{(\lambda-k)/2}}dx, \tag{0.7}$$

$$c = \frac{\Gamma((\lambda-k)/2)\,\Gamma(n/2)}{\Gamma(\lambda/2)\,\Gamma((n-k)/2)}, \qquad Re\,\lambda > k$$

(see Theorem 2.3 in [**Ru4**]). Choosing $\lambda = n$, we obtain

$$\int\limits_{\mathcal{G}_{n,k}} \frac{\hat{f}(\tau)}{(1+|\tau|^2)^{n/2}}d\tau = \int\limits_{\mathbb{R}^n} \frac{f(x)}{(1+|x|^2)^{(n-k)/2}}dx. \tag{0.8}$$

It is instructive to compare (0.8) with [**So**, Theorem 3.8] where, instead of explicit equality (0.8), one has an inequality (which is of independent interest!). Different modifications of (0.7) and (0.8), which control the Radon transform and its dual at the origin and near spherical surfaces, can be found in [**Ru4**].

It is worth noting that "Solmon's condition" $(1+|x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$ is not necessary for the a.e. existence of $\hat f(\tau)$ because, the Radon transform essentially has an exterior (or "right-sided") nature; see (0.11) below. For any rapidly decreasing function $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying $f(x) \sim |x|^{-n}$ as $|x| \to 0$, the Solmon condition fails, but $\hat f(\xi, t) < \infty$ a.e. (it is finite for all $\tau \not\ni 0$). However, if for $f \geq 0$, one also assumes $\hat f(\tau) \in L^1_{loc}$ (in addition to $\hat f(\tau) < \infty$ a.e.), then, necessarily, $(1+|x|)^{k-n} f(x) \in L^1$ [**So**, Theorem 3.9].

**0.2.2. The $k$-plane transform and the dual $k$-plane transform of radial functions.** There is an important connection between the transformations $f \to \hat f$ and $\varphi \to \check\varphi$, and the Riemann-Liouville (or Abel) fractional integrals

$$(0.9) \qquad (I^\alpha_+ f)(s) = \frac{1}{\Gamma(\alpha)} \int\limits_0^s \frac{f(r)}{(s-r)^{1-\alpha}} dr, \qquad (I^\alpha_- f)(s) = \frac{1}{\Gamma(\alpha)} \int\limits_s^\infty \frac{f(r)}{(r-s)^{1-\alpha}} dr,$$

$Re\,\alpha > 0$. Let $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ be the area of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. The following statement is known; see, e.g., [**Ru4**].

THEOREM 0.2. *For $x \in \mathbb{R}^n$ and $\tau \equiv (\eta, \lambda) \in \mathcal{G}_{n,k}$, let*

$$(0.10) \qquad r = |x| = \mathrm{dist}(o, x), \qquad s = |\lambda| = \mathrm{dist}(o, \tau)$$

*denote the corresponding euclidean distances from the origin. If $f(x)$ and $\varphi(\tau)$ are radial functions, i.e., $f(x) \equiv f_0(r)$ and $\varphi(\tau) \equiv \varphi_0(s)$, then $\hat f(\tau)$ and $\check\varphi(x)$ are represented by the Abel type integrals*

$$(0.11) \qquad \hat f(\tau) = \sigma_{k-1} \int\limits_s^\infty f_0(r)(r^2 - s^2)^{k/2-1} r\,dr,$$

$$(0.12) \qquad \check\varphi(x) = \frac{\sigma_{k-1}\,\sigma_{n-k-1}}{\sigma_{n-1}\, r^{n-2}} \int\limits_0^r \varphi_0(s)(r^2 - s^2)^{k/2-1} s^{n-k-1} ds,$$

*provided that these integrals exist in the Lebesgue sense.*

Obvious transformations reduce (0.11) and (0.12) to the corresponding fractional integrals (0.9).

**0.2.3. Inversion of the $k$-plane transform.** Abel type representations (0.11) and (0.12) enable us to invert the $k$-plane transform and its dual for radial functions. To this end, we utilize diverse methods of fractional calculus described, e.g., in [**Ru1**] and [**SKM**]. In the general case, the following approaches can be applied.

*The method of mean value operators.* The idea of the method amounts to original papers by Funk and Radon and employs the fact that the $k$-plane transform commutes with isometries of $\mathbb{R}^n$. Thanks to this property, the inversion problem reduces to the case of radial functions if one applies to $\hat f(\tau)$ the so-called *shifted dual $k$-plane transform* (this terminology is due to F. Rouvière [**Rou**]). The latter is defined on functions $\varphi(\tau)$, $\tau \in \mathcal{G}_{n,k}$, by the formula

$$(0.13) \qquad \check\varphi_r(x) = \int\limits_{SO(n)} \varphi(\gamma\tau_r + x)\, d\gamma,$$

where $\tau_r$ is an arbitrary fixed $k$-plane at distance $r$ from the origin. The integral (0.13) is the mean value of $\varphi(\tau)$ over all $\tau$ at distance $r$ from $x$. For $r = 0$, it coincides with the dual $k$-plane transform $\check{\varphi}(x)$. Modifications of (0.13) for Radon transforms on the $n$-dimensional unit sphere and the real hyperbolic space were introduced by Helgason; see [**H**] and references therein.

The core of the method is the following.

LEMMA 0.3. *Let $\varphi = \hat{f}$, $f \in L^p$, $1 \le p < n/k$, and let*

$$(\mathcal{M}_r f)(x) = \frac{1}{\sigma_{n-1}} \int\limits_{S^{n-1}} f(x + r\theta)\, d\theta, \qquad r > 0, \tag{0.14}$$

*be the spherical mean of $f$. If $g_x(s) = (\mathcal{M}_{\sqrt{s}} f)(x)$ and $\psi_x(s) = \pi^{-k/2} \check{\varphi}_{\sqrt{s}}(x)$, then*

$$(I_-^{k/2} g_x)(s) = \psi_x(s). \tag{0.15}$$

If $k$ is even, (0.15) yields

$$f(x) = \pi^{-k/2} \Big( -\frac{1}{2r} \frac{d}{dr} \Big)^{k/2} \check{\varphi}_r(x) \Big|_{r=0}, \tag{0.16}$$

where, for non-smooth $f$, the equality is understood in the almost everywhere sense. This inversion formula is of local nature because it assumes that $\check{\varphi}_r(x)$ is known only for arbitrary small $r$. The case of odd $k$ is more delicate. If $f$ is continuous and decays sufficiently fast at infinity, then (0.15) is inverted by the formula

$$f(x) = \Big( -\frac{d}{ds} \Big)^m (I_-^{m-k/2} \psi_x)(s) \Big|_{s=0}, \qquad \forall m \in \mathbb{N}, \quad m > k/2, \tag{0.17}$$

which is essentially non-local. For $f$ belonging to $L^p$, the integral $(I_-^{m-k/2} \psi_x)(s)$ can be divergent if $n/2m \le p < n/k$. It means that the inversion method should not increase the order of the fractional integral. This obstacle can be circumvented by making use of Marchaud's fractional derivatives; see [**Ru4**, Section 5] for details.

*The method of Riesz potentials and the method of plane waves.* It is instructive to describe these methods simultaneously because both employ the Riesz potentials

$$(I^\alpha f)(x) = \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} f(x - y) |y|^{\alpha - n}\, dy, \qquad x \in \mathbb{R}^n, \tag{0.18}$$

$$\gamma_n(\alpha) = \frac{2^\alpha \pi^{n/2} \Gamma(\alpha/2)}{\Gamma((n - \alpha)/2)}, \qquad Re\,\alpha > 0, \qquad \alpha \ne n, n+2, \dots\,. \tag{0.19}$$

The Fourier transform of the corresponding Riesz distribution $|x|^{\alpha-n}/\gamma_n(\alpha)$ is $|y|^{-\alpha}$ in a suitable sense. It means that $I^\alpha$ can be regarded as $(-\alpha/2)$th power of $-\Delta$, where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ is the Laplace operator . For $f \in L^p(\mathbb{R}^n)$, the integral (0.18) absolutely converges if and only if $1 \le p < n/Re\,\alpha$. For $Re\,\alpha \le 0$ and sufficiently smooth $f$, the Riesz potential $I^\alpha f$ is defined by analytic continuation so that $I^0 f = f$. Numerous inversion formulas for $I^\alpha f$ are known in diverse function spaces, see [**Ru1**], [**Ru4**], [**Sa**], [**SKM**], and references therein.

The method of plane waves is based on decomposition of the Riesz distribution in plane waves; see [**GSh1**] for the case $k = n-1$. Technically, this method realizes as follows. In order to recover $f$ at a point $x$ from the data $\hat{f}(\tau) \equiv \hat{f}(\xi, t)$, one first applies the Riesz potential operator of the negative order $-k$ (this is a differential or pseudo-differential operator) in the $t$-variable, and then integrates over all $k$-planes through $x$, i.e., applies the dual Radon transform. This method requires $f$ to be smooth enough. Another method, referred to as the method of Riesz potentials, is applicable to "rough" functions as well. Now we utilize the same operators but in the reverse order: first we apply the dual Radon transform to $\hat{f}(\tau)$, and then the Riesz potential operator of order $-k$ in the $x$-variable. This method allows us to reconstruct any function $f \in L^p$, $1 \le p < n/k$, for almost all $x \in \mathbb{R}^n$ [**Ru4**].

Both methods can be realized in the framework of analytic families of intertwining fractional integrals

$$(P^\alpha f)(\tau) = \frac{1}{\gamma_{n-k}(\alpha)} \int\limits_{\mathbb{R}^n} f(x)|x-\tau|^{\alpha+k-n}\, dx, \tag{0.20}$$

$$(\overset{*}{P}{}^{\alpha}\varphi)(x) = \frac{1}{\gamma_{n-k}(\alpha)} \int\limits_{\mathcal{G}_{n,k}} \varphi(\tau)|x-\tau|^{\alpha+k-n}\, d\tau, \tag{0.21}$$

$$Re\,\alpha > 0, \qquad \alpha + k - n \neq 0, 2, 4, \dots\,,$$

where $|x-\tau|$ denotes the euclidian distance between the point $x$ and the $k$-plane $\tau$. These operators were introduced by Semyanistyi [1] [**Se**] for $k = n-1$. They were generalized in [**Ru2**] and subsequent publications of B. Rubin to all $0 < k < n$ and totally geodesic Radon transform on spaces of constant curvature; see also [**Ru4**]. Semyanistyi's fractional integrals have a deep philosophical meaning. One can write

$$P^\alpha f = \tilde{I}^\alpha \hat{f}, \qquad \overset{*}{P}{}^{\alpha}\varphi = (\tilde{I}^\alpha \varphi)^\vee, \tag{0.22}$$

where for $\varphi(\tau) \equiv \varphi(\xi, t)$, $\tilde{I}^\alpha\varphi$ denotes the Riesz potential on $\mathbb{R}^{n-k}$ in the $t$-variable. If $f$ and $\varphi$ are smooth enough, then the equalities (0.22) extend definitions (0.20) and (0.21) to all complex $\alpha \neq n-k,\ n-k+2, \dots$. In particular, for $\alpha = 0$, we get the $k$-plane transform and the dual $k$-plane transform, respectively.

The following statement generalizes the well known formula of Fuglede

$$(\hat{f})^\vee = c_{k,n} I^k f, \qquad c_{k,n} = (2\pi)^k \sigma_{n-k-1}/\sigma_{n-1}; \tag{0.23}$$

see [**Fu**], [**So**, Theorem 3.6], [**H**, p. 29].

THEOREM 0.4. *Let $f \in L^p$, $1 \le p < n/(Re\,\alpha + k)$, $Re\,\alpha \ge 0$. Then*

$$\overset{*}{P}{}^{\alpha}\hat{f} = c_{k,n} I^{\alpha+k} f. \tag{0.24}$$

The equality (0.24) gives a family of inversion formulas (at least on a formal level):

$$c_{k,n} f = I^{-\alpha-k}\, \overset{*}{P}{}^{\alpha}\hat{f}. \tag{0.25}$$

In the case $\alpha = 0$, the last equality reads

$$c_{k,n} f = \mathbb{D}^k \check{\varphi}, \qquad \varphi = \hat{f}. \tag{0.26}$$

Here, $\mathbb{D}^k = I^{-k} = (-\Delta)^{k/2}$ denotes the Riesz fractional derivative that can be realized in different ways depending on a class of functions $f$ and the evenness of

[1] Vladimir Il'ich Semyanistyi (1925–1984), a talented mathematician whose papers have played an important role in integral geometry and related areas of analysis.

$k$. For example, if $f$ is good enough, the $k$-plane transform $\hat{f} = \varphi$ can be inverted by repeated application of the operator $-\Delta$. Namely, for $k$ even,

$$c_{k,n} f(x) = (-\Delta)^{k/2} \check{\varphi}(x). \tag{0.27}$$

If $k$ is odd and $1 \le k \le n-2$, then

$$c_{k,n} f(x) = (-\Delta)^{(k+1)/2} (I^1 \check{\varphi})(x). \tag{0.28}$$

Inversion formulas (0.26)–(0.28) are typical for the method of Riesz potentials. Furthermore, if $\alpha = -k$, then (0.25) gives

$$c_{k,n} f = \overset{*}{P}{}^{-k} \varphi, \qquad \varphi(\xi, t) = \hat{f}(\xi, t). \tag{0.29}$$

The last formula can be regarded as a decomposition of $f$ in plane waves. Different realizations of operators $\mathbb{D}^k$ and $\overset{*}{P}{}^{-k}$ can be found in [**Ru1**] and [**Ru4**]; see also [**Ru6**].

## 0.3. Main results

This section is organized so that the reader could compare our new results with those in the rank-one case. That was the reason why subsections below are entitled as the similar ones in Section 0.2. The general organization of the material is clear from the Contents.

**0.3.1. Definitions and the framework of the inversion problem.** In order to give precise meaning to the integral (0.2), we use the parameterization $\tau = \tau(\xi, t)$, $\xi \in V_{n,n-k}$, $t \in \mathfrak{M}_{n-k,m}$, of the matrix $k$-plane (0.1) and arrive at the formula

$$\hat{f}(\xi, t) = \int\limits_{\{y:\, y \in \mathfrak{M}_{n,m}\,;\, \xi' y = 0\}} f(y + \xi t)\, d_\xi y, \tag{0.30}$$

where $d_\xi y$ is the induced measure on the subspace $\{y : \, y \in \mathfrak{M}_{n,m} \,;\, \xi' y = 0\}$. This agrees with the formula (0.3) for the $k$-plane transform and can be also written in a different way, see (4.10). Another parameterization

$$\tau = \tau(\eta, \lambda), \qquad \eta \in G_{n,k}, \quad \lambda = [\lambda_1 \dots \lambda_m] \in \mathfrak{M}_{n,m}, \quad \lambda_i \in \eta^\perp,$$

which is one-to-one (unlike that in (0.30)), gives

$$\hat{f}(\tau) = \int\limits_\eta dy_1 \dots \int\limits_\eta f([y_1 + \lambda_1 \dots y_m + \lambda_m])\, dy_m, \tag{0.31}$$

cf. (0.4).

We first specify the set of all triples $(n, m, k)$ for which the Radon transform is injective. It is natural to assume that the transformed function should depend on at least as many variables as the original one, i.e.,

$$\dim \mathfrak{T} \ge \dim \mathfrak{M}_{n,m}.$$

By taking into account that $\dim V_{n,m} = m(2n-m-1)/2$ [**Mu**, p. 67], we have

$$\begin{aligned} \dim \mathfrak{T} &= \dim(V_{n,n-k} \times \mathfrak{M}_{n-k,m})/O(n-k) \\ &= (n-k)(n+k-1)/2 + m(n-k) - (n-k)(n-k-1)/2 \\ &= (n-k)(k+m). \end{aligned}$$

The inequality $(n-k)(k+m) \geq nm$ implies the natural framework of the inversion problem, namely,

$$1 \leq k \leq n-m, \qquad m \geq 1. \tag{0.32}$$

We show (Theorem 4.9) that the Radon transform $f \to \hat{f}$ is injective on the Schwartz space $\mathcal{S}(\mathfrak{M}_{n,m})$ if and only if the triple $(n,m,k)$ satisfies (0.32). The proof of this statement relies on the matrix generalization of *the projection-slice theorem* (Theorem 4.7) which establishes connection between the matrix Fourier transform and the Radon transform. For the case $m=1$, we refer to [**Na**, p. 11] $(k=n-1)$ and [**Ke**, p. 283] (any $0<k<n$); see also [**Sh1**], [**Sh2**], and [**OR**] for the higher rank case.

We also specify classes of functions for which the Radon transform is well defined. Let $I_m$ be the identity $m \times m$ matrix and let $|a|$ denote the absolute value of the determinant of a square matrix $a$. The following statement agrees with Theorem 0.1.

THEOREM 0.5.
(i) *Let $f(x)$ be a continuous function on $\mathfrak{M}_{n,m}$, satisfying $f(x) = O(|I_m + x'x|^{-a/2})$. If $a > k+m-1$, then the Radon transform $\hat{f}(\tau)$ is finite for all $\tau \in \mathfrak{T}$.*
(ii) *If $|I_m + x'x|^{\alpha/2} f(x) \in L^1(\mathfrak{M}_{n,m})$ for some $\alpha \geq k-n$, in particular, if $f \in L^p(\mathfrak{M}_{n,m})$, $1 \leq p < p_0$, $p_0 = (n+m-1)/(k+m-1)$, then $\hat{f}(\tau)$ is finite for almost all $\tau \in \mathfrak{T}$.*

As in Theorem 0.1, the conditions $a > k+m-1$, $\alpha \geq k-n$, and $p < p_0$ cannot be improved. For instance, if $p \geq p_0$ and $f(x) = |2I_m + x'x|^{-(n+m-1)/2p} (\log |2I_m + x'x|)^{-1}$ $(\in L^p(\mathfrak{M}_{n,m}))$, then $\hat{f}(\tau) \equiv \infty$.

In fact, we have proved the following generalization of (0.7) which implies (ii) above:

$$\begin{aligned} &\frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi,t) \, \frac{|t't|^{(\lambda-n)/2}}{|I_m + t't|^{\lambda/2}} \, dt \\ &= c \int\limits_{\mathfrak{M}_{n,m}} f(x) \, \frac{|x'x|^{(\lambda-n)/2}}{|I_m + x'x|^{(\lambda-k)/2}} \, dx, \qquad Re\, \lambda > k+m-1, \end{aligned} \tag{0.33}$$

$1 \leq k \leq n-m$, where the constant $c$ is explicitly evaluated; see (4.48), (4.42).

We introduce *the dual Radon transform* which assigns to a function $\varphi(\tau)$ on $\mathfrak{T}$ the integral over all matrix $k$-planes through $x \in \mathfrak{M}_{n,m}$:

$$\check{\varphi}(x) = \int\limits_{\tau \ni x} \varphi(\tau). \tag{0.34}$$

In terms of the parameterization $\tau = \tau(\xi, t)$, $\xi \in V_{n,n-k}$, $t \in \mathfrak{M}_{n-k,m}$, this integral reads as follows:

$$\check{\varphi}(x) = \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \varphi(\xi, \xi' x) \, d\xi, \tag{0.35}$$

where $\sigma_{n,n-k}$ is the volume of the Stiefel manifold $V_{n,n-k}$; see (1.37). The integral (0.35) is well defined for any locally integrable function $\varphi$. We do not study the inversion problem for the dual Radon transform here. However, it is natural to conjecture that the dual Radon transform is injective if and only if $k \geq n-m$, and the relevant inversion formula can be derived from that for the Radon transform. The respective results were obtained in [**Ru5**] for Radon transforms on affine Grassmann manifolds. We plan to study this topic in forthcoming publications.

**0.3.2. The Radon transform and the dual Radon transform of radial functions.** We call functions $f$ on $\mathfrak{M}_{n,m}$ and $\varphi$ on $\mathfrak{T}$ *radial* if they are $O(n)$ left-invariant. Each radial function $f(x)$ on $\mathfrak{M}_{n,m}$ ($\varphi(\xi,t)$ on $\mathfrak{T}$, resp.) has the form $f(x) = f_0(x'x)$ ($\varphi(\xi,t) = \varphi_0(t't)$, resp.). We establish connection between the Radon transforms of radial functions and the *Gårding-Gindikin fractional integrals*

$$(I_+^\alpha f)(s) = \frac{1}{\Gamma_m(\alpha)} \int_0^s f(r)|s-r|^{\alpha-d} dr, \tag{0.36}$$

$$(I_-^\alpha f)(s) = \frac{1}{\Gamma_m(\alpha)} \int_s^\infty f(r)|r-s|^{\alpha-d} dr \tag{0.37}$$

associated to the cone $\mathcal{P}_m$ of positive definite symmetric $m \times m$ matrices. Here, $d = (m+1)/2$ and $\Gamma_m(\alpha)$ is the generalized gamma function (1.6) which is also known as the Siegel gamma function [**Si**]. In (0.36), $s$ belongs to $\mathcal{P}_m$, and integration is performed over the "interval" $(0,s) = \{r : \; r \in \mathcal{P}_m, \; s-r \in \mathcal{P}_m\}$. In (0.37), $s$ belongs to the closure $\overline{\mathcal{P}}_m$, and one integrates over the shifted cone $s + \mathcal{P}_m = \{r : r \in \mathcal{P}_m, \; r-s \in \mathcal{P}_m\}$. For $m = 1$, the integrals (0.36) and (0.37) coincide with the Riemann-Liouville fractional integrals (0.9). Section 2.1 contains historical notes related to integrals $I_\pm^\alpha f$ in the higher rank case.

For sufficiently good $f$, the integrals $I_\pm^\alpha f$ converge absolutely if $Re\,\alpha > d-1$ and extend to all $\alpha \in \mathbb{C}$ as entire functions of $\alpha$. If $\alpha$ is real and belongs to *the Wallach-like set*

$$\mathcal{W}_m = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{m-1}{2}\right\} \cup \left\{\alpha : \; Re\,\alpha > \frac{m-1}{2}\right\},$$

then $I_\pm^\alpha f$ are convolutions with positive measures; see (2.12), (2.22). The converse is also true; cf. [**FK**, p. 137]. Owing to this property, if $\alpha \in \mathcal{W}_m$, then one can define the Gårding-Gindikin fractional integrals for all locally integrable functions which in the case of $I_-^\alpha$ obey some extra conditions at infinity. Different explicit formulas for $I_\pm^\alpha f$, $\alpha \in \mathcal{W}_m$, are obtained in Section 2.2.

A natural analog of Theorem 0.2 for the higher rank case employs fractional integrals (0.36)–(0.37) and reads as follows.

THEOREM 0.6. *For $x \in \mathfrak{M}_{n,m}$ and $t \in \mathfrak{M}_{n-k,m}$, let $r = x'x$, $s = t't$.*
(i) *If $f(x)$ is radial, i.e., $f(x) = f_0(x'x)$, then*

$$\hat{f}(\xi,t) = \pi^{km/2}(I_-^{k/2} f_0)(s). \tag{0.38}$$

(ii) *Let $\varphi(\tau)$ be radial, i.e., $\varphi(\xi,t) = \varphi_0(t't)$. We denote*

$$\Phi_0(s) = |s|^\delta \varphi_0(s), \qquad \delta = (n-k)/2 - d, \quad d = (m+1)/2.$$

*If $1 \leq k \leq n-m$, then for any matrix $x \in \mathfrak{M}_{n,m}$ of rank $m$,*

$$\check{\varphi}(x) = c\,|r|^{d-n/2}(I_+^{k/2}\Phi_0)(r), \quad c = \pi^{km/2}\sigma_{n-k,m}/\sigma_{n,m}, \tag{0.39}$$

*$\sigma_{n,m}$ being the volume of the Stiefel manifold $V_{n,m}$. Equalities (0.38) and (0.39) hold for any locally integrable functions provided that either side exists in the Lebesgue sense.*

**0.3.3. Inversion of the Radon transform.** Theorem 0.6 reduces inversion of the Radon transform and the dual Radon transform of radial functions to the similar problem for the Gårding-Gindikin fractional integrals. If $f$ belongs to the space $\mathcal{D}(\mathcal{P}_m)$ of infinitely differentiable functions compactly supported away from the boundary $\partial\mathcal{P}_m$, we proceed as follows. For $r = (r_{i,j}) \in \mathcal{P}_m$, we consider the differential operators $D_\pm$ defined by

$$D_+ = \det\left(\eta_{i,j}\,\frac{\partial}{\partial r_{i,j}}\right), \quad \eta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 1/2 & \text{if } i \neq j \end{cases}, \qquad D_- = (-1)^m D_+.$$

If $\alpha \in \mathbb{C}$ and $j \in \mathbb{N}$, then

$$D_\pm^j I_\pm^\alpha f = I_\pm^\alpha D_\pm^j f = I_\pm^{\alpha-j} f, \qquad f \in \mathcal{D}(\mathcal{P}_m).$$

Hence $I_\pm^\alpha f$ can be inverted by the formula

$$f = D_\pm^j I_\pm^{j-\alpha} g, \quad g = I_\pm^\alpha f, \tag{0.40}$$

where $j \geq \alpha$ if $\alpha = 1/2, 1, \dots$ , and $j > Re\,\alpha + d - 1$ otherwise. For "rough" functions $f$, we interpret (0.40) in the sense of distributions; see Lemmas 2.23 and 2.24. A challenging open problem is to obtain "pointwise" inversion formulas for the Gårding-Gindikin fractional integrals similar to those for the classical Riemann-Liouville integrals [**Ru1**], [**SKM**]. It is desirable these formulas would not contain neither operations in the sense of distributions nor the Laplace (or the Fourier) transform, i.e., the problem would be resolved in the same language as it has been stated.

In the general case, we develop the following inversion methods for the Radon transform which extend the classical ones to the higher rank case.

*The method of mean value operators.* The Radon transform $\hat{f}(\tau)$ commutes with the group $M(n,m)$ of matrix motions sending $x \in \mathfrak{M}_{n,m}$ to $\gamma x \beta + b$, where $\gamma \in O(n)$ and $\beta \in O(m)$ are orthogonal matrices, and $b \in \mathfrak{M}_{n,m}$. This property allows us to apply a certain mean value operator to $\hat{f}(\tau)$ and thus reduce the inversion problem to the case of radial functions.

Let us make the following definitions. We supply the matrix space $\mathfrak{M}_{n,m}$ and the manifold $\mathfrak{T}$ of matrix $k$-planes with an entity that serves as a substitute for the euclidean distance. Specifically, given two points $x$ and $y$ in $\mathfrak{M}_{n,m}$, *the matrix distance* $d(x,y)$ is defined by

$$d(x,y) = [(x-y)'(x-y)]^{1/2}.$$

Accordingly, the "distance" between $x \in \mathfrak{M}_{n,m}$ and a matrix $k$-plane $\tau = \tau(\xi, t) \in \mathfrak{T}$ is defined by

$$d(x,\tau) = [(\xi' x - t)'(\xi' x - t)]^{1/2}.$$

Unlike the rank-one case, these quantities are not scalar-valued and represented by positive semi-definite matrices. Radial functions $f(x)$ on $\mathfrak{M}_{n,m}$ and $\varphi(\tau)$ on $\mathfrak{T}$ virtually depend on the matrix distance to the 0-matrix from $x$ and $\tau$, respectively.

For functions $\varphi(\tau)$, $\tau \in \mathfrak{T}$, we define *the shifted dual Radon transform*

$$\check{\varphi}_s(x) = \int_{d(x,\tau)=s^{1/2}} \varphi(\tau), \qquad s \in \mathcal{P}_m. \tag{0.41}$$

This is the mean value of $\varphi(\tau)$ over all matrix planes at distance $s^{1/2}$ from $x$. For $s = 0$, it coincides with the dual Radon transform $\check{\varphi}(x)$. A matrix generalization of the classical spherical mean is

$$(M_r f)(x) = \frac{1}{\sigma_{n,m}} \int\limits_{V_{n,m}} f(x + vr^{1/2}) dv, \qquad r \in \mathcal{P}_m.$$

The following statement generalizes Lemma 0.3 to the case $m > 1$.

LEMMA 0.7. *For fixed $x \in \mathfrak{M}_{n,m}$, let $F_x(r) = (M_r f)(x)$. If $f \in L^p(\mathfrak{M}_{n,m})$, $1 \le p < (n+m-1)/(k+m-1)$, then*

$$(\hat{f})^\vee_s(x) = \pi^{km/2}(I_-^{k/2} F_x)(s), \qquad s \in \mathcal{P}_m. \tag{0.42}$$

Owing to (0.42) and the inversion formula for the Gårding-Gindikin fractional integrals, one can recover $f(x)$ from the Radon transform $\hat{f}(\tau)$ as follows.

THEOREM 0.8. *Let $f \in L^p(\mathfrak{M}_{n,m})$,*

$$1 \le k \le n - m, \qquad 1 \le p < \frac{n+m-1}{k+m-1}.$$

*If $\varphi(\tau) = \hat{f}(\tau)$, then*

$$f(x) = \pi^{-km/2} \overset{(L^p)}{\lim_{r \to 0}} (D_-^{k/2} \Phi_x)(r), \qquad \Phi_x(s) = \check{\varphi}_s(x), \tag{0.43}$$

*where $D_-^{k/2} = I_-^{-k/2}$ is understood in the sense of $\mathcal{D}'$-distributions.*

*The method of Riesz potentials and the method of plane waves.* We extend the method of Riesz potentials to the higher rank case and give a new account of the method of plane waves. The second method was developed by E.E. Petrov [**P1**] for the case $k = n - m$, and outlined by L.P. Shibasov [**Sh2**] for all $1 \le k \le n - m$. Both methods employ the following matrix analog of the Riesz potential (0.18):

$$(I^\alpha f)(x) = \frac{1}{\gamma_{n,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,m}} f(x-y)\, |y|_m^{\alpha-n}\, dy, \quad Re\, \alpha > m-1, \tag{0.44}$$

where $|y|_m = \det(y'y)^{1/2}$,

$$\gamma_{n,m}(\alpha) = \frac{2^{\alpha m}\, \pi^{nm/2}\, \Gamma_m(\alpha/2)}{\Gamma_m((n-\alpha)/2)}, \qquad \alpha \ne n-m+1,\, n-m+2, \dots. \tag{0.45}$$

We consider *the Cayley-Laplace operator*

$$(\Delta f)(x) = (\det(\partial'\partial) f)(x), \qquad x = (x_{i,j}) \in \mathfrak{M}_{n,m},$$

where $\partial$ is the $n \times m$ matrix whose entries are partial derivatives $\partial/\partial x_{i,j}$. The name "Cayley-Laplace" was attributed to this operator by S.P. Khekalo [**Kh**] and suits it admirably. On the Fourier transform side, the action of $\Delta$ represents a multiplication by $(-1)^m |y|_m^2$. Since the Fourier transform of the Riesz distribution $|x|_m^{\alpha-n}/\gamma_{n,m}(\alpha)$ is $|y|_m^{-\alpha}$ in a suitable sense, then $I^\alpha$ can be regarded as $(-\alpha/2)$th power of the operator $(-1)^m\Delta$. For $Re\, \alpha \le m-1$, $\alpha \ne n-m+1,\, n-m+2, \dots$ and sufficiently smooth $f$, the Riesz potential is defined as analytic continuation of the integral (0.44). If $\alpha$ is real and belongs to the corresponding Wallach-like set

$$\mathbf{W}_{n,m} = \{0, 1, 2, \dots, k_0\} \cup \{\alpha : Re\, \alpha > m-1;\ \alpha \ne n-m+1,\, n-m+2, \dots\},$$

$$k_0 = \min(m-1, n-m),$$

then $I^\alpha f$ is a convolution with a positive measure; see (3.54)–(3.56). This result allows us to extend the original definition of the Riesz potential to locally integrable functions provided $\alpha \in \mathbf{W}_{n,m}$. It was shown in [**Ru7**], that if

$$f \in L^p(\mathfrak{M}_{n,m}), \qquad 1 \le p < \frac{n}{Re\,\alpha + m - 1}, \qquad \alpha \in \mathbf{W}_{n,m}, \tag{0.46}$$

then $(I^\alpha f)(x)$ is finite for almost all $x \in \mathfrak{M}_{n,m}$. On the other hand, if

$$p \ge \frac{n+m-1}{k+m-1}$$

then there is a function $f \in L^p(\mathfrak{M}_{n,m})$ such that $(I^k f)(x) \equiv \infty$ (see (6.16), Theorem 4.27, and Appendix B). It is still not clear what happens if

$$\frac{n}{Re\,\alpha + m - 1} \le p < \frac{n+m-1}{Re\,\alpha + m - 1}\ . \tag{0.47}$$

This gap represents an open problem.

As in the rank-one case, the method of Riesz potentials and the method of plane waves employ intertwining operators

$$P^\alpha f = \tilde{I}^\alpha \hat{f}, \qquad \overset{*}{P}{}^\alpha \varphi = (\tilde{I}^\alpha \varphi)^\vee, \tag{0.48}$$

$$\alpha \in \mathbb{C}, \qquad \alpha \ne n-k-m+1,\ n-k-m+2, \dots\ .$$

Here, $1 \le k \le n-m$ and $\tilde{I}^\alpha$ is the Riesz potential on $\mathfrak{M}_{n-k,m}$ acting in the $t$-variable. If

$$Re\,\alpha > m-1, \qquad \alpha \ne n-k-m+1,\ n-k-m+2, \dots\ ,$$

then

$$(P^\alpha f)(\xi, t) = \frac{1}{\gamma_{n-k,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,m}} f(x)\, |\xi' x - t|_m^{\alpha+k-n}\, dx, \tag{0.49}$$

$$(\overset{*}{P}{}^\alpha \varphi)(x) = \frac{1}{\gamma_{n-k,m}(\alpha)} \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi, t)\, |\xi' x - t|_m^{\alpha+k-n}\, dt, \tag{0.50}$$

where $d_*\xi = \sigma_{n,n-k}^{-1}\, d\xi$ is the normalized measure on $V_{n,n-k}$ and $\gamma_{n-k,m}(\alpha)$ is the normalizing constant in the definition of the Riesz potential on $\mathfrak{M}_{n-k,m}$; cf. (0.45). We call $P^\alpha f$ and $\overset{*}{P}{}^\alpha \varphi$ *the generalized Semyanistyi fractional integrals*. For $\alpha = 0$, operators (0.48) include the Radon transform and its dual, respectively. In the case $m = 1$, the integrals (0.49) and (0.50) coincide with those in (0.20) and (0.21).

THEOREM 0.9. *Let* $1 \le k \le n-m$, $\alpha \in \mathbf{W}_{n,m}$. *Suppose that*

$$f \in L^p, \qquad 1 \le p < \frac{n}{Re\,\alpha + k + m - 1}\ .$$

*Then*

$$(\overset{*}{P}{}^\alpha \hat{f})(x) = c_{n,k,m} (I^{\alpha+k} f)(x), \tag{0.51}$$

$$c_{n,k,m} = 2^{km} \pi^{km/2} \Gamma_m\left(\frac{n}{2}\right) / \Gamma_m\left(\frac{n-k}{2}\right).$$

*In particular,*

$$(\hat{f})^\vee(x) = c_{n,k,m} (I^k f)(x) \tag{0.52}$$

*(the generalized Fuglede formula).*

Theorem 0.9 is a higher rank copy of Theorem 0.4. Thus, as in the case $m = 1$, inversion of the Radon transform reduces to inversion of the Riesz potential. Unlike the rank-one case, we have not succeeded to obtain a pointwise inversion formula for the higher rank Riesz potential so far, and use the theory of distributions. The properly defined space $\Phi(\mathfrak{M}_{n,m})$ of test functions is invariant under the action of Riesz potentials, and

$$(I^{-\alpha}\phi)(x) = (\mathcal{F}^{-1}\, |y|_m^\alpha\, \mathcal{F}\phi)(x), \qquad \phi \in \Phi, \qquad \alpha \in \mathbb{C}, \tag{0.53}$$

in terms of the Fourier transform.

THEOREM 0.10. *Let $f \in L^p(\mathfrak{M}_{n,m})$, $1 \le p < n/(k+m-1)$, so that the Radon transform $\varphi(\tau) = \hat{f}(\tau)$ is finite for almost all matrix $k$-planes $\tau$. The function $f$ can be recovered from $\varphi$ in the sense of $\Phi'$-distributions by the formula*

$$c_{n,k,m}(f, \phi) = (\check{\varphi}, I^{-k}\phi), \qquad \phi \in \Phi, \tag{0.54}$$

*where $I^{-k}$ is the operator (0.53). In particular, for $k$ even,*

$$c_{n,k,m}(f, \phi) = (-1)^{mk/2}(\check{\varphi}, \Delta^{k/2}\phi), \qquad \phi \in \Phi. \tag{0.55}$$

This theorem reflects the essence of the method of Riesz potentials. The case $\alpha = -k$ in (0.51) provides the following inversion result in the framework of the method of plane waves.

THEOREM 0.11. *Let $1 \le k \le n - m$. If $f$ belongs to the Schwartz space $\mathcal{S}(\mathfrak{M}_{n,m})$, then the Radon transform $\varphi = \hat{f}$ can be inverted by the formula*

$$c_{n,k,m} f(x) = (\overset{*}{P}{}^{-k}\varphi)(x). \tag{0.56}$$

Explicit expressions for $\overset{*}{P}{}^{-k}\varphi$ are given by (6.21)-(6.23).

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CHAPTER 1

# Preliminaries

In this chapter, we establish our notation and recall some basic facts important for the sequel. The main references are [**FZ**], [**Herz**], [**Mu**], [**T**].

## 1.1. Matrix spaces. Notation

Let $\mathfrak{M}_{n,m}$ be the space of real matrices having $n$ rows and $m$ columns; $\mathfrak{M}^{(k)}_{n,m} \subset \mathfrak{M}_{n,m}$ is the subset of all matrices of rank $k$. We identify $\mathfrak{M}_{n,m}$ with the real Euclidean space $\mathbb{R}^{nm}$. The latin letters $x$, $y$, $r$, $s$, etc. stand for both the matrices and the points since it is always clear from the context which is meant. We use a standard notation $O(n)$ and $SO(n)$ for the group of real orthogonal $n \times n$ matrices and its connected component of the identity, respectively. The corresponding invariant measures on $O(n)$ and $SO(n)$ are normalized to be of total mass 1. We denote by $M(n, m)$ the group of transformations sending $x \in \mathfrak{M}_{n,m}$ to $\gamma x \beta + b$, where $\gamma \in O(n)$, $\beta \in O(m)$, and $b \in \mathfrak{M}_{n,m}$. We call $M(n, m)$ *the group of matrix motions*.

If $x = (x_{i,j}) \in \mathfrak{M}_{n,m}$, we write $dx = \prod_{i=1}^{n} \prod_{j=1}^{m} dx_{i,j}$ for the elementary volume in $\mathfrak{M}_{n,m}$. In the following, $x'$ denotes the transpose of $x$, $I_m$ is the identity $m \times m$ matrix, and 0 stands for zero entries. Given a square matrix $a$, we denote by $\det(a)$ the determinant of $a$, and by $|a|$ the absolute value of $\det(a)$; $\mathrm{tr}(a)$ stands for the trace of $a$.

Let $\mathcal{S}_m$ be the space of $m \times m$ real symmetric matrices $s = (s_{i,j})$, $s_{i,j} = s_{j,i}$. It is a measure space isomorphic to $\mathbb{R}^{m(m+1)/2}$ with the volume element $ds = \prod_{i \le j} ds_{i,j}$. We denote by $\mathcal{P}_m$ the cone of positive definite matrices in $\mathcal{S}_m$; $\overline{\mathcal{P}}_m$ is the closure of $\mathcal{P}_m$, that is the set of all positive semi-definite $m \times m$ matrices. If $m = 1$, then $\mathcal{P}_m$ is the open half-line $(0, \infty)$ and $\overline{\mathcal{P}}_m = [0, \infty)$. For $r \in \mathcal{P}_m$ ($r \in \overline{\mathcal{P}}_m$) we write $r > 0$ ($r \ge 0$). Given $s_1$ and $s_2$ in $S_m$, the inequality $s_1 > s_2$ means $s_1 - s_2 \in \mathcal{P}_m$. If $a \in \overline{\mathcal{P}}_m$ and $b \in \mathcal{P}_m$, then the symbol $\int_a^b f(s)ds$ denotes the integral over the set

$$\{s : s \in \mathcal{P}_m,\, a < s < b\} = \{s : s - a \in \mathcal{P}_m,\, b - s \in \mathcal{P}_m\}.$$

For $s \in S_m$, we denote by $s_+^\lambda$ the function which equals $|s|^\lambda$ for $s \in \mathcal{P}_m$ and zero otherwise; $s_-^\lambda$ equals $(-s)_+^\lambda$.

The group $G = GL(m, \mathbb{R})$ of real non-singular $m \times m$ matrices $g$ acts transitively on $\mathcal{P}_m$ by the rule $r \to g'rg$. The corresponding $G$-invariant measure on $\mathcal{P}_m$ is

$$d_*r = |r|^{-d} dr, \qquad |r| = \det(r), \qquad d = (m+1)/2 \tag{1.1}$$

[**T**, p. 18]. The cone $\mathcal{P}_m$ is a $G$-orbit in $\mathcal{S}_m$ of the identity matrix $I_m$. The boundary $\partial\mathcal{P}_m$ of $\mathcal{P}_m$ is a union of $G$-orbits of $m \times m$ matrices

$$e_k = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}, \qquad k = 0, 1, \ldots, m-1.$$

More information about the boundary structure of $\mathcal{P}_m$ can be found in [**FK**, p. 72] and [**Bar**, p. 78].

Let $T_m$ be the subgroup of $GL(m,\mathbb{R})$ consisting of upper triangular matrices

$$t = \begin{bmatrix} t_{1,1} & & & \\ & \cdot & t_* & \\ & & \cdot & \\ & 0 & \cdot & \\ & & & t_{m,m} \end{bmatrix}, \qquad t_{i,i} > 0, \tag{1.2}$$

$$t_* = \{t_{i,j} : i < j\} \in \mathbb{R}^{m(m-1)/2}.$$

Each $r \in \mathcal{P}_m$ has a unique representation $r = t't,\ t \in T_m$, so that

$$\begin{aligned} \int\limits_{\mathcal{P}_m} f(r)dr &= \int\limits_0^\infty t_{1,1}^m \, dt_{1,1} \int\limits_0^\infty t_{2,2}^{m-1} \, dt_{2,2} \, \ldots \\ &\times \int\limits_0^\infty t_{m,m} \tilde{f}(t_{1,1}, \ldots, t_{m,m}) \, dt_{m,m}, \end{aligned} \tag{1.3}$$

$$\tilde{f}(t_{1,1}, \ldots, t_{m,m}) = 2^m \int\limits_{\mathbb{R}^{m(m-1)/2}} f(t't) \, dt_*, \quad dt_* = \prod_{i<j} dt_{i,j},$$

[**T**, p. 22] , [**Mu**, p. 592]. In the last integration, the diagonal entries of the matrix $t$ are given by the arguments of $\tilde{f}$, and the strictly upper triangular entries of $t$ are variables of integration.

We recall some useful formulas for Jacobians; see, e.g., [**Mu**, pp. 57–59].

Lemma 1.1.
(i) *If* $x = ayb$ *where* $y \in \mathfrak{M}_{n,m}$, $a \in GL(n,\mathbb{R})$, *and* $b \in GL(m,\mathbb{R})$, *then* $dx = |a|^m |b|^n dy$.
(ii) *If* $r = q'sq$ *where* $s \in S_m$, *and* $q \in GL(m,\mathbb{R})$, *then* $dr = |q|^{m+1} ds$.
(iii) *If* $r = s^{-1}$ *where* $s \in \mathcal{P}_m$, *then* $r \in \mathcal{P}_m$, *and* $dr = |s|^{-m-1} ds$.

Some more notation are in order: $\mathbb{N}$ denotes the set of all positive integers; $\mathbb{R}_+ = (0,\infty)$; $\delta_{ij}$ is the usual Kronecker delta. The space $C(\mathfrak{M}_{n,m})$ of continuous functions, the Lebesgue space $L^p(\mathfrak{M}_{n,m})$, and the Schwartz space $\mathcal{S}(\mathfrak{M}_{n,m})$ of $C^\infty$ rapidly decreasing functions are identified with the corresponding spaces on $\mathbb{R}^{nm}$. We denote by $C_c(\mathfrak{M}_{n,m})$ the space of compactly supported continuous functions on $\mathfrak{M}_{n,m}$. The spaces $C^\infty(\mathcal{P}_m)$, $C^k(\mathcal{P}_m)$ and $\mathcal{S}(\mathcal{P}_m)$ consist of restrictions onto $\mathcal{P}_m$ of the corresponding functions on $\mathcal{S}_m \sim \mathbb{R}^N$, $N = m(m+1)/2$. We denote by $\mathcal{D}(\mathcal{P}_m)$ the space of functions $f \in C^\infty(\mathcal{P}_m)$ with supp$f \subset \mathcal{P}_m$; $L^1_{loc}(\mathcal{P}_m)$ is the space of locally integrable functions on $\mathcal{P}_m$. The Fourier transform of a function $f \in L^1(\mathfrak{M}_{n,m})$ is defined by

$$(\mathcal{F}f)(y) = \int\limits_{\mathfrak{M}_{n,m}} \exp(\mathrm{tr}(iy'x)) f(x) dx, \qquad y \in \mathfrak{M}_{n,m} \,. \tag{1.4}$$

This is the usual Fourier transform on $\mathbb{R}^{nm}$ so that the relevant Parseval formula reads

$$(\mathcal{F}f, \mathcal{F}\varphi) = (2\pi)^{nm} \, (f, \varphi), \tag{1.5}$$

where

$$(f,\varphi) = \int\limits_{\mathfrak{M}_{n,m}} f(x)\varphi(x)\,dx.$$

We write $c$, $c_1$, $c_2, \dots$ for different constants the meaning of which is clear from the context.

## 1.2. Gamma and beta functions of the cone of positive definite matrices

The generalized *gamma function* associated to the cone $\mathcal{P}_m$ of positive definite symmetric matrices is defined by

$$\Gamma_m(\alpha) = \int\limits_{\mathcal{P}_m} \exp(-\mathrm{tr}(r))|r|^{\alpha-d}dr, \qquad d=(m+1)/2. \tag{1.6}$$

This is also known as the *Siegel integral* [**Si**] (1935) arising in number theory. Actually it came up earlier in statistics. Substantial generalizations of (1.6) and historical comments can be found in [**Gi1**], [**FK**, Chapter VII], [**T**, p. 41], [**GrRi**, p. 800].

Using (1.3), it is easy to check [**Mu**, p. 62] that the integral (1.6) converges absolutely if and only if $Re\,\alpha > d-1$. Moreover, $\Gamma_m(\alpha)$ is a product of ordinary gamma functions:

$$\Gamma_m(\alpha) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(\alpha - j/2)\ . \tag{1.7}$$

This implies a number of useful formulas:

$$\frac{\Gamma_m(d-\alpha+1)}{\Gamma_m(d-\alpha)} = (-1)^m \frac{\Gamma_m(\alpha)}{\Gamma_m(\alpha-1)}, \qquad d=(m+1)/2, \tag{1.8}$$

$$(-1)^m \frac{\Gamma_m(1-\alpha/2)}{\Gamma_m(-\alpha/2)} = 2^{-m}\frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} = 2^{-m}(\alpha,m), \tag{1.9}$$

where $(\alpha,m) = \alpha(\alpha+1)\cdots(\alpha+m-1)$ is the Pochhammer symbol. Furthermore, for $1\le k<m,\ k\in\mathbb{N}$, the equality (1.7) yields

$$\frac{\Gamma_m(\alpha)}{\Gamma_m(\alpha+k/2)} = \frac{\Gamma_k(\alpha+(k-m)/2)}{\Gamma_k(\alpha+k/2)}, \tag{1.10}$$

$$\Gamma_m(\alpha) = \pi^{k(m-k)/2}\Gamma_k(\alpha)\Gamma_{m-k}(\alpha-k/2). \tag{1.11}$$

In particular, for $k=m-1$,

$$\Gamma_m(\alpha) = \pi^{(m-1)/2}\Gamma_{m-1}(\alpha)\Gamma(\alpha-(m-1)/2). \tag{1.12}$$

The *beta function* of the cone $\mathcal{P}_m$ is defined by

$$B_m(\alpha,\beta) = \int\limits_0^{I_m} |r|^{\alpha-d}|I_m-r|^{\beta-d}dr, \tag{1.13}$$

where, as above, $d=(m+1)/2$. This integral converges absolutely if and only if $Re\,\alpha, Re\,\beta > d-1$. The following classical relation holds

$$B_m(\alpha,\beta) = \frac{\Gamma_m(\alpha)\Gamma_m(\beta)}{\Gamma_m(\alpha+\beta)} \tag{1.14}$$

[**FK**, p. 130]. In Appendix A, we present a table of some integrals that can be easily evaluated in terms of the generalized gamma and beta functions. Most of these integrals are known but scattered all over the different sources. This table will be repeatedly referred to in the sequel. For the sake of completeness, we mention the paper [**Ner**] containing many other formulas of this type in the very general set-up.

## 1.3. The Laplace transform

Let $\mathcal{T}_m = \mathcal{P}_m + i\mathcal{S}_m$ be the generalized half-plane in the space $\mathcal{S}_m^{\mathbb{C}} = \mathcal{S}_m + i\mathcal{S}_m$, the complexification of $\mathcal{S}_m$. The domain $\mathcal{T}_m$ consists of all complex symmetric matrices $z = \sigma + i\omega$ such that $\sigma = Re\ z \in \mathcal{P}_m$, and $\omega = Im\ z \in \mathcal{S}_m$. Let $f$ be a locally integrable function on $\mathcal{S}_m$, $f(r) = 0$ if $r \notin \overline{\mathcal{P}}_m$, and $\exp(-\mathrm{tr}(\sigma_0 r))f(r) \in L^1(\mathcal{S}_m)$ for some $\sigma_0 \in \mathcal{P}_m$. Then the integral

$$(Lf)(z) = \int_{\mathcal{P}_m} \exp(-\mathrm{tr}(zr))f(r)dr \tag{1.15}$$

is absolutely convergent in the (generalized) half-plane $Re\ z > \sigma_0$ and represents a complex analytic function there. $Lf$ is called the Laplace transform of $f$. If

$$(\mathcal{F}g)(\omega) = \int_{\mathcal{S}_m} \exp(\mathrm{tr}(i\omega s))g(s)ds, \qquad \omega \in \mathcal{S}_m, \tag{1.16}$$

is the Fourier transform of a function $g$ on $\mathcal{S}_m$, then

$$(Lf)(z) = (\mathcal{F}g_\sigma)(-\omega),$$

where $g_\sigma(r) = \exp(-\mathrm{tr}(\sigma r))f(r) \in L^1(\mathcal{S}_m)$ for $\sigma > \sigma_0$. Thus, all properties of the Laplace transform are obtained from the general Fourier transform theory for Euclidean spaces [**Herz**], [**Vl**, p. 126]. In particular, we have the following statement.

THEOREM 1.2. *Let* $\exp(-\mathrm{tr}(\sigma_0 r))f(r) \in L^1(\mathcal{P}_m)$, $\sigma_0 \in \mathcal{P}_m$, *and let*

$$\int_{\mathcal{S}_m} |(Lf)(\sigma + i\omega)|d\omega < \infty$$

*for some* $\sigma > \sigma_0$. *Then the Cauchy inversion formula holds:*

$$\frac{1}{(2\pi i)^N} \int_{Re\, z=\sigma} \exp(\mathrm{tr}(sz)(Lf)(z)dz = \begin{cases} f(s) & \text{if } s \in \mathcal{P}_m, \\ 0 & \text{if } s \notin \mathcal{P}_m, \end{cases} \tag{1.17}$$

$N = m(m+1)/2$. *The integration in (1.17) is performed over all* $z = \sigma + i\omega$ *with fixed* $\sigma > \sigma_0$ *and* $\omega$ *ranging over* $\mathcal{S}_m$.

The following uniqueness result for the Laplace transform immediately follows from injectivity of the Fourier transform of tempered distributions.

LEMMA 1.3. *If* $f_1(r)$ *and* $f_2(r)$ *satisfy*

$$\exp(-\mathrm{tr}(\sigma_0 r))f_j(r) \in L^1(\mathcal{P}_m), \quad j = 1, 2, \tag{1.18}$$

*for some* $\sigma_0 \in \mathcal{P}_m$, *and* $(Lf_1)(z) = (Lf_2)(z)$ *whenever* $Re\ z > \sigma_0$, *then* $f_1(r) = f_2(r)$ *almost everywhere on* $\mathcal{S}_m$.

The convolution theorem for the Laplace transform reads as follows: If $f_1$ and $f_2$ obey (1.18), and

$$(f_1 * f_2)(r) = \int_0^r f_1(r-s) f_2(s) ds$$

is the Laplace convolution, then

$$L(f_1 * f_2)(z) = (Lf_1)(z)(Lf_2)(z), \quad Re\, z > \sigma_0. \tag{1.19}$$

LEMMA 1.4. *Let $z \in \mathcal{T}_m = \mathcal{P}_m + i\mathcal{S}_m$, $d = (m+1)/2$. Then*

$$\int_{\mathcal{P}_m} \exp(-\mathrm{tr}(zr)) |r|^{\alpha-d} dr = \Gamma_m(\alpha) \det(z)^{-\alpha}, \qquad Re\, \alpha > d-1. \tag{1.20}$$

Here and throughout the paper, we set $\det(z)^{-\alpha} = \exp(-\alpha \log \det(z))$ where the branch of $\log \det(z)$ is chosen so that $\det(z) = |\sigma|$ for real $z = \sigma \in \mathcal{P}_m$.

The equality (1.20) is well known, see, e.g., [**Herz**, p. 479]. For real $z = \sigma \in \mathcal{P}_m$, it can be easily obtained by changing variable $r \to z^{-1/2} r z^{-1/2}$. For complex $z$, it follows by analytic continuation.

## 1.4. Differential operators $D_\pm$

Let $r = (r_{i,j}) \in \mathcal{P}_m$. We define the following differential operators acting in the $r$-variable:

$$D_+ \equiv D_{+,\,r} = \det\left(\eta_{i,j} \frac{\partial}{\partial r_{i,j}}\right), \quad \eta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 1/2 & \text{if } i \neq j, \end{cases} \tag{1.21}$$

$$D_- \equiv D_{-,\,r} = (-1)^m D_{+,\,r}. \tag{1.22}$$

LEMMA 1.5. *If $f \in \mathcal{D}(\mathcal{P}_m)$, and the derivatives $D_\mp g$ exist in a neighborhood of the support of $f$, then*

$$\int_{\mathcal{P}_m} (D_\pm f)(r) g(r) dr = \int_{\mathcal{P}_m} f(r)(D_\mp g)(r) dr. \tag{1.23}$$

PROOF. Since $\operatorname{supp} f \subset \mathcal{P}_m$, one can replace integrals over $\mathcal{P}_m$ by those over the whole space $\mathcal{S}_m \sim \mathbb{R}^{m(m+1)/2}$. The function $g$ can be multiplied by a smooth cut-off function (in necessary) which $\equiv 1$ on the support of $f$. Then the ordinary integration by parts yields (1.23). □

LEMMA 1.6.
(i) *For $s \in \mathcal{P}_m$ and $z \in \mathcal{S}_m^{\mathbb{C}}$,*

$$D_{+,\,s}[\exp(-\mathrm{tr}(sz))] = (-1)^m \det(z) \exp(-\mathrm{tr}(sz)). \tag{1.24}$$

(ii) *For $s \in \mathcal{P}_m$ and $\alpha \in \mathbb{C}$,*

$$D_+(|s|^{\alpha-d}/\Gamma_m(\alpha)) = |s|^{\alpha-1-d}/\Gamma_m(\alpha-1), \qquad d = (m+1)/2. \tag{1.25}$$

PROOF. These statements are well known; see, e.g., [**Gå**, p. 813], [**Herz**, p. 481], [**FK**, p. 125]. We recall the proof for convenience of the reader. The equality (1.24) is verified by direct calculation. Furthermore, by (1.20),

$$\Gamma_m(\beta)|s|^{-\beta} = \int_{\mathcal{P}_m} \exp(-\mathrm{tr}(rs)) |r|^{\beta-d} dr, \qquad Re\, \beta > d-1. \tag{1.26}$$

Applying $D_+$ to (1.26) and using (1.24), we get

$$D_+(|s|^{-\beta}\Gamma_m(\beta)) = (-1)^m|s|^{-\beta-1}\Gamma_m(\beta+1), \qquad Re\,\beta > d-1.$$

Now we set $\beta = d-\alpha$. Since

$$\Gamma_m(d-\alpha+1)/\Gamma_m(d-\alpha) = (-1)^m\Gamma_m(\alpha)/\Gamma_m(\alpha-1),$$

(1.25) follows for $Re\,\alpha < 1$. By analyticity, it holds generally. □

REMARK 1.7. By changing notation and using (1.9), one can write (1.25) as

$$D_+|s|^\alpha = b(\alpha)|s|^{\alpha-1}, \tag{1.27}$$

where

$$b(\alpha) = \alpha(\alpha+1/2)\cdots(\alpha+d-1), \qquad d = (m+1)/2, \tag{1.28}$$

is the so-called *Bernstein polynomial* of the determinant [**FK**]. It is worth noting that $b(\alpha)$ can be written in different forms, namely,

$$b(\alpha) = (-1)^m b(1-d-\alpha) = 2^{-m}\Gamma(2\alpha+m)/\Gamma(2\alpha) = 2^{-m}(2\alpha, m) \tag{1.29}$$

or

$$b(\alpha) = (-1)^m\Gamma_m(1-\alpha)/\Gamma_m(-\alpha) = \Gamma_m(\alpha+d)/\Gamma_m(\alpha+d-1). \tag{1.30}$$

LEMMA 1.8. *For $f \in \mathcal{D}(\mathcal{P}_m)$ and $z \in \mathcal{S}_m^{\mathbb{C}}$,*

$$(LD_+f)(z) = \det(z)(Lf)(z). \tag{1.31}$$

PROOF. By (1.23), (1.22) and (1.24),

$$(LD_+f)(z) = \int\limits_{\mathcal{P}_m} D_{-,\,r}[\exp(-\mathrm{tr}(rz))]f(r)dr = \det(z)(Lf)(z),$$

as desired. □

## 1.5. Bessel functions of matrix argument

We recall some facts from [**Herz**] and [**FK**]. *The $\mathcal{J}$-Bessel function $\mathcal{J}_\nu(r)$,* $r \in \mathcal{P}_m$, can be defined in terms of the Laplace transform by the property

$$\int\limits_{\mathcal{P}_m} \exp(-\mathrm{tr}(zr))\mathcal{J}_\nu(r)|r|^{\nu-d}dr = \Gamma_m(\nu)\exp(-\mathrm{tr}(z^{-1}))\det(z)^{-\nu}, \tag{1.32}$$

$$d = (m+1)/2, \qquad z \in \mathcal{T}_m = \mathcal{P}_m + i\mathcal{S}_m.$$

This gives

$$\frac{1}{\Gamma_m(\nu)}\,\mathcal{J}_\nu(r) = \frac{1}{(2\pi i)^N}\int\limits_{Re\,z=\sigma_0} \exp(\mathrm{tr}(z - rz^{-1}))\det(z)^{-\nu}dz, \quad \sigma_0 \in \mathcal{P}_m, \tag{1.33}$$

$N = m(m+1)/2$. This integral is absolutely convergent for $Re\,\nu > m$ and extends analytically to all $\nu \in \mathbb{C}$. Moreover, the formula (1.33) allows us to extend $\mathcal{J}_\nu(r)$ in the $r$-variable as an entire function $\tilde{\mathcal{J}}_\nu(z)$, $z = (z_{i,j})_{m\times m} \in \mathbb{C}^{m^2}$, so that $\tilde{\mathcal{J}}_\nu(z)\big|_{z=r} = \mathcal{J}_\nu(r)$ and

$$\tilde{\mathcal{J}}_\nu(\gamma z\gamma') = \tilde{\mathcal{J}}_\nu(z) \quad \text{for all} \quad \gamma \in O(m). \tag{1.34}$$

Functions satisfying (1.34) are called symmetric. If $\lambda_1, \dots, \lambda_m$ are eigenvalues of $z$, and

$$\sigma_1 = \lambda_1 + \ldots + \lambda_m, \quad \sigma_2 = \lambda_1\lambda_2 + \ldots + \lambda_{m-1}\lambda_m, \quad \ldots\;, \quad \sigma_m = \lambda_1 \ldots \lambda_m$$

are the corresponding elementary symmetric functions (which are polynomials of $z_{i,j}$), then each analytic symmetric function of $z \in \mathbb{C}^{m^2}$ is an analytic function of $m$ variables $\sigma_1, \dots, \sigma_m$. Since for any $r \in \mathcal{P}_m$ and $s \in \mathcal{P}_m$, the matrices

$$rs, \qquad sr, \qquad s^{1/2}rs^{1/2}, \qquad r^{1/2}sr^{1/2}$$

have the same eigenvalues, then

$$\tilde{\mathcal{J}}_\nu(rs) = \tilde{\mathcal{J}}_\nu(sr) = \tilde{\mathcal{J}}_\nu(s^{1/2}rs^{1/2}) = \tilde{\mathcal{J}}_\nu(r^{1/2}sr^{1/2}).$$

In the following, we keep the usual notation $\mathcal{J}_\nu(z)$ for the extended function $\tilde{\mathcal{J}}_\nu(z)$.

For $m = 1$, the classical Bessel function $J_\nu(r)$ expresses through $\mathcal{J}_\nu(r)$ by the formula

$$J_\nu(r) = \frac{1}{\Gamma(\nu+1)} \left(\frac{r}{2}\right)^\nu \mathcal{J}_{\nu+1}\left(\frac{r^2}{4}\right).$$

There is an intimate connection between the Fourier transform and $\mathcal{J}$-Bessel functions.

Theorem 1.9. [**Herz**, p. 492], [**FK**, p. 355] *If $f(x)$ is an integrable function of the form $f(x) = f_0(x'x)$ where $f_0$ is a function on $\mathcal{P}_m$, then*

$$(\mathcal{F}f)(y) \equiv \int\limits_{\mathfrak{M}_{n,m}} \exp(\mathrm{tr}(iy'x)) f_0(x'x)dx = \frac{\pi^{nm/2}}{\Gamma_m(n/2)} \tilde{f}_0\left(\frac{y'y}{4}\right), \tag{1.35}$$

*where*

$$\tilde{f}_0(s) = \int\limits_{\mathcal{P}_m} \mathcal{J}_{n/2}(rs)|r|^{n/2-d} f_0(r)dr, \qquad d = (m+1)/2 \tag{1.36}$$

*(the Hankel transform of $f_0$).*

This statement represents a matrix generalization of the classical result of Bochner, see, e.g., [**SW**, Chapter IV, Theorem 3.3] for the case $m = 1$. Since the product $rs$ in (1.36) may not be a positive definite matrix, the expression $\mathcal{J}_{n/2}(rs)$ is understood in the sense of analytic continuation explained above.

Note that one can meet a different notation for the $\mathcal{J}$-Bessel function of matrix argument in the literature. We follow the notation from [**FK**]. It relates to the notation $A_\delta(r)$ in [**Herz**] by the formula $\mathcal{J}_\nu(r) = \Gamma_m(\nu)A_\delta(r)$, $\delta = \nu - d$. More information about $\mathcal{J}$-Bessel functions and their generalizations can be found in [**Dib**], [**FK**], [**FT**], [**GK1**], [**GK2**].

## 1.6. Stiefel manifolds

For $n \geq m$, let $V_{n,m} = \{v \in \mathfrak{M}_{n,m} : v'v = I_m\}$ be *the Stiefel manifold* of orthonormal $m$-frames in $\mathbb{R}^n$. If $n = m$, then $V_{n,n} = O(n)$ is the orthogonal group in $\mathbb{R}^n$. It is known that $\dim V_{n,m} = m(2n-m-1)/2$ [**Mu**, p. 67]. The group $O(n)$ acts on $V_{n,m}$ transitively by the rule $g : v \to gv$, $\quad g \in O(n)$, in the sense of matrix multiplication. The same is true for the special orthogonal group $SO(n)$ provided $n > m$. We fix the corresponding invariant measure $dv$ on $V_{n,m}$ normalized by

$$\sigma_{n,m} \equiv \int\limits_{V_{n,m}} dv = \frac{2^m \pi^{nm/2}}{\Gamma_m(n/2)}, \tag{1.37}$$

[**Mu**, p. 70], [**J**, p. 57], [**FK**, p. 351]. This measure is also $O(m)$ right-invariant.

LEMMA 1.10. [**FK**, p. 354], [**Herz**, p. 495]. *The Fourier transform of the invariant measure $dv$ on $V_{n,m}$ represents the $\mathcal{J}$-Bessel function:*

$$\int\limits_{V_{n,m}} \exp(\mathrm{tr}(iy'v))dv = \sigma_{n,m}\mathcal{J}_{n/2}\left(\frac{y'y}{4}\right). \tag{1.38}$$

LEMMA 1.11. (polar decomposition). *Let $x \in \mathfrak{M}_{n,m}$, $n \geq m$. If* $\mathrm{rank}(x) = m$, *then*

$$x = vr^{1/2}, \qquad v \in V_{n,m}, \qquad r = x'x \in \mathcal{P}_m,$$

*and* $dx = 2^{-m}|r|^{(n-m-1)/2}dr dv$.

This statement can be found in many sources, see, e.g., [**Herz**, p. 482], [**GK1**, p. 93], [**Mu**, pp. 66, 591], [**FT**, p. 130]. A modification of Lemma 1.11 in terms of upper triangular matrices $t \in T_m$ (see (1.2)) reads as follows.

LEMMA 1.12. *Let $x \in \mathfrak{M}_{n,m}$, $n \geq m$. If* $\mathrm{rank}(x) = m$, *then*

$$x = vt, \qquad v \in V_{n,m}, \qquad t \in T_m,$$

*so that*

$$dx = \prod_{j=1}^{m} t_{j,j}^{n-j} dt_{j,j}\, dt_*\, dv, \qquad dt_* = \prod_{i<j} dt_{i,j}.$$

This statement is also well known and has different proofs. For instance, it can be easily derived from Lemma 1.11 and (1.3); see Lemma 2.7 in [**Ru7**].

LEMMA 1.13. (bi-Stiefel decomposition). *Let $k$, $m$, and $n$ be positive integers satisfying*

$$1 \leq k \leq n-1, \qquad 1 \leq m \leq n-1, \qquad k + m \leq n.$$

(i) *Almost all matrices $v \in V_{n,m}$ can be represented in the form*

$$v = \begin{bmatrix} a \\ u(I_m - a'a)^{1/2} \end{bmatrix}, \qquad a \in \mathfrak{M}_{k,m}, \quad u \in V_{n-k,m}, \tag{1.39}$$

*so that*

$$\int\limits_{V_{n,m}} f(v)dv = \int\limits_{0<a'a<I_m} d\mu(a) \int\limits_{V_{n-k,m}} f\left(\begin{bmatrix} a \\ u(I_m - a'a)^{1/2} \end{bmatrix}\right) du, \tag{1.40}$$

$$d\mu(a) = |I_m - a'a|^{\delta} da, \quad \delta = (n-k)/2 - d, \quad d = (m+1)/2.$$

(ii) *If, moreover, $k \geq m$, then*

$$\int\limits_{V_{n,m}} f(v)dv = \int\limits_{0}^{I_m} d\nu(r) \int\limits_{V_{k,m}} dw \int\limits_{V_{n-k,m}} f\left(\begin{bmatrix} wr^{1/2} \\ u(I_m - r)^{1/2} \end{bmatrix}\right) du, \tag{1.41}$$

$$d\nu(r) = 2^{-m}|r|^{\gamma}|I_m - r|^{\delta}dr, \qquad \gamma = k/2 - d.$$

PROOF. For m=1, this is a well known bispherical decomposition [**VK**, pp. 12, 22]. For $k = m$, this statement is due to [**Herz**, p. 495]. The proof of Herz was extended in [**GR**] to all $k + m \le n$. For convenience of the reader, we reproduce this proof in our notations which differ from those in [**GR**].

Let us check (1.39). If $v = \begin{bmatrix} a \\ b \end{bmatrix} \in V_{n,m}$, $\quad a \in \mathfrak{M}_{k,m}$, $b \in \mathfrak{M}_{n-k,m}$, then $I_m = v'v = a'a + b'b$, i.e., $b'b = I_m - a'a$. By Lemma 1.11, for almost all $b$ (specifically, for all $b$ of rank $m$), we have $b = u(I_m - a'a)^{1/2}$ where $u \in V_{n-k,m}$. This gives (1.39). In order to prove (1.40), we show the coincidence of the two measures, $dv$ and $\tilde{d}v = |I_m - a'a|^\delta dadu$. Following [**Herz**], we consider the Fourier transforms

$$F_1(y) = \int\limits_{V_{n,m}} \exp(\mathrm{tr}(iy'v))dv \quad \text{and} \quad F_2(y) = \int\limits_{V_{n,m}} \exp(\mathrm{tr}(iy'v))\tilde{d}v,$$

$y \in \mathfrak{M}_{n,m}$, and show that $F_1 = F_2$. Let

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad v = \begin{bmatrix} a \\ u(I_m - a'a)^{1/2} \end{bmatrix},$$

where

$$y_1 \in \mathfrak{M}_{k,m}, \qquad y_2 \in \mathfrak{M}_{n-k,m}; \qquad a \in \mathfrak{M}_{k,m}.$$

Then $y'v = y_1'a + y_2'u(I_m - a'a)^{1/2}$, and we have

$$\begin{aligned} F_2(y) &= \int\limits_{a'a<I_m} \exp(\mathrm{tr}(iy_1'a))|I_m - a'a|^\delta da \\ &\times \int\limits_{V_{n-k,m}} \exp(\mathrm{tr}(iy_2'u(I_m - a'a)^{1/2}))du. \end{aligned}$$

By (1.38), the inner integral is evaluated as

$$\sigma_{n-k,m}\mathcal{J}_{(n-k)/2}(\frac{1}{4}y_2'y_2(I_m - a'a)).$$

Thus

$$F_2(y) = \int\limits_{a'a<I_m} \exp(\mathrm{tr}(iy_1'a))\varphi(a'a)da,$$

$$\varphi(r) = \sigma_{n-k,m}|I_m - r|^{(n-k)/2-d}\mathcal{J}_{(n-k)/2}(\frac{1}{4}y_2'y_2(I_m - r)).$$

The function $F_2(y)$ can be transformed by the generalized Bochner formula (1.35) as follows

$$\begin{aligned} F_2(y) &= \frac{\pi^{km/2}\sigma_{n-k,m}}{\Gamma_m(k/2)} \int\limits_0^{I_m} \mathcal{J}_{k/2}(\frac{1}{4}y_1'y_1 r)|r|^{k/2-d} \\ &\times \mathcal{J}_{(n-k)/2}(\frac{1}{4}y_2'y_2(I_m - r))|I_m - r|^{(n-k)/2-d}dr. \end{aligned}$$

The last integral can be evaluated by the formula

$$\int\limits_0^s \mathcal{J}_\alpha(pr)|r|^{\alpha-d}\mathcal{J}_\beta(q(s-r))|s-r|^{\beta-d}dr = B_m(\alpha,\beta)\mathcal{J}_{\alpha+\beta}((p+q)s)|s|^{\alpha+\beta-d}, \tag{1.42}$$

$$Re\,\alpha, Re\,\beta > d-1, \qquad d=(m+1)/2, \qquad p,q \in \overline{\mathcal{P}}_m, \qquad s \in \mathcal{P}_m,$$

cf. [**Herz**, formula (2.6)]. Applying this formula with $\alpha = k/2$, $\beta = (n-k)/2$, and $s = I_m$, we obtain

$$\begin{aligned} F_2(y) &= \frac{\pi^{km/2}\sigma_{n-k,m}}{\Gamma_m(k/2)} B_m\left(\frac{k}{2}, \frac{n-k}{2}\right) \mathcal{J}_{n/2}(\frac{1}{4}(y_1'y_1 + y_2'y_2)) \\ &= \sigma_{n,m}\mathcal{J}_{n/2}(\frac{1}{4}y'y). \end{aligned}$$

By (1.38), this coincides with $F_1(y)$, and (1.40) follows. The equality (1.41) is a consequence of (1.40) by Lemma 1.11. □

## 1.7. Radial functions and the Cayley-Laplace operator

The notion of a radial function usually alludes to invariance under rotations. In the classical analysis on $\mathbb{R}^n$ the radial function $f(x)$ is virtually a function of a non-negative number $r = |x|$. In the matrix case the situation is similar *but now $r$ is a positive semi-definite matrix.*

DEFINITION 1.14. A function $f(x)$ on $\mathfrak{M}_{n,m}$ is called radial, if it is $O(n)$ left-invariant, i.e.,

$$f(\gamma x) = f(x), \quad \forall \gamma \in O(n). \tag{1.43}$$

If $f$ is continuous, this equality is understood "for all $x$". For generic measurable functions it is interpreted in the almost everywhere sense.

LEMMA 1.15. *Let $x \in \mathfrak{M}_{n,m}$. Each function of the form $f(x) = f_0(x'x)$ is radial. Conversely, if $n \ge m$, and $f(x)$ is a radial function, then there exists $f_0(r)$ on $\mathcal{P}_m$ such that $f(x) = f_0(x'x)$ for all (or almost all) $x$.*

PROOF. The first statement follows immediately from Definition 1.14. To prove the second one, we note that for $n \ge m$, any matrix $x \in \mathfrak{M}_{n,m}$ admits representation $x = vr^{1/2}$, $v \in V_{n,m}$, $r = x'x \in \overline{\mathcal{P}}_m$; see Appendix C, **11**. Fix any frame $v_0 \in V_{n,m}$ and choose $\gamma \in O(n)$ so that $\gamma v_0 = v$. Owing to (1.43),

$$f(x) = f(vr^{1/2}) = f(\gamma v_0 r^{1/2}) = f(v_0 r^{1/2}) \stackrel{\text{def}}{=} f_0(r),$$

as desired. Clearly, the result is independent of the choice of $v_0 \in V_{n,m}$. □

*The Cayley-Laplace operator* $\Delta$ on the space of matrices $x = (x_{i,j}) \in \mathfrak{M}_{n,m}$ is defined by

$$\Delta = \det(\partial'\partial). \tag{1.44}$$

Here, $\partial$ is an $n \times m$ matrix whose entries are partial derivatives $\partial/\partial x_{i,j}$. In terms of the Fourier transform, the action of $\Delta$ represents a multiplication by the polynomial $(-1)^m P(y)$, $y \in \mathfrak{M}_{n,m}$, where

$$P(y) = |y'y| = \det \begin{bmatrix} y_1 \cdot y_1 & . & . & y_1 \cdot y_m \\ . & . & . & . \\ . & . & . & . \\ y_m \cdot y_1 & . & . & y_m \cdot y_m \end{bmatrix},$$

$y_1, \ldots, y_m$ are column-vectors of $y$, and " $\cdot$ " stands for the usual inner product in $\mathbb{R}^n$. Clearly, $P(y)$ is a homogeneous polynomial of degree $2m$ of $nm$ variables $y_{i,j}$, and $\Delta$ is a homogeneous differential operator of order $2m$. For $m = 1$, it coincides with the Laplace operator on $\mathbb{R}^n$.

Operators (1.44) and their generalizations were studied by S.P. Khekalo [**Kh**]. For $m > 1$, the operator $\Delta$ is not elliptic because $P(y) = 0$ for all non-zero matrices $y$ of rank $< m$. Moreover, $\Delta$ is not hyperbolic, [**Ho**, p. 132], although, for some $n, m$ and $\ell$, its power $\Delta^\ell$ enjoys the strengthened Huygens' principle; see [**Kh**] for details.

The following statement gives explicit representation of the radial part of the Cayley-Laplace operator.

THEOREM 1.16. [**Ru7**]. *Let $f_0(r) \in C^{2m}(\mathcal{P}_m)$, $f(x) = f_0(x'x)$, $d = (m+1)/2$. Then for each matrix $x \in \mathfrak{M}_{n,m}$ of rank $m$,*

$$(\Delta f)(x) = (Lf_0)(x'x), \tag{1.45}$$

*where*

$$L = 4^m |r|^{d-n/2} D_+ |r|^{n/2-d+1} D_+, \tag{1.46}$$

*$D_+$ being the operator (1.21).*

EXAMPLE 1.17. Let $f(x) = |x|_m^\lambda$, $|x|_m = \det(x'x)^{1/2}$. By Theorem 1.16, we have $(\Delta f)(x) = \varphi(x'x)$, where $\varphi(r) = 4^m |r|^{d-n/2} D_+ |r|^{n/2-d+1} D_+ |r|^{\lambda/2}$. This expression can be evaluated using (1.27):

$$\begin{aligned}\varphi(r) &= 4^m b(\lambda/2) |r|^{d-n/2} D_+ |r|^{(n+\lambda)/2-d} \\ &= 4^m b(\lambda/2) b((n+\lambda)/2-d) |r|^{\lambda/2-1}.\end{aligned}$$

Thus, we have arrived at the following identity of the Bernstein type

$$\Delta |x|_m^\lambda = \mathcal{B}(\lambda) |x|_m^{\lambda-2}, \tag{1.47}$$

(cf. [**FK**, p. 125]) where, owing to (1.28), the polynomial $\mathcal{B}(\lambda)$ has the form

$$\mathcal{B}(\lambda) = (-1)^m \prod_{i=0}^{m-1} (\lambda + i)(2 - n - \lambda + i). \tag{1.48}$$

An obvious consequence of (1.47) in a slightly different notation reads

$$\Delta^k |x|_m^{\alpha+2k-n} = B_k(\alpha) |x|_m^{\alpha-n}, \tag{1.49}$$

where

$$\begin{aligned}B_k(\alpha) &= \prod_{i=0}^{m-1} \prod_{j=0}^{k-1} (\alpha - i + 2j)(\alpha - n + 2 + 2j + i) \\ &= B_k(n - \alpha - 2k).\end{aligned} \tag{1.50}$$

CHAPTER 2

# The Gårding-Gindikin fractional integrals

## 2.1. Definitions and comments

In this section, we begin detailed investigation of fractional integrals associated to the cone $\mathcal{P}_m$ of positive definite $m \times m$ matrices. These integrals were introduced by L. Gårding [**Gå**] and substantially generalized by S. Gindikin [**Gi1**].

Let $f(r)$ be a smooth function on $\mathcal{P}_m$ which is rapidly decreasing at infinity and bounded with all its derivatives when $r$ approaches the boundary $\partial\mathcal{P}_m$. Consider the integral

$$I_f(\alpha) = \int_{\mathcal{P}_m} f(r)|r|^{\alpha-d} dr, \qquad d = (m+1)/2. \tag{2.1}$$

If $f(r) = \exp(-\mathrm{tr}(r))$ then

$$I_f(\alpha) = \Gamma_m(\alpha) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(\alpha - j/2)\ ;$$

see (1.6), (1.7). In the general case, by passing to upper triangular matrices and using (1.3), we have

$$\begin{aligned} I_f(\alpha) &= \int_0^\infty t_{1,1}^{2\alpha-1}\, dt_{1,1} \int_0^\infty t_{2,2}^{2\alpha-2}\, dt_{2,2} \ldots \int_0^\infty t_{m,m}^{2\alpha-m} \tilde{f}(t_{1,1}, \ldots, t_{m,m})\, dt_{m,m} \\ &= 2^{-m} \int_0^\infty y_1^{\alpha-1}\, dy_1 \int_0^\infty y_2^{\alpha-3/2}\, dy_2 \ldots \int_0^\infty y_m^{\alpha-(m+1)/2} \tilde{f}(y_1^{1/2}, \ldots, y_m^{1/2})\, dy_m, \end{aligned}$$

$$\tilde{f}(t_{1,1}, \ldots, t_{m,m}) = 2^m \int_{\mathbb{R}^{m(m-1)/2}} f(t't)\, dt_*, \quad dt_* = \prod_{i<j} dt_{i,j}.$$

The function $\tilde{f}$ extends to the whole space $\mathbb{R}^m$ as an even function in each argument, and the function

$$f_0(y_1, \ldots, y_m) \equiv 2^{-m} \tilde{f}(y_1^{1/2}, \ldots, y_m^{1/2})$$

belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^m)$. Hence $I_f(\alpha)$ can be regarded as a direct product of one-dimensional distributions

$$I_f(\alpha) = \left( \prod_{j=1}^m (y_j)_+^{\alpha-(j+1)/2}, f_0(y_1, \ldots, y_m) \right). \tag{2.2}$$

It follows [**GSh1**, Chapter 1, Section 3.5] that the integral (2.1) absolutely converges if and only if $Re\,\alpha > (m-1)/2$ and extends as a meromorphic function of $\alpha$ with the only poles $(m-1)/2,\ (m-2)/2, \dots$ . These poles and their orders coincide with those of the gamma function $\Gamma_m(\alpha)$ so that $I(\alpha)/\Gamma_m(\alpha)$ is an entire function of $\alpha$.

These preliminaries motivate the following definitions. We first consider "good" functions $f \in \mathcal{D}(\mathcal{P}_m)$ which are infinitely differentiable and compactly supported away form the boundary $\partial\mathcal{P}_m$. Then we proceed to the case of arbitrary locally integrable functions.

DEFINITION 2.1. Let $f \in \mathcal{D}(\mathcal{P}_m)$, $\alpha \in \mathbb{C}$, $d = (m+1)/2$. The *left-sided Gårding-Gindikin fractional integral* is defined by

$$(I_+^\alpha f)(s) = \begin{cases} \dfrac{1}{\Gamma_m(\alpha)} \displaystyle\int_0^s f(r)|s-r|^{\alpha-d} dr & \text{if } Re\,\alpha > d-1, \\ \\ (I_+^{\alpha+\ell} D_+^\ell f)(s) & \text{if } d-1-\ell < Re\,\alpha \le d-\ell; \\ & \ell = 1, 2, \dots . \end{cases} \tag{2.3}$$

Here, $s \in \mathcal{P}_m$, $D_+$ is the differential operator (1.21), and we integrate over the "interval" $(0, s) = \{r :\ r \in \mathcal{P}_m,\ s-r \in \mathcal{P}_m\}$.

DEFINITION 2.2. For $f \in \mathcal{D}(\mathcal{P}_m)$ and $\alpha \in \mathbb{C}$, the *right-sided Gårding-Gindikin fractional integral* is defined by

$$(I_-^\alpha f)(s) = \begin{cases} \dfrac{1}{\Gamma_m(\alpha)} \displaystyle\int_s^\infty f(r)|r-s|^{\alpha-d} dr & \text{if } Re\,\alpha > d-1, \\ \\ (I_-^{\alpha+\ell} D_-^\ell f)(s) & \text{if } d-1-\ell < Re\,\alpha \le d-\ell; \\ & \ell = 1, 2, \dots . \end{cases} \tag{2.4}$$

Here, $s$ is a positive semi-definite matrix in $\overline{\mathcal{P}}_m$, $D_-$ is defined by (1.22), and $\int_s^\infty$ denotes integration over the shifted cone $s + \mathcal{P}_m = \{r :\ r \in \mathcal{P}_m : r-s \in \mathcal{P}_m\}$. The vertex $s$ of this cone may be an inner point of $\mathcal{P}_m$ or its boundary point.

In the rank-one case $m = 1$, when $\mathcal{P}_m$ is a positive half-line, (2.3) and (2.4) are ordinary Riemann-Liouville (or Abel) fractional integrals [**Ru1**], [**SKM**].

REMARK 2.3. Analogous definitions can be given for functions on the ambient space $\mathcal{S}_m \supset \mathcal{P}_m$, for instance, for $f \in \mathcal{S}(\mathcal{S}_m)$. We do not do this for two reasons. Firstly, because our purposes are exclusively connected with analysis on the cone $\mathcal{P}_m$, and the space $\mathcal{D}(\mathcal{P}_m)$ fits our needs completely. Secondly, dealing with the space $\mathcal{D}(\mathcal{P}_m)$, we can still reveal basic features of our objects and give much simpler proofs then for other classes of functions.

According to Definitions 2.1 and 2.2, it is convenient to regard the complex plane as a union $\mathbb{C} = \cup_{\ell=0}^\infty \Omega_\ell$, where

$$\Omega_\ell = \begin{cases} \{\alpha : Re\,\alpha > d-1\} & \text{if } \ell = 0, \\ \\ \{\alpha : d-1-\ell < Re\,\alpha \le d-\ell\} & \text{if } \ell = 1, 2, \dots . \end{cases} \tag{2.5}$$

LEMMA 2.4. *For $f \in \mathcal{D}(\mathcal{P}_m)$, the integrals $(I_\pm^\alpha f)(s)$ are entire functions of $\alpha$.*

PROOF. Let us consider the integrals $(I_+^\alpha f)(s)$. Given an integer $j > 0$, we set $F_j(\alpha) = I_+^{\alpha+j} D_+^j f$. This function is analytic in the half-plane $Re\,\alpha > d-1-j$. Owing to uniqueness property of analytic functions, it suffices to show that $F_j(\alpha) = I_+^\alpha f$ on each strip $\Omega_\ell$ provided that $j = j(\ell)$ is large enough. For $\alpha \in \Omega_\ell$, we have $\alpha + j - d > j - \ell - 1$, and therefore, by (1.23) and (1.25),

$$\begin{aligned} F_j(\alpha) &= \frac{1}{\Gamma_m(\alpha+j)} \int\limits_{\mathcal{S}_m} (s-r)_+^{\alpha+j-d} (D_+^j f)(r) dr \\ &= \frac{1}{\Gamma_m(\alpha+j)} \int\limits_{\mathcal{S}_m} D_{-,\,r}^{j-\ell} (s-r)_+^{\alpha+j-d} (D_+^\ell f)(r) dr \\ &= \frac{1}{\Gamma_m(\alpha+\ell)} \int\limits_0^s |s-r|^{\alpha+\ell-d} (D_+^\ell f)(r) dr \\ &= (I_+^{\alpha+\ell} D_+^\ell f)(s). \end{aligned}$$

By Definition 2.1, this means that $F_j(\alpha) = I_+^\alpha f$ for $\alpha \in \Omega_\ell$, and we are done. For operators $I_-^\alpha$ the proof is similar. □

LEMMA 2.5. *Let $f \in \mathcal{D}(\mathcal{P}_m)$. Then*

$$D_\pm^j I_\pm^\alpha f = I_\pm^\alpha D_\pm^j f = I_\pm^{\alpha-j} f, \tag{2.6}$$

*for any $\alpha \in \mathbb{C}$ and $j \in \mathbb{N}$.*

PROOF. Fix $j \in \mathbb{N}$. For $Re\,\alpha > j + d - 1$, the equality $I_\pm^\alpha D_\pm^j f = I_\pm^{\alpha-j} f$ can be obtained using integration by parts (see the proof of Lemma 2.4). For all $\alpha \in \mathbb{C}$ it then follows by analytic continuation. Let us show that $D_\pm^j I_\pm^\alpha f = I_\pm^{\alpha-j} f$. Let $Re\,\alpha - j \in \Omega_\ell$; see (2.5). If $\ell = 0$, then

$$\begin{aligned} I_+^{\alpha-j} f &= \frac{1}{\Gamma_m(\alpha-j)} \int\limits_0^s |s-r|^{\alpha-j-d} f(r) dr \\ &= \frac{1}{\Gamma_m(\alpha)} \int\limits_{\mathcal{S}_m} D_{+,\,s}^j (s-r)_+^{\alpha-d} f(r) dr \\ &= (D_+^j I_+^\alpha f)(s). \end{aligned}$$

If $\ell \geq 1$, then, by Definition 2.1, $I_+^{\alpha-j} f = I_+^{\alpha-j+\ell} D_+^\ell f$. By the previous case, this coincides with $D_+^j I_+^{\alpha+\ell} D_+^\ell f = D_+^j I_+^\alpha f$ (here we used Definition 2.1 again). For the right-sided integrals the argument follows the same lines. □

Some historical notes and comments are in order. Fractional integrals similar to (2.3)–(2.4) and associated to the light cone were introduced by M. Riesz [**Ri1**], [**Ri2**] in 1936. The investigation was continued by N.E. Fremberg [**Fr1**], [**Fr2**], and by Riesz himself in [**Ri3**], [**Ri4**]; see also B.B. Baker and E.T. Copson [**BC**], and E.T. Copson [**Co**]. A key motivation of this research was to find a simple form of the solution to the Cauchy problem for the wave equation. M. Riesz' argument was essentially simplified by J.J. Duistermaat [**Dui**]; see also J. A.C. Kolk and V.S. Varadarajan [**KV**].

The results of Riesz were extended by L. Gårding [**Gå**] (1947), who replaced the light cone (of rank 2) by the higher rank cone $\mathcal{P}_m$ of positive definite symmetric matrices, and studied the corresponding hyperbolic equations.[1] The Riesz-Gårding fractional integrals are very close to those in (2.3) but do not coincide with them completely. The difference is as follows. In the above-mentioned papers, mainly devoted to PDE, the upper limit $s$ in (2.3) varies in the whole space $\mathbb{R}^N$, $N = m(m+1)/2$ (not only in $\overline{\mathcal{P}}_m$), and the domain of integration is bounded by the retrograde cone $s - \mathcal{P}_m$ and a smooth surface on which the Cauchy data are given. For $s \in \overline{\mathcal{P}}_m$, this means that the part of the domain of integration in (2.3) containing the origin is cut off smoothly.

Further progress is connected with fundamental results of S.G. Gindikin [**Gi1**] (1964), who developed a general theory of Riemann-Liouville integrals associated to homogeneous cones. The light cone of Riesz and the cone $\mathcal{P}_m$ of Gårding fall into the scope of his theory. In the case of $\mathcal{P}_m$, Gindikin's fractional integrals are convolutions of the form

$$J^\alpha_\pm f = \frac{s_\pm^{\alpha-d}}{\Gamma_m(\alpha)} * f, \qquad f \in \mathcal{S}(\mathbb{R}^N), \quad N = \frac{m(m+1)}{2}, \tag{2.7}$$

where the function $s_+^{\alpha-d}$ equals $|s|^{\alpha-d}$ for $s \in \mathcal{P}_m$, zero otherwise, and $s_-^{\alpha-d}$ equals $(-s)_+^{\alpha-d}$. Operators (2.7) are defined on the whole space $\mathbb{R}^N$, and can be treated using the Fourier transform technique and the theory of distributions; see [**Gi1**], [**VG**], [**Rab**], [**Wa**], [**FK**]. See also [**Rich**] concerning application of such operators in multivariate statistics.

Definitions 2.1 and 2.2 are motivated by consideration of Radon transforms of functions of matrix argument in the next chapters. It is important that the variable of integration $r$ and the exterior variable $s$ do not range in the whole space and are restricted to $\mathcal{P}_m$. In the following, the class of functions $f$ will be essentially enlarged. We shall define and investigate fractional integrals $I^\alpha_\pm f$ of arbitrary locally integrable functions $f$ with possibly minimal decay at infinity. The main emphasis is made upon $\alpha$ belonging to the Wallach-like set $\mathcal{W}_m = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{m-1}{2}\right\} \cup \left\{\alpha : \ Re\,\alpha > \frac{m-1}{2}\right\}$, cf. [**FK**, p. 137].

Thus, one can see that most of the facts for fractional integrals from [**Gå**] and [**Gi1**] cannot be automatically transferred to our case, and a careful consideration is required. The case $\alpha = k/2$, $k \in \mathbb{N}$, is of primary importance because it arises in our treatment of the Radon transform.

## 2.2. Fractional integrals of half-integral order

An important feature of the higher rank fractional integrals $I^\alpha_\pm f$ is that for $\alpha = k/2$, $k = 0, 1, \dots, m-1$, they are convolutions with the corresponding positive measures supported on the boundary $\partial\mathcal{P}_m$. In the rank-one case $m = 1$, this phenomenon becomes trivial (we have only one value $\alpha = 0$ corresponding to the delta function). For $m > 1$, when dimension of the boundary $\partial\mathcal{P}_m$ is positive we deal with many "delta functions" supported by manifolds of different dimensions. This phenomenon was drawn considerable attention in analysis on symmetric domains; see [**FK**, Chapter VII], [**Gi2**], [**Ishi**], and references therein.

Our nearest goal is to find precise form of $I^{k/2}_\pm f$ for $k = 1, \dots, m-1$. We start with the following simple observation. For $k \geq m$, by Lemma 1.11 and (1.37), one

[1] Prof. Lars Gårding kindly informed the second-named author that his construction of fractional integrals was inspired by close developments in statistics.

can write

$$(I_+^{k/2}f)(s) = \frac{1}{\Gamma_m(k/2)} \int_0^s f(s-r)|r|^{k/2-d} dr$$

$$= \pi^{-km/2} \int_{\{\omega \in \mathfrak{M}_{k,m}:\, \omega'\omega < s\}} f(s-\omega'\omega) d\omega, \tag{2.8}$$

$$(I_-^{k/2}f)(s) = \frac{1}{\Gamma_m(k/2)} \int_{\mathcal{P}_m} f(s+r)|r|^{k/2-d} dr$$

$$= \pi^{-km/2} \int_{\mathfrak{M}_{k,m}} f(s+\omega'\omega) d\omega. \tag{2.9}$$

The expressions (2.8) and (2.9) are meaningful for all $k \in \mathbb{N}$, and it is natural to conjecture that for $k = 1, \dots, m-1$, the integrals $I_\pm^{k/2} f$ are represented by (2.8) and (2.9) too. Let us prove this conjecture. We start with the left-sided integrals $I_+^{k/2} f$ and make use of the Laplace transform.

LEMMA 2.6. *For $f \in \mathcal{D}(\mathcal{P}_m)$ and $\alpha \in \mathbb{C}$,*

$$(LI_+^\alpha f)(z) = \det(z)^{-\alpha}(Lf)(z), \qquad Re\, z > 0. \tag{2.10}$$

PROOF. Let $\alpha \in \Omega_\ell$; see (2.5). For $\ell = 0$, (2.10) is an immediate consequence of (1.19) and (1.20). The desired result for $\ell > 1$ follows by Definition 2.1 and (1.31):

$$(LI_+^\alpha f)(z) = (LI_+^{\alpha+\ell} D_+^\ell f)(z) = \det(z)^{-\alpha-\ell}(LD_+^\ell f)(z) = \det(z)^{-\alpha}(Lf)(z).$$

□

COROLLARY 2.7. *For $f \in \mathcal{D}(\mathcal{P}_m)$,*

$$(D_+^\ell f)(s) = (I_+^{-\ell} f)(s), \qquad \ell = 0, 1, 2, \dots . \tag{2.11}$$

PROOF. By (1.31) and (2.10), $(LD_+^\ell f)(z) = (LI_+^{-\ell} f)(z)$, $Re\, z > 0$. Since, by Definition 2.1, $I_+^{-\ell} f = I_+^d D_+^{\ell+d} f$ if $d = (m+1)/2$ is an integer, and $I_+^{-\ell} f = I_+^{d-1/2} D_+^{\ell+d-1/2} f$ otherwise, the result follows owing to the uniqueness property of the Laplace transform (see Lemma 1.3).

□

THEOREM 2.8. *If $f \in \mathcal{D}(\mathcal{P}_m)$, then for all $k \in \mathbb{N}$,*

$$(I_+^{k/2} f)(s) = \pi^{-km/2} \int_{\{\omega \in \mathfrak{M}_{k,m}:\, \omega'\omega < s\}} f(s-\omega'\omega) d\omega. \tag{2.12}$$

*Moreover,*

$$(I_+^0 f)(s) = f(s). \tag{2.13}$$

PROOF. Denote by $\varphi(s)$ the right-hand side of (2.12). By changing the order of integration, we have

$$
\begin{aligned}
(L\varphi)(z) &= \pi^{-km/2}\int\limits_{\mathcal{P}_m}\exp(-\mathrm{tr}(zs))ds\int\limits_{\{y\in\mathfrak{M}_{k,m}:\, y'y<s\}} f(s-y'y)dy \\
&= \pi^{-km/2}(Lf)(z)\int\limits_{\mathfrak{M}_{k,m}}\exp(-\mathrm{tr}(zy'y))dy. \qquad (2.14)
\end{aligned}
$$

The integral in (2.14) is easily evaluated: if $z=r\in\mathcal{P}_m$, we replace $y$ by $yr^{-1/2}$ and get

$$
\begin{aligned}
\int\limits_{\mathfrak{M}_{k,m}}\exp(-\mathrm{tr}(ry'y))dy &= |r|^{-k/2}\int\limits_{\mathfrak{M}_{k,m}}\exp(-\mathrm{tr}(y'y))dy \\
&= |r|^{-k/2}\int\limits_{\mathbb{R}^{km}}\exp(-y_{1,1}^2-\ldots-y_{k,m}^2)dy \\
&= \pi^{km/2}|r|^{-k/2}.
\end{aligned}
$$

By analytic continuation,

$$
\int\limits_{\mathfrak{M}_{k,m}}\exp(-\mathrm{tr}(zy'y))dy=\pi^{km/2}\det(z)^{-k/2}, \qquad Re\, z>0.
$$

Thus, $(L\varphi)(z)=\det(z)^{-k/2}(Lf)(z)$, and (2.10) yields $(L\varphi)(z)=(LI_+^{k/2}f)(z)$, $Re\, z>0$. Since for each $\sigma_0\in\mathcal{P}_m$, the integrals

$$
\int_{\mathcal{P}_m}\exp(-\mathrm{tr}(\sigma_0 s)|\varphi(s)|ds, \qquad \int_{\mathcal{P}_m}\exp(-\mathrm{tr}(\sigma_0 s)|(I_+^{k/2}f)(s)|ds
$$

are finite (both are dominated by $|\sigma_0|^{-k/2}(L|f|)(\sigma_0))$, by Lemma 1.3 we get $\varphi(s)=(I_+^{k/2}f)(s)$, as desired.

$\square$

REMARK 2.9. For $0<k<m$, the integral (2.12) can be written as

$$
(I_+^{k/2}f)(s)=c_{k,m}|s|^{k/2}\int\limits_{V_{m,k}}dv\int\limits_0^{I_k}f(s-s^{1/2}vqv's^{1/2})|q|^{(m-k-1)/2}dq, \qquad (2.15)
$$

$$
c_{k,m}=\pi^{-km/2}2^{-k}.
$$

Indeed, if we transform (2.12) by setting $\omega=hs^{1/2}$, where $h\in\mathfrak{M}_{k,m},\ h'h<I_m$, and pass to polar coordinates, we obtain (2.15).

The integral (2.12) can also be written as the Laplace convolution

$$
(I_+^{k/2}f)(s)=\int\limits_0^s f(s-r)d\mu_k(r), \qquad (2.16)
$$

where $\mu_k$ is a positive measure defined by

$$(\mu_k, \psi) = \pi^{-km/2} \int\limits_{\mathfrak{M}_{k,m}} \psi(\omega'\omega) d\omega, \qquad \psi \in C_c(\mathcal{S}_m). \tag{2.17}$$

Here, $C_c(\mathcal{S}_m)$ denotes the space of compactly supported continuous functions on $\mathcal{S}_m$. For $k \geq m$, Lemma 1.11 yields

$$(\mu_k, \psi) = \frac{1}{\Gamma_m(k/2)} \int\limits_{\mathcal{P}_m} \psi(r) |r|^{k/2-d} dr, \qquad d = (m+1)/2.$$

In order to clarify geometric structure of $\mu_k$ for $k < m$, we denote

$$\Lambda_k = \{s : s \in \overline{\mathcal{P}_m}, \ \mathrm{rank}(s) = k\}, \qquad k = 0, 1, \ldots, m-1, \tag{2.18}$$

$$G = GL(m, \mathbb{R}), \qquad e_k = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \in \partial\mathcal{P}_m.$$

The manifold $\Lambda_k$ is an orbit of $e_k$ under $G$ in $\mathcal{S}_m$, i.e., $\Lambda_k = Ge_k$. Indeed, $\mathrm{rank}(g'e_kg) = k$ for all $g \in G$, and conversely, each matrix $s$ of rank $k$ is representable as $s = g'e_kg$ for some $g \in G$; see, e.g., [**Mu**, A6 (V) and Theorem A9.4(ii)] . The closure $\overline{\Lambda_k}$ is a union of $\Lambda_1, \ldots, \Lambda_k$. Since $\mathrm{rank}(\omega'\omega) \leq k < m$, then $(\mu_k, \psi) = 0$ for all $\psi$ supported away from $\overline{\Lambda_k} = \overline{Ge_k}$, and therefore $\mathrm{supp}\, \mu_k = \overline{\Lambda_k}$. Note also that $\mu_k(\overline{\Lambda_{k-1}}) = 0$ because

$$\int\limits_{\overline{\Lambda_{k-1}}} \psi(s) d\mu_k(s) = \int\limits_{\mathfrak{A}_{k-1}} \psi(\omega'\omega) d\omega = 0,$$

$\mathfrak{A}_{k-1} = \{\omega \in \mathfrak{M}_{k,m} : \mathrm{rank}(\omega) \leq k-1\}$ (the set $\mathfrak{A}_{k-1}$ has measure 0 in $\mathfrak{M}_{k,m}$). Thus the following statement holds.

THEOREM 2.10. *For $0 < k < m$, $I_+^{k/2} f$ is the Laplace convolution (2.16) with a positive measure $\mu_k$ defined by (2.17) and such that (a) $\mathrm{supp}\, \mu_k = \overline{Ge_k}$, and (b) $\mu_k(\overline{Ge_{k-1}}) = 0$.*

Let us consider the right-sided integrals $I_-^\alpha f$. They can be expressed through the left-sided ones as follows.

LEMMA 2.11. *For $f \in \mathcal{D}(\mathcal{P}_m)$ and $\alpha \in \mathbb{C}$,*

$$(I_-^\alpha f)(s) = |s|^{\alpha-d} (I_+^\alpha g)(s^{-1}), \qquad g(r) = |r|^{-\alpha-d} f(r^{-1}). \tag{2.19}$$

*In particular,*

$$(I_-^0 f)(s) = f(s). \tag{2.20}$$

PROOF. Since the integrals $(I_+^\alpha f)(s)$ and $(I_-^\alpha f)(s)$ are entire functions of $\alpha$ (see Lemma 2.4), it suffices to prove (2.19) for $Re\, \alpha > d-1$. This can be easily done by changing variables according to Lemma 1.1 (iii). □

COROLLARY 2.12. *For $f \in \mathcal{D}(\mathcal{P}_m)$ and $\ell \in \mathbb{N}$,*

$$(D_-^\ell f)(s) = |s|^{-\ell-d} (D_+^\ell g)(s^{-1}), \qquad g(r) = |r|^{\ell-d} f(r^{-1}). \tag{2.21}$$

PROOF. The formula (2.21) follows from (2.11) and (2.19).

□

Now we can justify (2.9) for all $k \in \mathbb{N}$.

THEOREM 2.13. *If $f \in \mathcal{D}(\mathcal{P}_m)$, $k \in \mathbb{N}$, then*

$$(I_-^{k/2} f)(s) = \pi^{-km/2} \int\limits_{\mathfrak{M}_{k,m}} f(s+\omega'\omega) d\omega. \tag{2.22}$$

PROOF. According to (2.19),

$$(I_-^{k/2} f)(s) = |s|^{k/2-d} (I_+^{k/2} g)(s^{-1}), \qquad g(r) = |r|^{-k/2-d} f(r^{-1}). \tag{2.23}$$

By (2.15), $(I_+^{k/2} g)(s^{-1})$ is represented as

$$c_{k,m} |s|^{-k/2} \int\limits_{V_{m,k}} dv \int\limits_0^{I_k} g(s^{-1} - s^{-1/2} vqv' s^{-1/2}) |q|^{(m-k-1)/2} \, dq$$

$$= c_{k,m} |s|^d \int\limits_{V_{m,k}} dv \int\limits_0^{I_k} f((s^{-1} - s^{-1/2} vqv' s^{-1/2})^{-1}) \frac{|q|^{(m-k-1)/2} \, dq}{|I_m - vqv'|^{k/2+d}}.$$

Using the property

$$|I_m - vqv'| = |I_k - q|, \qquad q \in \mathcal{P}_k, \qquad v \in V_{m,k}, \tag{2.24}$$

(see [**Mu**, p. 575] or Appendix C, **1** ), this can be written as

$$c_{k,m} |s|^d \int\limits_{V_{m,k}} dv \int\limits_0^{I_k} f(s^{1/2} (I_m - vqv')^{-1} s^{1/2}) \frac{|q|^{(m-k-1)/2} \, dq}{|I_k - q|^{k/2+d}}$$

or

$$c_{k,m} \sigma_{m,k} |s|^d \int\limits_{O(m)} d\gamma \int\limits_0^{I_k} f(s^{1/2} \gamma (I_m - v_0 q v_0')^{-1} \gamma' s^{1/2}) \frac{|q|^{(m-k-1)/2} \, dq}{|I_k - q|^{k/2+d}},$$

$$v_0 = \left[ \begin{array}{c} 0 \\ I_k \end{array} \right] \in V_{m,k}.$$

One can readily check that

$$(I_m - v_0 q v_0')^{-1} = \left[ \begin{array}{ll} I_{m-k} & 0 \\ 0 & I_k - q \end{array} \right]^{-1} = \left[ \begin{array}{ll} I_{m-k} & 0 \\ 0 & (I_k - q)^{-1} \end{array} \right].$$

After changing variable $(I_k - q)^{-1} \to q$, the integral $(I_+^{k/2} g)(s^{-1})$ becomes

$$c_{k,m} \sigma_{m,k} |s|^d \int\limits_{O(m)} d\gamma \int\limits_{I_k}^{\infty} f\left( s^{1/2} \gamma \left[ \begin{array}{ll} I_{m-k} & 0 \\ 0 & q \end{array} \right] \gamma' s^{1/2} \right) |I_k - q|^{(m-k-1)/2} dq,$$

and therefore (replace $q$ by $I_k + q$),

$$\begin{aligned}
(I_+^{k/2} g)(s^{-1}) &= c_{k,m}|s|^d \int\limits_{V_{m,k}} dv \int\limits_{\mathcal{P}_k} f(s^{1/2}(I_m + vqv')s^{1/2})|q|^{(m-k-1)/2} dq \\
&= c_{k,m} 2^k |s|^d \int\limits_{\mathfrak{M}_{m,k}} f(s + s^{1/2}yy's^{1/2})\, dy \qquad (y = s^{-1/2}z) \\
&= c_{k,m} 2^k |s|^{d-k/2} \int\limits_{\mathfrak{M}_{m,k}} f(s + zz')\, dz \\
&= \pi^{-km/2} |s|^{d-k/2} \int\limits_{\mathfrak{M}_{k,m}} f(s + \omega'\omega) d\omega.
\end{aligned}$$

Owing to (2.23), this gives the desired equality. □

Remark 2.14. For $0 < k < m$, the integral (2.22) can be written in polar coordinates as

$$(I_-^{k/2} f)(s) = c_{k,m} \int\limits_{V_{m,k}} dv \int\limits_{\mathcal{P}_k} f(vqv' + s)|q|^{(m-k-1)/2} dq, \tag{2.25}$$

$$c_{k,m} = \pi^{-km/2} 2^{-k}.$$

Furthermore,

$$(I_-^{k/2} f)(s) = \int\limits_{\mathcal{P}_m} f(s + r) d\mu_k(r),$$

$\mu_k$ being a positive measure defined by (2.17).

## 2.3. The Gårding-Gindikin distributions

It is instructive to give an alternative proof of the formula (2.22). This proof follows the argument from [**FK**, p. 134] and is of independent interest. It can be useful in different occurrences. Consider the Gårding-Gindikin distribution

$$\mathcal{G}_\alpha(f) = \frac{1}{\Gamma_m(\alpha)} \int\limits_{\mathcal{P}_m} f(r)|r|^{\alpha - d} dr, \qquad d = (m+1)/2, \tag{2.26}$$

where $f$ belongs to the Schwartz space $\mathcal{S}(\mathcal{S}_m)$. The integral (2.26) converges absolutely for $Re\,\alpha > d - 1$ and admits analytic continuation as an entire function of $\alpha$ so that

$$\mathcal{G}_0(f) = f(0), \tag{2.27}$$

see [**FK**, pp. 132–133]. The following statement implies (2.22) and extends it for $f \in \mathcal{S}(\mathcal{S}_m)$.

Lemma 2.15. *For $f \in \mathcal{S}(\mathcal{S}_m)$ and $0 < k < m$,*

$$\mathcal{G}_{k/2}(f) = \pi^{-km/2} \int\limits_{\mathfrak{M}_{k,m}} f(\omega'\omega) d\omega. \tag{2.28}$$

PROOF. Let us transform $\mathcal{G}_\alpha(f)$ to upper triangular matrices. By (1.3),

$$\mathcal{G}_\alpha(f) = \frac{2^m}{\Gamma_m(\alpha)} \int\limits_{T_m} f(t't) \prod_{i=1}^m t_{i,i}^{2\alpha-i} \prod_{i\le j} dt_{i,j} \qquad Re\,\alpha > d-1.$$

We write $t = a + b$, where $a = (a_{i,j})$ and $b = (b_{i,j})$ are upper triangular matrices so that the lower $n-k$ rows of $a$ and the upper $k$ rows of $b$ consist of zeros. We denote by $A$ and $B$ the sets of all matrices of the form $a$ and $b$, respectively. Since $t't = a'a + b'b$, then

$$\mathcal{G}_\alpha(f) = \frac{2^k \Gamma_{m-k}(\alpha - k/2)}{\Gamma_m(\alpha)} \int\limits_A g_\alpha(a'a) \prod_{i=1}^k a_{i,i}^{2\alpha-i} \prod_{i\le j} da_{i,j}\ , \tag{2.29}$$

where

$$g_\alpha(a'a) = \frac{2^{m-k}}{\Gamma_{m-k}(\alpha-k/2)} \int\limits_B f(a'a+b'b) \prod_{i=1}^{m-k} b_{k+i,k+i}^{2(\alpha-k/2)-i} \prod_{i\le j} db_{k+i,k+j}\ .$$

Note that $g_\alpha(a'a) = \mathcal{G}_{\alpha-k/2}\left(f\left(\begin{bmatrix} * & * \\ * & \bullet \end{bmatrix}\right)\right)$ is the Gårding-Gindikin distribution acting on the $(\bullet)$ matrix variable belonging to $\mathcal{S}_{m-k}$. By (2.29), $\mathcal{G}_\alpha(f)$ is a direct product of two distributions which are analytic in $\alpha$. By taking into account (2.27), we get $g_{k/2}(a'a) = f(a'a)$, and therefore,

$$\mathcal{G}_{k/2}(f) = c \int\limits_A f(a'a) \prod_{i=1}^k a_{i,i}^{k-i} \prod_{i\le j} da_{i,j}\ . \tag{2.30}$$

Here, by (1.7),

$$c = 2^k \lim_{\alpha\to k/2} \frac{2^k \Gamma_{m-k}(\alpha-k/2)}{\Gamma_m(\alpha)} = \frac{2^k \pi^{k(k-m)/2}}{\Gamma_k(k/2)}\ .$$

This representation was established in [**FK**, p. 134]. Let us show that it coincides with (2.28). We replace $\omega$ in (2.28) by $[\eta, \zeta]$, where $\eta \in \mathfrak{M}_{k,k}$, $\zeta \in \mathfrak{M}_{k,m-k}$. Then

$$\pi^{-km/2} \int\limits_{\mathfrak{M}_{k,m}} f(\omega'\omega) d\omega = \pi^{-km/2} \int\limits_{\mathfrak{M}_{k,k}} d\eta \int\limits_{\mathfrak{M}_{k,m-k}} f\left(\begin{bmatrix} \eta'\eta & \eta'\zeta \\ \zeta'\eta & \zeta'\zeta \end{bmatrix}\right) d\zeta =$$

( set $\eta = \gamma q$, $\gamma \in O(k)$, $q \in T_k$ and use Lemma 1.12)

$$= \quad \pi^{-km/2} \sigma_{k,k} \int\limits_{T_k} \prod_{i=1}^k q_{i,i}^{k-i} \prod_{i\le j} dt_{i,j} \int\limits_{\mathfrak{M}_{k,m-k}} f\left(\begin{bmatrix} q'q & q'y \\ y'q & y'y \end{bmatrix}\right) dy$$

$$\overset{(1.37)}{=} \quad \frac{2^k \pi^{k(k-m)/2}}{\Gamma_k(k/2)} \int\limits_A f(a'a) \prod_{i=1}^k a_{i,i}^{k-i} \prod_{i\le j} da_{i,j}$$

where $a = \begin{bmatrix} q & y \\ 0 & 0 \end{bmatrix}$. This proves the statement.

□

## 2.4. Fractional integrals of locally integrable functions

In the previous sections, we studied fractional integrals $I^\alpha_\pm f$ assuming $f$ is very nice, namely, $f \in \mathcal{D}(\mathcal{P}_m)$. Below we explore these integrals for arbitrary locally integrable functions $f$ provided that $\alpha$ belongs to the Wallach-like set

$$\mathcal{W}_m = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{m-1}{2}\right\} \cup \left\{\alpha : \; Re\,\alpha > \frac{m-1}{2}\right\}. \tag{2.31}$$

The key question is for which $f$ the integrals $I^\alpha_\pm f$ do exist. We start with the left-sided integrals.

Definition 2.16. For a locally integrable function $f$ on $\mathcal{P}_m$ and $\alpha \in \mathcal{W}_m$, we define

$$(I^\alpha_+ f)(s) = \begin{cases} \dfrac{1}{\Gamma_m(\alpha)} \displaystyle\int_0^s f(r)|s-r|^{\alpha-d} dr & \text{if } Re\,\alpha > d-1, \\ \pi^{-km/2} \displaystyle\int\limits_{\{\omega \in \mathfrak{M}_{k,m} :\, \omega'\omega < s\}} f(s-\omega'\omega) d\omega & \text{if } \alpha = k/2. \end{cases} \tag{2.32}$$

Here, $s \in \mathcal{P}_m$, $d = (m+1)/2$, and $k = 1, 2, \dots, m-1$. For $\alpha = 0$, we set $(I^0_+ f)(s) = f(s)$.

This definition agrees with consideration in Sections 2.1 and 2.2. We recall that the second line in (2.32) coincides with the first one if $k \geq m$, $\alpha = k/2$.

Theorem 2.17. *If $f \in L^1_{loc}(\mathcal{P}_m)$ and $\alpha \in \mathcal{W}_m$, then $(I^\alpha_+ f)(s)$ converges absolutely for almost all $s \in \mathcal{P}_m$.*

Proof. It suffices to show that the integral $\int_0^a (I^\alpha_+ f)(s) ds$ is finite for any $a \in \mathcal{P}_m$ provided that $\alpha \in \mathcal{W}_m$ and $f \in L^1_{loc}(\mathcal{P}_m)$ is nonnegative. For $\alpha > d-1$, changing the order of integration, and evaluating the inner integral according to (A.1), we have

$$\begin{aligned} \int_0^a (I^\alpha_+ f)(s) ds &= \frac{1}{\Gamma_m(\alpha)} \int_0^a f(r) dr \int_r^a |s-r|^{\alpha-d} ds \\ &= \frac{\Gamma_m(d)}{\Gamma_m(\alpha+d)} \int_0^a f(r)|a-r|^\alpha dr \\ &\leq \frac{\Gamma_m(d)|a|^\alpha}{\Gamma_m(\alpha+d)} \int_0^a f(r) dr < \infty \end{aligned}$$

(note that $|a-r|<|a|$; see, e.g., [**Mu**, p. 586] ). For the second line of (2.32), by changing the order of integration, we obtain

$$\begin{aligned}
\int_0^a (I_+^{k/2} f)(s)ds &= \pi^{-km/2} \int\limits_{\omega'\omega<a} d\omega \int\limits_{\omega'\omega}^{a} f(s-\omega'\omega)ds \\
&= \pi^{-km/2} \int\limits_{\omega'\omega<a} d\omega \int\limits_{0}^{a-\omega'\omega} f(r)dr \\
&\le \pi^{-km/2} c_a \int\limits_0^a f(r)dr,
\end{aligned}$$

where

$$c_a = \int\limits_{\omega'\omega<a} d\omega \overset{(A.7)}{=} \frac{\pi^{km/2}\,\Gamma_m(d)}{\Gamma_m(k/2+d)}\,|a|^{k/2} < \infty,$$

as required. □

The following equality can be easily obtained by changing the order of integration and using (A.4)–(A.6):

$$\int\limits_{\mathcal{P}_m} \frac{(I_+^\alpha f)(s)\,ds}{|\varepsilon I_m + s|^\gamma} = \frac{\Gamma_m(\gamma-\alpha)}{\Gamma_m(\gamma)} \int\limits_{\mathcal{P}_m} \frac{f(r)\,dr}{|\varepsilon I_m + r|^{\gamma-\alpha}}\ , \tag{2.33}$$

$$\alpha \in \mathcal{W}_m, \qquad Re\,(\gamma-\alpha) > (m-1)/2, \qquad \varepsilon = 0, 1.$$

Let us consider the right-sided integrals $I_-^\alpha f$. An idea of the following definition is the same as above.

DEFINITION 2.18. For a locally integrable function $f$ on $\mathcal{P}_m$ and $\alpha \in \mathcal{W}_m$, we define fractional integrals $(I_-^\alpha f)(s),\ s \in \overline{\mathcal{P}_m}$, by

$$(I_-^\alpha f)(s) = \begin{cases} \dfrac{1}{\Gamma_m(\alpha)} \displaystyle\int\limits_s^\infty f(r)|r-s|^{\alpha-d}dr & \text{if } Re\,\alpha > d-1, \\[2ex] \pi^{-km/2} \displaystyle\int\limits_{\mathfrak{M}_{k,m}} f(s+\omega'\omega)d\omega & \text{if } \alpha = k/2. \end{cases} \tag{2.34}$$

For $\alpha = 0$, we set $(I_-^0 f)(s) = f(s)$. As before, here we assume $d = (m+1)/2$ and $k = 1, 2, \dots, m-1$.

Note that unlike Definition 2.16, here $s$ may be a boundary point of $\mathcal{P}_m$. The following statement extends Lemma 2.11 (on interrelation between $I_+^\alpha$ and $I_-^\alpha$) to locally integrable functions.

THEOREM 2.19. *Let $\alpha \in \mathcal{W}_m$, $d = (m+1)/2$. Suppose that $f$ is a measurable function on $\mathcal{P}_m$ satisfying*

$$\int\limits_R^\infty |r|^{Re\,\alpha - d}|f(r)|\,dr < \infty, \quad \forall R \in \mathcal{P}_m, \tag{2.35}$$

*or, equivalently,*

$$g(r) \equiv |r|^{-\alpha-d} f(r^{-1}) \in L^1_{loc}(\mathcal{P}_m). \tag{2.36}$$

*Then the integral* $(I^\alpha_- f)(s)$ *defined by (2.34) converges absolutely for almost all* $s \in \mathcal{P}_m$ *and*

$$(I^\alpha_- f)(s) = |s|^{\alpha-d}(I^\alpha_+ g)(s^{-1}). \tag{2.37}$$

PROOF. Equivalence of (2.35) and (2.36) is obvious in view of Lemma 1.1 (iii). In fact, we have already proved (2.37) provided that either side of it exists in the Lebesgue sense. This was done in Lemma 2.11 for $Re\,\alpha > d-1$, and in Theorem 2.13 for $\alpha = k/2$, $0 < k < m$. By Theorem 2.17, the condition $g(r) \equiv |r|^{-\alpha-d} f(r^{-1}) \in L^1_{loc}(\mathcal{P}_m)$ guarantees finiteness of the right-hand side of (2.37) for almost all $s$. This completes the proof.

□

REMARK 2.20. Condition (2.35) is best possible in the sense that there is a function $f \geq 0$ for which (2.35) fails and $(I^\alpha_- f)(s) \equiv \infty$. Let

$$f_\lambda(r) = |I_m + r|^{-\lambda/2}, \tag{2.38}$$

where $\lambda \leq 2\alpha + m - 1$, $\alpha$ is real. Then for $\alpha > d-1$,

$$\int\limits_R^\infty |r|^{\alpha-d} f_\lambda(r)\,dr = \int\limits_0^{R^{-1}} |r|^{\lambda/2-\alpha-d}|I_m + r|^{-\lambda/2}\,dr \equiv \infty,$$

and by (A.5),

$$\begin{aligned}(I^\alpha_- f_\lambda)(s) &= \frac{|I_m + s|^{\alpha-\lambda/2}}{\Gamma_m(\alpha)} \int\limits_0^{I_m} |r|^{\lambda/2-\alpha-d}|I_m - r|^{\alpha-d}\,dr \\ &= |I_m + s|^{\alpha-\lambda/2} B_m(\alpha, \lambda/2 - \alpha)/\Gamma_m(\alpha) \equiv \infty.\end{aligned} \tag{2.39}$$

In the case $\alpha = k/2$, $k < m$, we have

$$\begin{aligned}(I^{k/2}_- f_\lambda)(s) &= \pi^{-km/2} \int\limits_{\mathfrak{M}_{k,m}} f_\lambda(s + \omega'\omega)d\omega \\ &= \pi^{-km/2} \int\limits_{\mathfrak{M}_{m,k}} |b + yy'|^{-\lambda/2}\,dy, \qquad b = I_m + s.\end{aligned} \tag{2.40}$$

Owing to (A.6), the last integral coincides with (2.39) (with $\alpha = k/2$) and is infinite.

The following formula is a consequence of (2.33) and (2.37):

$$\int\limits_{\mathcal{P}_m} (I^\alpha_- f)(s)\,\frac{|s|^{\gamma-a-d}}{|I_m + \varepsilon s|^\gamma}\,ds = \frac{\Gamma_m(\gamma-\alpha)}{\Gamma_m(\gamma)} \int\limits_{\mathcal{P}_m} f(r)\,\frac{|r|^{\gamma-d}}{|I_m + \varepsilon r|^{\gamma-\alpha}}\,dr\,, \tag{2.41}$$

$$\alpha \in \mathcal{W}_m, \qquad d = (m+1)/2, \qquad Re\,(\gamma-\alpha) > (m-1)/2, \qquad \varepsilon = 0, 1.$$

## 2.5. The semigroup property

By the Fubini theorem, the equality (A.1) yields

$$I^\alpha_\pm I^\beta_\pm f = I^{\alpha+\beta}_\pm f, \qquad Re\,\alpha > d-1, \quad Re\,\beta > d-1, \tag{2.42}$$

provided the corresponding integral $I^{\alpha+\beta}_+ f$ or $I^{\alpha+\beta}_- f$ is absolutely convergent. The equality (2.42) is usually referred to as the semigroup property of fractional integrals. It extends to all complex $\alpha$ and $\beta$ if $f$ is good enough and fractional integrals are interpreted as multiplier operators in the framework of the relevant Fourier-Laplace analysis. This approach allows us to formulate the semigroup property in the language of the theory of distributions; see [**Gi1**], [**FK**], [**VG**], [**Rab**] for details.

Below we extend (2.42) to the important special case when $\alpha$ and $\beta$ lie in the Wallach-like set $\mathcal{W}_m$ defined by (2.31). Definitions (2.32) and (2.34) enable us to do this under minimal assumptions for $f$ so that all integrals are understood in the classical Lebesgue sense.

LEMMA 2.21. *If $f \in L^1_{loc}(\mathcal{P}_m)$, then*

$$I^\alpha_\pm I^\beta_\pm f = I^{\alpha+\beta}_\pm f, \qquad \alpha, \beta \in \mathcal{W}_m, \tag{2.43}$$

*provided the corresponding integral on the right-hand side is absolutely convergent.*

PROOF. In view of (2.42), it suffices to prove that

$$I^\alpha_\pm I^{k/2}_\pm f = I^{k/2}_\pm I^\alpha_\pm f = I^{\alpha+k/2}_\pm f, \qquad Re\,\alpha > d-1, \tag{2.44}$$

and

$$I^{k/2}_\pm I^{\ell/2}_\pm f = I^{(k+\ell)/2}_\pm f, \tag{2.45}$$

where $k$ and $\ell$ are positive integers less than $m$. By changing the order of integration, we have

$$\begin{aligned}
(I^\alpha_- I^{k/2}_- f)(s) &= \frac{\pi^{-km/2}}{\Gamma_m(\alpha)} \int_s^\infty |r-s|^{\alpha-d}\,dr \int_{\mathfrak{M}_{k,m}} f(r+\omega'\omega)\,d\omega \\
&= \frac{\pi^{-km/2}}{\Gamma_m(\alpha)} \int_{\mathfrak{M}_{k,m}} d\omega \int_{s+\omega'\omega}^\infty f(t)|t-s-\omega'\omega|^{\alpha-d}\,dt \\
&= \frac{\pi^{-km/2}}{\Gamma_m(\alpha)} \int_s^\infty f(t) h(t-s)\,dt,
\end{aligned}$$

where

$$h(z) = \int_{\{\omega\in\mathfrak{M}_{k,m}:\, \omega'\omega < z\}} |z-\omega'\omega|^{\alpha-d} d\omega.$$

The same expression comes out if we transform $I^{k/2}_- I^\alpha_- f$. By (A.6),

$$h(z) = \frac{\pi^{km/2}\Gamma_m(\alpha)}{\Gamma_m(\alpha+k/2)} |z|^{\alpha+k/2-d}, \qquad Re\,\alpha > d-1, \tag{2.46}$$

and we are done. For the left-sided integrals, the argument is similar. Namely,

$$\begin{aligned}(I_+^{\alpha} I_+^{k/2} f)(s) &= \frac{\pi^{-km/2}}{\Gamma_m(\alpha)} \int_0^s |s-r|^{\alpha-d} dr \int\limits_{\{\omega\in\mathfrak{M}_{k,m}\,:\,\omega'\omega<r\}} f(r-\omega'\omega)d\omega \\ &= \frac{\pi^{-km/2}}{\Gamma_m(\alpha)} \int_0^s f(t)h(s-t)\,dt \\ &= (I_+^{\alpha+k/2} f)(s).\end{aligned}$$

Let us prove (2.45). By (2.34),

$$(I_-^{k/2} I_-^{\ell/2} f)(s) = \pi^{-m(k+\ell)/2} \int\limits_{\mathfrak{M}_{k,m}} d\omega \int\limits_{\mathfrak{M}_{\ell,m}} f(y'y+\omega'\omega+s)dy.$$

The change of variables $z = \begin{bmatrix} y \\ \omega \end{bmatrix} \in \mathfrak{M}_{k+\ell,m}$ yields $z'z = y'y + \omega'\omega$, and therefore

$$(I_-^{k/2} I_-^{\ell/2} f)(s) = \pi^{-m(k+\ell)/2} \int\limits_{\mathfrak{M}_{k+\ell,m}} f(z'z+s)dz = (I_-^{(k+\ell)/2} f)(s).$$

For the left-sided integrals, owing to (2.32), we have

$$(I_+^{(k+\ell)/2} f)(s) = \pi^{-m(k+\ell)/2} \int\limits_{\{z\in\mathfrak{M}_{k+\ell,m}\,:\,z'z<s\}} f(s-z'z)\,dz.$$

By the Fubini theorem, this reads

$$\pi^{-m(k+\ell)/2} \int\limits_{\{\omega\in\mathfrak{M}_{k,m}\,:\,\omega'\omega<s\}} d\omega \int\limits_{\{y\in\mathfrak{M}_{\ell,m}:\,y'y<s-\omega'\omega\}} f(s-y'y-\omega'\omega)\,dy$$

and coincides with $(I_+^{k/2} I_+^{\ell/2} f)(s)$. □

## 2.6. Inversion formulas

Below we obtain inversion formulas for the integrals $(I_\pm^\alpha f)(r)$, $r \in \mathcal{P}_m$. For our purposes, it suffices to restrict to the case $\alpha \in \mathcal{W}_m$. We recall that $k \in \mathbb{N}$ and $d = (m+1)/2$. For "good" functions $f$, the following statement holds.

LEMMA 2.22. *If* $\varphi = I_\pm^\alpha f$, $f \in \mathcal{D}(\mathcal{P}_m)$, $\alpha \in \mathcal{W}_m$, *then*

$$f = D_\pm^j I_\pm^{j-\alpha} \varphi, \tag{2.47}$$

*provided that* $j \ge \alpha$ *if* $\alpha = k/2$ *and* $j > Re\,\alpha + d - 1$ *otherwise.*

PROOF. By the semigroup property (2.43), we have $I_\pm^{j-\alpha}\varphi = I_\pm^j f$. Then application of (2.6) and (2.13) (or (2.20)) gives (2.47). □

Unfortunately, we are not able to obtain pointwise inversion formulas for $I_\pm^\alpha f$ if $f$ is an arbitrary locally integrable function. To get around this difficulty, we utilize the theory of distributions.

LEMMA 2.23. *If* $\varphi = I_+^\alpha f,\ f \in L^1_{loc}(\mathcal{P}_m),\ \alpha \in \mathcal{W}_m$, *then*

$$f = D_+^j I_+^{j-\alpha} \varphi, \tag{2.48}$$

*for any integer* $j$ *subject to the condition that* $j \ge \alpha$ *if* $\alpha = k/2$ *and* $j > Re\,\alpha + d - 1$ *otherwise. The differentiation in (2.48) is understood in the sense of distributions, so that*

$$(f,\ \phi) = (I_+^{j-\alpha}\varphi,\ D_-^j \phi), \qquad \phi \in \mathcal{D}(\mathcal{P}_m). \tag{2.49}$$

*In particular, for* $\alpha = j$,

$$(f,\ \phi) = (\varphi,\ D_-^j \phi). \tag{2.50}$$

PROOF. As above, we have $I_+^{j-\alpha}\varphi = I_+^j f$, and therefore,

$$(f,\ \phi) \overset{(1)}{=} (f,\ I_-^j D_-^j \phi) \overset{(2)}{=} (I_+^j f,\ D_-^j \phi) \overset{(3)}{=} (I_+^{j-\alpha}\varphi,\ D_-^j \phi).$$

The equality (1) holds thanks to (2.6), in (2) we apply the Fubini theorem; (3) holds owing to the semigroup property (2.44). □

Let us invert the equation $I_-^\alpha f = \varphi$. Formally, $f = D_-^\alpha \varphi$ where $D_-^\alpha = I_-^{-\alpha}$, or $(f,\ \phi) = (\varphi,\ I_+^{-\alpha}\phi)$, $\phi \in \mathcal{D}(\mathcal{P}_m)$ in the sense of distributions. The next statement justifies this equality.

LEMMA 2.24. *Let* $\alpha \in \mathcal{W}_m$, *and let* $f$ *be a locally integrable function on* $\mathcal{P}_m$ *subject to the decay condition (2.35), so that* $(I_-^\alpha f)(s)$ *exists for almost all* $s \in \mathcal{P}_m$. *If* $\varphi = I_-^\alpha f$ *then the integral*

$$(\varphi,\ I_+^{-\alpha}\phi) = \int\limits_{\mathcal{P}_m} \varphi(r)(I_+^{-\alpha}\phi)(r)dr, \qquad \phi \in \mathcal{D}(\mathcal{P}_m),$$

*is finite, and* $f$ *can be recovered by the formula*

$$f = D_-^\alpha \varphi, \tag{2.51}$$

*where* $D_-^\alpha \varphi$ *is defined as a distribution*

$$(D_-^\alpha \varphi,\ \phi) = (\varphi,\ I_+^{-\alpha}\phi). \tag{2.52}$$

*In particular, for* $\alpha = j$,

$$(f,\ \phi) = (\varphi,\ D_+^j \phi). \tag{2.53}$$

PROOF. Let $\phi_1(s) = |s|^{\alpha-d}\phi(s^{-1})$. This function belongs to $\mathcal{D}(\mathcal{P}_m)$. By (2.19),

$$(I_+^{-\alpha}\phi)(s) = |s|^{-\alpha-d}(I_-^{-\alpha}\phi_1)(s^{-1}).$$

If $\varphi = I_-^\alpha f$ then

$$\begin{aligned}
(\varphi,\ I_+^{-\alpha}\phi) \quad &= \quad ((I_-^\alpha f)(s),\ |s|^{-\alpha-d}(I_-^{-\alpha}\phi_1)(s^{-1})) \\
&\qquad \text{(use Theorem 2.19 with } g(r) = |r|^{-\alpha-d} f(r^{-1})) \\
&= \quad ((I_+^\alpha g)(s^{-1}),\ |s|^{-2d}(I_-^{-\alpha}\phi_1)(s^{-1})) \\
&= \quad ((I_+^\alpha g)(s),\ (I_-^{-\alpha}\phi_1)(s)) \\
&\overset{(1)}{=} \quad (g,\ I_-^\alpha I_-^{-\alpha}\phi_1) \\
&\overset{(2)}{=} \quad (g,\ \phi_1) \\
&= \quad (f, \phi).
\end{aligned}$$

Let us comment on these calculations. The equality (1) holds by the Fubini theorem. Here we take into account that $I_-^{-\alpha}\phi_1 \in \mathcal{D}(\mathcal{P}_m)$, because by the second equality in (2.6),

$$I_-^{-\alpha}\phi_1 = I_-^{j-\alpha} D_-^j \phi_1 \tag{2.54}$$

if $j$ is big enough. A rigorous justification of the obvious equality $I_-^{\alpha} I_-^{-\alpha}\phi_1 = \phi_1$ used in (2) is as follows:

$$I_-^{\alpha} I_-^{-\alpha}\phi_1 \overset{(2.54)}{=} I_-^{\alpha} I_-^{j-\alpha} D_-^j \phi_1 \overset{(2.43)}{=} I_-^{j} D_-^j \phi_1 \overset{(2.6)}{=} I_-^{0} \phi_1 \overset{(2.20)}{=} \phi_1.$$

For $\alpha = j$, the result follows by Corollary 2.7, according to which $I_+^{-j}\phi = D_+^j \phi$. The proof is complete. $\square$

CHAPTER 3

# Riesz potentials on matrix spaces

Riesz potentials of functions of matrix argument and their generalizations arise in different contexts in harmonic analysis, integral geometry, and PDE; see [**Far**], [**Ge**], [**Kh**], [**St2**], [**Sh1**]–[**Sh3**]. They are intimately connected with representations of Jordan algebras and zeta functions (or zeta distributions) extensively studied in the last three decades; see [**FK**], [**Cl**], [**Ig**], [**SS**], [**Sh**] and references therein. In this chapter, we explore basic properties of matrix Riesz potentials which will be used in Chapters 5 and 6 in our study of Radon transforms. Numerous results, bibliography, and historical notes related to ordinary Riesz potentials on $\mathbb{R}^n$ can be found in [**Ru1**], [**Sa**], [**SKM**].

## 3.1. Zeta integrals

**3.1.1. Definition and examples.** We denote

$$|x|_m = \det(x'x)^{1/2}. \tag{3.1}$$

For $m = 1$, this is the usual euclidean norm on $\mathbb{R}^n$. If $m > 1$, then $|x|_m$ is the volume of the parallelepiped spanned by the column-vectors of the matrix $x$ [**G**, p. 251]. Let $f(x)$ be a Schwartz function on the matrix space $\mathfrak{M}_{n,m}$, $n \geq m$. Let us consider the following distribution

$$\mathcal{Z}(f, \alpha - n) = \int\limits_{\mathfrak{M}_{n,m}} f(x)|x|_m^{\alpha-n} dx, \qquad \alpha \in \mathbb{C}. \tag{3.2}$$

For $Re\,\alpha > m-1$, the integral (3.2) absolutely converges, and for $Re\,\alpha \leq m-1$, it is understood in the sense of analytic continuation (see Lemma 3.2). Integrals like (3.2) are also known as the *zeta integrals* or *zeta distributions.*

The case $n = m$ deserves special consideration. In this case, we denote $\mathfrak{M}_m = \mathfrak{M}_{m,m}$ and define two zeta integrals

$$\mathcal{Z}_+(f, \alpha - m) = \int\limits_{\mathfrak{M}_m} f(x)\,|\det(x)|^{\alpha-m}\,dx \qquad (\equiv \mathcal{Z}(f, \alpha - m)), \tag{3.3}$$

$$\mathcal{Z}_-(f, \alpha - m) = \int\limits_{\mathfrak{M}_m} f(x)\,|\det(x)|^{\alpha-m}\,\mathrm{sgn}\det(x)\,dx. \tag{3.4}$$

We call (3.4) *the conjugate zeta integral* (or distribution) by analogy with the case $\alpha = 0$, $m = 1$, where

$$\mathcal{Z}_-(f, -1) = \text{p.v.} \int\limits_{-\infty}^{\infty} \frac{f(x)}{x}\,dx.$$

A convolution of $f$ with the distribution p.v.$\frac{1}{x}$ represents the Hilbert transform [**Ne**], [**SW**], and is called a conjugate function.

EXAMPLE 3.1. It is instructive to evaluate zeta integrals for the Gaussian functions. Let

$$e(x) = \exp(-\mathrm{tr}(x'x)). \tag{3.5}$$

By Lemma 1.11, for $Re\,\alpha > m-1$ we have

$$\mathcal{Z}(e, \alpha - n) = 2^{-m}\sigma_{n,m} \int\limits_{\mathcal{P}_m} |r|^{\alpha/2-d} \exp(-\mathrm{tr}(r))dr = c_{n,m}\Gamma_m(\alpha/2), \tag{3.6}$$

where $\sigma_{n,m}$ is the constant (1.37), $d = (m+1)/2$,

$$c_{n,m} = \frac{\pi^{nm/2}}{\Gamma_m(n/2)}\,, \quad n \ge m.$$

In the case $n = m$, for the function

$$e_1(x) = e(x)\det(x) \tag{3.7}$$

we have

$$\mathcal{Z}_-(e_1, \alpha - m) = \mathcal{Z}_+(e, \alpha + 1 - m) = c_m\Gamma_m((\alpha+1)/2), \tag{3.8}$$

$$c_m \equiv c_{m,m} = \frac{\pi^{m^2/2}}{\Gamma_m(m/2)}, \qquad Re\,\alpha > m-2.$$

Formulas (3.6) and (3.8) extend meromorphically to all complex $\alpha$, excluding $\alpha = m-1, m-2, \ldots$ for (3.6), and $\alpha = m-2, m-3, \ldots$ for (3.8). Excluded values are poles of the corresponding gamma functions written for $m \ge 2$. If $m = 1$, these poles proceed with step 2, namely, $\alpha = 0, -2, -4, \ldots$ and $\alpha = -1, -3, \ldots$, respectively.

**3.1.2. Analytic continuation.** In the following, throughout the paper, we assume $m \ge 2$.

LEMMA 3.2. *Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$. For $Re\,\alpha > m-1$, the integrals (3.2)-(3.4) are absolutely convergent. For $Re\,\alpha \le m-1$, they extend as meromorphic functions of $\alpha$ with the only poles*

$$\begin{aligned} &m-1,\ m-2, \ldots, &&\text{for (3.2) and (3.3)},\\ &m-2,\ m-3, \ldots, &&\text{for (3.4)}. \end{aligned}$$

*These poles and their orders are exactly the same as those of the gamma functions $\Gamma_m(\alpha/2)$ and $\Gamma_m((\alpha+1)/2)$, respectively. The normalized zeta integrals*

$$\frac{\mathcal{Z}(f, \alpha-n)}{\Gamma_m(\alpha/2)}, \qquad \frac{\mathcal{Z}_+(f, \alpha-m)}{\Gamma_m(\alpha/2)}, \qquad \frac{\mathcal{Z}_-(f, \alpha-m)}{\Gamma_m((\alpha+1)/2)} \tag{3.9}$$

*are entire functions of $\alpha$.*

PROOF. This statement is known; see, e.g., [**P5**], [**Kh**], [**Sh1**]. We present the proof for the sake of completeness. The equalities (3.6) and (3.8) say that the functions $\alpha \to \mathcal{Z}(f, \alpha-n)$ and $\alpha \to \mathcal{Z}_-(f, \alpha-m)$ have poles at least at the same points and of the same order as the gamma functions $\Gamma_m(\alpha/2)$ and $\Gamma_m((\alpha+1)/2)$, respectively. Our aim is to show that no other poles occur, and the orders cannot exceed those of $\Gamma_m(\alpha/2)$ and $\Gamma_m((\alpha+1)/2)$. Let us transform (3.2) by passing to upper triangular matrices $t \in T_m$ according to Lemma 1.12. We have

$$\mathcal{Z}(f, \alpha - n) = \int_{\mathbb{R}_+^m} F(t_{1,1}, \ldots, t_{m,m}) \prod_{i=1}^{m} t_{i,i}^{\alpha-i} dt_{i,i} \ , \tag{3.10}$$

$$F(t_{1,1}, \ldots, t_{m,m}) = \int_{\mathbb{R}^{m(m-1)/2}} dt_* \int_{V_{n,m}} f(vt) dv, \quad dt_* = \prod_{i<j} dt_{i,j}.$$

Since $F$ extends as an even Schwartz function in each argument, it can be written as

$$F(t_{1,1}, \ldots, t_{m,m}) = F_0(t_{1,1}^2, \ldots, t_{m,m}^2),$$

where $F_0 \in \mathcal{S}(\mathbb{R}^m)$ (use, e.g., Lemma 5.4 from [**Tr**, p. 56]). Replacing $t_{i,i}^2$ by $s_{i,i}$, we represent (3.10) as a direct product of one-dimensional distributions

$$\mathcal{Z}(f, \alpha - n) = 2^{-m} (\prod_{i=1}^{m} (s_{i,i})_+^{(\alpha-i-1)/2}, \ F_0(s_{1,1}, \ldots, s_{m,m})), \tag{3.11}$$

which is a meromorphic function of $\alpha$ with the poles $m-1, m-2, \ldots$, see [**GSh1**]. These poles and their orders coincide with those of the gamma function $\Gamma_m(\alpha/2)$. To normalize the function (3.11), following [**GSh1**], we divide it by the product

$$\prod_{i=1}^{m} \Gamma((\alpha - i + 1)/2) = \prod_{i=0}^{m-1} \Gamma((\alpha - i)/2) = \Gamma_m(\alpha/2)/\pi^{m(m-1)/4}.$$

As a result, we obtain an entire function.

For the distribution $\mathcal{Z}_-(f, \alpha - m)$, the argument is almost the same. Namely, by Lemma 1.12, the integral (3.4) can be written as

$$\int_{\mathbb{R}_+^m} \Phi(t_{1,1}, \ldots, t_{m,m}) \prod_{i=1}^{m} t_{i,i}^{\alpha-i} \, dt_{i,i}$$

where

$$\Phi(t_{1,1}, \ldots, t_{m,m}) = \sigma_{m,m} \int_{\mathbb{R}^{m(m-1)/2}} dt_* \int_{O(m)} f(vt) \operatorname{sgn} \det(v) \, dv. \tag{3.12}$$

Since $\Phi$ extends to $\mathbb{R}^m$ as an odd Schwartz function in each argument, then

$$\mathcal{Z}_-(f, \alpha - m) = \frac{1}{2^m} \int_{\mathbb{R}^m} \Phi(t_{1,1}, \ldots, t_{m,m}) \prod_{i=1}^{m} |t_{i,i}|^{\alpha-i} \operatorname{sgn}(t_{i,i}) \, dt_{i,i}. \tag{3.13}$$

By the well known theory [**GSh1**, Chapter 1, Section 3.5], this integral extends as a meromorphic function of $\alpha$ with the only poles $m-2, m-3, \ldots$. To normalize this function, we divide it by the product

$$\prod_{i=1}^{m} \Gamma\left(\frac{\alpha - i + 2}{2}\right) = \prod_{i=0}^{m-1} \Gamma\left(\frac{\alpha - i + 1}{2}\right) = \Gamma_m\left(\frac{\alpha + 1}{2}\right) / \pi^{m(m-1)/4}.$$

As a result, we get an entire function. $\square$

Analytic continuation of integrals (3.2)-(3.4) can be performed with the aid of the corresponding differential operators. In particular, for $\mathcal{Z}_{\pm}(f, \alpha - m)$, one can utilize *the Cayley differential operator*

$$\mathcal{D} = \det(\partial/\partial x_{i,j}), \quad x = (x_{i,j}) \in \mathfrak{M}_m, \tag{3.14}$$

that enjoys the following obvious relations in the Fourier terms:

$$(\mathcal{F}[\mathcal{D}f])(y) = (-i)^m \det(y)\,(\mathcal{F}f)(y), \tag{3.15}$$

$$\mathcal{D}(\mathcal{F}f)(y) = i^m\,(\mathcal{F}[f(x)\det(x)])(y). \tag{3.16}$$

LEMMA 3.3. *Let $x \in \mathfrak{M}_m$, $\operatorname{rank}(x) = m$. For any $\lambda \in \mathbb{C}$,*

$$\mathcal{D}\,[|\det(x)|^{\lambda}] = (\lambda, m)\,|\det(x)|^{\lambda-1}\operatorname{sgn}\det(x), \tag{3.17}$$

$$\mathcal{D}\,[|\det(x)|^{\lambda}\operatorname{sgn}\det(x)] = (\lambda, m)\,|\det(x)|^{\lambda-1}. \tag{3.18}$$

PROOF. Different proofs of these important formulas can be found in [**Ra**] and [**P2**]; see also [**Tu**, p. 114]. All these proofs are very involved. Below we give an alternative proof which is elementary. Note that (3.17) and (3.18) follow one from another. Furthermore, it suffices to assume that $\lambda$ is not an integer ($\lambda \notin \mathbb{Z}$). Once (3.17) and (3.18) are proved for such $\lambda$, the result for $\lambda \in \mathbb{Z}$ then follows by continuity.

We start with the formula

$$\mathcal{D}_x[f(ax)] = \det(a)\,(\mathcal{D}f)(ax), \qquad a \in GL(m, \mathbb{R}), \tag{3.19}$$

which can be easily checked by applying the Fourier transform to both sides. Indeed, if $f$ is good enough at infinity (otherwise we can multiply $f$ by a smooth cut-off function) then by (3.15), the Fourier transform of the left-hand side of (3.19) reads

$$(-i)^m \det(y)\,\mathcal{F}[f(ax)](y) = \frac{(-i)^m \det(y)}{|\det(a)|^m}(\mathcal{F}f)((a^{-1})'y)$$

which coincides with the Fourier transform of the right-hand side. If $f(x) = |\det(x)|^{\lambda}$ then (3.19) yields

$$|\det(a)|^{\lambda}\,\mathcal{D}\,|\det(x)|^{\lambda} = \det(a)\,[\mathcal{D}\,|\det(\cdot)|^{\lambda}](ax).$$

By setting $a = x^{-1}$ (recall that $\operatorname{rank}(x) = m$ so that $x$ is non-singular) we obtain

$$\mathcal{D}\,|\det(x)|^{\lambda} = A|\det(x)|^{\lambda-1}\operatorname{sgn}\det(x), \quad A = [\mathcal{D}\,|\det(x)|^{\lambda}](I_m),$$

and therefore

$$\mathcal{D}\,[|\det(x)|^{\lambda}\operatorname{sgn}\det(x)] = A|\det(x)|^{\lambda-1}. \tag{3.20}$$

In order to evaluate $A$, we make use of the Gaussian functions

$$e(x) = \exp(-\operatorname{tr}(x'x)) \qquad \text{and} \qquad e_1(x) = e(x)\det(x) = (-2)^{-m}(\mathcal{D}e)(x),$$

see (3.5) and (3.7). By (3.8),

$$\mathcal{Z}_-(e_1, \lambda) = \mathcal{Z}_+(e, \lambda + 1) = c_m\Gamma_m((\lambda + 1 + m)/2)$$

(here and on we do not care about the poles because we assumed in advance that $\lambda \notin \mathbb{Z}$). On the other hand, by (3.20) and (3.6),

$$\begin{aligned}
\mathcal{Z}_-(e_1,\lambda) &= (-2)^{-m}\mathcal{Z}_-(\mathcal{D}e,\lambda) \\
&= 2^{-m}(e(x),\mathcal{D}\,[|\det(x)|^{\lambda}\operatorname{sgn}\det(x)]) \\
&= 2^{-m}A\mathcal{Z}_+(e,\lambda-1) \\
&= c_m 2^{-m}A\,\Gamma_m((\lambda-1+m)/2).
\end{aligned}$$

Hence, owing to (1.9), we obtain

$$A=\frac{2^m\,\Gamma_m((\lambda+1+m)/2)}{\Gamma_m(\lambda-1+m)/2)}=(-1)^m\frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda-m)}=(\lambda,m).$$

□

Formulas (3.17) and (3.18) imply the following connection between zeta integrals $\mathcal{Z}_+(\cdot)$ and $\mathcal{Z}_-(\cdot)$.

Corollary 3.4.

$$\mathcal{Z}_{\mp}(f,\alpha-m)=c_\alpha\,\mathcal{Z}_{\pm}(\mathcal{D}f,\alpha+1-m),\quad c_\alpha=(-1)^m\frac{\Gamma(\alpha+1-m)}{\Gamma(\alpha+1)}. \tag{3.21}$$

The formula (3.21) can be used for analytic continuation of the zeta integrals $\mathcal{Z}_{\mp}(f,\alpha-m)$.

**3.1.3. Functional equations for the zeta integrals.** The following statement is a core of the theory of zeta integrals (3.2)-(3.4).

Theorem 3.5. *Let $f\in\mathcal{S}(\mathfrak{M}_{n,m})$, $n\ge m$. Then*

$$\frac{\mathcal{Z}(f,\alpha-n)}{\Gamma_m(\alpha/2)}=\pi^{-nm/2}\,2^{m(\alpha-n)}\,\frac{\mathcal{Z}(\mathcal{F}f,-\alpha)}{\Gamma_m((n-\alpha)/2)}, \tag{3.22}$$

$$\frac{\mathcal{Z}_+(f,\alpha-m)}{\Gamma_m(\alpha/2)}=\pi^{-m^2/2}\,2^{m(\alpha-m)}\,\frac{\mathcal{Z}_+(\mathcal{F}f,-\alpha)}{\Gamma_m((m-\alpha)/2)}, \tag{3.23}$$

$$\frac{\mathcal{Z}_-(f,\alpha-m)}{\Gamma_m((\alpha+1)/2)}=(-i)^m\,\pi^{-m^2/2}\,2^{m(\alpha-m)}\,\frac{\mathcal{Z}_-(\mathcal{F}f,-\alpha)}{\Gamma_m((m-\alpha+1)/2)}. \tag{3.24}$$

Proof. We recall that both sides of each equality are understood in the sense of analytic continuation and represent entire functions of $\alpha$. The equalities (3.22) and (3.23) were obtained in [**Far**], [**FK**], [**Ge**], [**Ra**], [**P2**] in the framework of more general considerations. A self-contained proof of them and detailed discussion can be found in [**Ru7**]. The equality (3.24) was implicitly presented in [**P2**, p. 289]. In fact, it follows from (3.23) owing to the formulas (3.21) and (3.15). We have

$$\begin{aligned}
\frac{\mathcal{Z}_-(f,\alpha-m)}{\Gamma_m((\alpha+1)/2)} &= \frac{c_\alpha\,\mathcal{Z}_+(\mathcal{D}f,\alpha+1-m)}{\Gamma_m((\alpha+1)/2)} \\
&= c_\alpha\,\pi^{-m^2/2}\,2^{m(\alpha+1-m)}\,\frac{\mathcal{Z}_+(\mathcal{F}[\mathcal{D}f],-\alpha-1)}{\Gamma_m((m-\alpha-1)/2)} \\
&= (-i)^m\,c_\alpha\,\pi^{-m^2/2}\,2^{m(\alpha+1-m)}\,\frac{((\mathcal{F}f)(y),\det(y)\,|\det(y)|^{-\alpha-1})}{\Gamma_m((m-\alpha-1)/2)} \\
&= c\,\frac{\mathcal{Z}_-(\mathcal{F}f,-\alpha)}{\Gamma_m((m-\alpha+1)/2)}\ ,
\end{aligned}$$

where (use (3.21) and (1.9))

$$c = (-i)^m\, c_\alpha\, \pi^{-m^2/2}\, 2^{m(\alpha+1-m)}\, \frac{\Gamma_m((m-\alpha+1)/2)}{\Gamma_m((m-\alpha-1)/2)} = (-i)^m\, \pi^{-m^2/2}\, 2^{m(\alpha-m)}.$$

□

## 3.2. Normalized zeta distributions

It is convenient to introduce a special notation for the normalized zeta integrals (3.9) which are entire functions of $\alpha$. Let

$$\zeta_\alpha(x) = \frac{|x|_m^{\alpha-n}}{\Gamma_m(\alpha/2)}, \qquad x \in \mathfrak{M}_{n,m}, \quad n \geq m, \tag{3.25}$$

$$\zeta_\alpha^+(x) = \frac{|\det(x)|^{\alpha-m}}{\Gamma_m(\alpha/2)}, \quad \zeta_\alpha^-(x) = \frac{|\det(x)|^{\alpha-m}\,\mathrm{sgn}\det(x)}{\Gamma_m((\alpha+1)/2)}, \quad x \in \mathfrak{M}_m. \tag{3.26}$$

Given a Schwartz function $f$, we denote

$$(\zeta_\alpha, f) = a.c. \int\limits_{\mathfrak{M}_{n,m}} f(x)\zeta_\alpha(x)\,dx, \quad (\zeta_\alpha^\pm, f) = a.c. \int\limits_{\mathfrak{M}_m} f(x)\zeta_\alpha^\pm(x)\,dx \tag{3.27}$$

where"*a.c.*" abbreviates analytic continuation. We call $\zeta_\alpha$ and $\zeta_\alpha^\pm$ *normalized zeta distributions of order* $\alpha$ on $\mathfrak{M}_{n,m}$ and $\mathfrak{M}_m$, respectively. By (3.21),

$$(\zeta_\alpha^-, f) = c_\alpha\, (\zeta_{\alpha+1}^+, \mathcal{D}f), \qquad (\zeta_\alpha^+, f) = d_\alpha\, (\zeta_{\alpha+1}^-, \mathcal{D}f), \tag{3.28}$$

where

$$\begin{aligned} c_\alpha &= (-1)^m \frac{\Gamma(\alpha+1-m)}{\Gamma(\alpha+1)}, \\ d_\alpha &= \frac{c_\alpha\, \Gamma_m(\alpha/2+1)}{\Gamma_m(\alpha/2)} = \frac{\Gamma(\alpha+1-m)\,\Gamma(m-\alpha)}{2^m\,\Gamma(\alpha+1)\,\Gamma(-\alpha)} \qquad \text{(use (1.9))}. \end{aligned}$$

Our next goal is to obtain explicit representations of the distributions (3.27) when the corresponding integrals are not absolutely convergent. Evaluation of $(\zeta_\alpha, f)$ when $\alpha$ is a non-negative integer is of primary importance in view of subsequent application to the Radon transform in Chapters 5 and 6.

THEOREM 3.6. *Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$. For $\alpha = k$, $k = 1, 2, \dots, n$,*

$$(\zeta_k, f) = \frac{\pi^{(n-k)m/2}}{\Gamma_m(n/2)} \int\limits_{O(n)} d\gamma \int\limits_{\mathfrak{M}_{k,m}} f\left(\gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix}\right) d\omega. \tag{3.29}$$

*Furthermore, in the case $\alpha = 0$ we have*

$$(\zeta_0, f) = \frac{\pi^{nm/2}}{\Gamma_m(n/2)} f(0). \tag{3.30}$$

PROOF. STEP 1. Let first $k > m-1$ . In polar coordinates we have

$$\begin{aligned} (\zeta_k, f) &= \frac{1}{\Gamma_m(k/2)} \int\limits_{\mathfrak{M}_{n,m}} f(x)|x|_m^{k-n} dx \\ &= \frac{2^{-m}\sigma_{n,m}}{\Gamma_m(k/2)} \int\limits_{\mathcal{P}_m} |r|^{k/2-d} dr \int\limits_{O(n)} f\left(\gamma \begin{bmatrix} r^{1/2} \\ 0 \end{bmatrix}\right) d\gamma. \end{aligned}$$

Now we replace $\gamma$ by $\gamma\begin{bmatrix}\beta & 0\\ 0 & I_{n-k}\end{bmatrix}$, $\beta\in O(k)$, then integrate in $\beta\in O(k)$, and replace the integration over $O(k)$ by that over $V_{k,m}$. We get

$$\begin{aligned}(\zeta_k,f) &= \frac{2^{-m}\sigma_{n,m}}{\sigma_{k,m}\Gamma_m(k/2)}\int\limits_{O(n)}d\gamma\int\limits_{\mathcal{P}_m}|r|^{k/2-d}dr\int\limits_{V_{k,m}}f\left(\gamma\begin{bmatrix}vr^{1/2}\\ 0\end{bmatrix}\right)dv\\ &\quad(\text{set}\quad \omega=vr^{1/2}\in\mathfrak{M}_{k,m})\\ &= \frac{\sigma_{n,m}}{\sigma_{k,m}\Gamma_m(k/2)}\int\limits_{\mathfrak{M}_{k,m}}d\omega\int\limits_{O(n)}f\left(\gamma\begin{bmatrix}\omega\\ 0\end{bmatrix}\right)d\gamma.\end{aligned}$$

This coincides with (3.29).

STEP 2. Our next task is to prove that analytic continuation of $(\zeta_\alpha,f)$ at the point $\alpha=k$ $(\le m-1)$ has the form (3.29). To this end, we express $\zeta_\alpha$ through the Gårding-Gindikin distribution; see Section 2.3. For $Re\ \alpha>m-1$, by passing to polar coordinates, we have $(\zeta_\alpha,f)=2^{-m}\sigma_{n,m}\mathcal{G}_{\alpha/2}(F)$, where

$$\mathcal{G}_{\alpha/2}(F)=\frac{1}{\Gamma_m(\alpha/2)}\int\limits_{\mathcal{P}_m}F(r)|r|^{\alpha/2-d}dr,\qquad F(r)=\frac{1}{\sigma_{n,m}}\int\limits_{V_{n,m}}f(vr^{1/2})dv.$$

To continue the proof, we need the following

LEMMA 3.7. *Let $\mathcal{S}(\overline{\mathcal{P}}_m)$ be the space of restrictions onto $\overline{\mathcal{P}}_m$ of the Schwartz functions on $S_m\supset\overline{\mathcal{P}}_m$ with the induced topology. The map*

$$\mathcal{S}(\overline{\mathcal{P}}_m)\ni F\ \to f(x)=F(x'x)$$

*is an isomorphism of $\mathcal{S}(\overline{\mathcal{P}}_m)$ onto the space $\mathcal{S}(\mathfrak{M}_{n,m})^\natural$ of $O(n)$ left-invariant functions on $\mathfrak{M}_{n,m}$.*

This important statement, which is well known for $m=1$ (see, e.g., Lemma 5.4 in [**Tr**, p. 56]), was presented in a slightly different form by J. Farau [**Far**, Prop. 3] and derived from the more general result of G. W. Schwarz [**Sc**, Theorem 1]. According to (2.28), analytic continuation of $\mathcal{G}_{\alpha/2}(F)$ at the point $\alpha=k$, $k=1,2,\dots,m-1$, is evaluated as follows:

$$\begin{aligned}\mathcal{G}_{k/2}(F) &= \pi^{-km/2}\int\limits_{\mathfrak{M}_{k,m}}F(\omega'\omega)d\omega\\ &= \pi^{-km/2}\int\limits_{\mathfrak{M}_{k,m}}d\omega\int\limits_{O(n)}f\left(\gamma\begin{bmatrix}(\omega'\omega)^{1/2}\\ 0\end{bmatrix}\right)d\gamma.\end{aligned}$$

By making use of the polar coordinates, one can write $\omega'\in\mathfrak{M}_{m,k}$ as

$$\omega'=\beta u_0(\omega\omega')^{1/2},\quad \beta\in O(m),\quad u_0=\begin{bmatrix}I_k\\ 0\end{bmatrix}\in V_{m,k}.$$

Hence, $\omega=(\omega\omega')^{1/2}u_0'\beta'$, and

$$(\omega'\omega)^{1/2}=(\beta u_0\omega\omega' u_0'\beta')^{1/2}=\beta u_0(\omega\omega')^{1/2}u_0'\beta'=\beta u_0\omega.$$

By changing variable $\gamma \to \gamma \begin{bmatrix} \beta' & 0 \\ 0 & I_{n-m} \end{bmatrix}$, we obtain

$$\begin{aligned} \mathcal{G}_{k/2}(F) &= \pi^{-km/2} \int\limits_{\mathfrak{M}_{k,m}} d\omega \int\limits_{O(n)} f\left(\gamma \begin{bmatrix} \beta u_0 \omega \\ 0 \end{bmatrix}\right) d\gamma \\ &= \pi^{-km/2} \int\limits_{\mathfrak{M}_{k,m}} d\omega \int\limits_{O(n)} f\left(\gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix}\right) d\gamma, \end{aligned}$$

and (3.29) follows. For $\alpha = 0$, owing to (2.27), we have

$$(\zeta_0, f) = 2^{-m} \sigma_{n,m} F(0) = \frac{\pi^{nm/2}}{\Gamma_m(n/2)} f(0).$$

$\square$

Remark 3.8. For $n > m$, the integration in (3.29) over $O(n)$ can be replaced by that over $SO(n)$.

The following formulas for $(\zeta_k, f)$ are consequences of (3.29). We denote

$$c_1 = \frac{\pi^{(nm-km-nk)/2}\,\Gamma_k(n/2)}{2^k\,\Gamma_m(n/2)}, \qquad c_2 = \frac{\pi^{(m-k)(n/2-k)}}{\Gamma_k(k/2)\,\Gamma_{m-k}((n-k)/2)}. \tag{3.31}$$

Corollary 3.9. *For all $k = 1, 2, \dots, n$,*

$$(\zeta_k, f) = c_1 \int\limits_{V_{n,k}} dv \int\limits_{\mathfrak{M}_{k,m}} f(v\omega) d\omega. \tag{3.32}$$

*Moreover, if $k = 1, 2, \dots, m-1$, then*

$$(\zeta_k, f) = c_1 \int\limits_{V_{m,k}} du \int\limits_{\mathfrak{M}_{n,k}} f(yu')|y|_k^{m-n} dy \tag{3.33}$$

$$= c_2 \int\limits_{\mathfrak{M}_{n,k}} \frac{dy}{|y|_k^{n-m}} \int\limits_{\mathfrak{M}_{k,m-k}} f([y; yz]) dz. \tag{3.34}$$

Proof. From (3.29) we have

$$\begin{aligned} (\zeta_k, f) &= \frac{\pi^{(n-k)m/2}}{\Gamma_m(n/2)} \int\limits_{\mathfrak{M}_{k,m}} d\omega \int\limits_{O(n)} f(\gamma \lambda_0 \omega) d\gamma \quad \left(\lambda_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in V_{n,k}\right) \\ &= \frac{\pi^{(n-k)m/2}}{\sigma_{n,k}\,\Gamma_m(n/2)} \int\limits_{\mathfrak{M}_{k,m}} d\omega \int\limits_{V_{n,k}} f(v\omega) dv \end{aligned}$$

which coincides with (3.32). To prove (3.33), we pass to polar coordinates in (3.32) by setting $\omega' = ur^{1/2},\ u \in V_{m,k},\ r \in \mathcal{P}_k$. This gives

$$\begin{aligned}(\zeta_k, f) &= 2^{-k} c_1 \int\limits_{V_{n,k}} dv \int\limits_{\mathcal{P}_k} |r|^{(m-k-1)/2} dr \int\limits_{V_{m,k}} f(vr^{1/2}u')du \\ &= c_1 \int\limits_{V_{m,k}} du \int\limits_{\mathfrak{M}_{n,k}} f(yu')|y|_k^{m-n} dy.\end{aligned}$$

To prove (3.34), we represent $\omega$ in (3.32) in the block form $u = [\eta; \zeta],\ \eta \in \mathfrak{M}_{k,k},\ \zeta \in \mathfrak{M}_{k,m-k}$, and change the variable $\zeta = \eta z$. This gives

$$(\zeta_k, f) = c_1 \int\limits_{\mathfrak{M}_{k,k}} |\eta|^{m-k} d\eta \int\limits_{\mathfrak{M}_{k,m-k}} dz \int\limits_{V_{n,k}} f(v[\eta; \eta z])dv.$$

Using Lemma 1.11 repeatedly, and changing variables, we obtain

$$\begin{aligned}(\zeta_k, f) &= 2^{-k} c_1\, \sigma_{k,k} \int\limits_{\mathcal{P}_k} |r|^{(m-k-1)/2} dr \int\limits_{\mathfrak{M}_{k,m-k}} dz \int\limits_{V_{n,k}} f(v[r^{1/2}, r^{1/2}z])dv \\ &= c_1\, \sigma_{k,k} \int\limits_{\mathfrak{M}_{n,k}} \frac{dy}{|y|_k^{n-m}} \int\limits_{\mathfrak{M}_{k,m-k}} f([y; yz])dz.\end{aligned}$$

By (3.31), (1.37) and (1.10), this coincides with (3.34). □

The representation (3.34) was obtained in [**Sh1**] and [**Kh**] in a different way.

REMARK 3.10. Another proof of Theorem 3.6, which does not utilize Faraut's Lemma 3.7, was given in [**Ru7**]. The proof given above mimics the classical argument (cf. [**GSh1**, Chapter 1, Section 3.9] in the rank-one case) and is very instructive.

One can also write $(\zeta_k, f)$ as

$$(\zeta_k, f) = \int\limits_{\mathfrak{M}_{n,m}} f(x) d\nu_k(x), \qquad f \in \mathcal{S}(\mathfrak{M}_{n,m}), \tag{3.35}$$

where $\nu_k$ is a positive locally finite measure $\nu_k$ defined by

$$(\nu_k, \varphi) \equiv c_1 \int\limits_{V_{n,k}} dv \int\limits_{\mathfrak{M}_{k,m}} \varphi(v\omega)d\omega, \quad \varphi \in C_c(\mathfrak{M}_{n,m}), \tag{3.36}$$

$C_c(\mathfrak{M}_{n,m})$ being the space of compactly supported continuous functions on $\mathfrak{M}_{n,m}$; cf. (3.32). In order to characterize the support of $\nu_k$, we recall that $\mathfrak{M}_{n,m}^{(k)}$ stands for the submanifold of matrices $x \in \mathfrak{M}_{n,m}$ having rank $k$, and denote

$$\overline{\mathfrak{M}}_{n,m}^{(k)} = \bigcup_{j=0}^{k} \mathfrak{M}_{n,m}^{(j)} \qquad \text{(the closure of } \mathfrak{M}_{n,m}^{(k)}). \tag{3.37}$$

LEMMA 3.11. *The following statements hold.*
(i) $\operatorname{supp} \nu_k = \overline{\mathfrak{M}}_{n,m}^{(k)}$.

(ii) *The manifold* $\mathfrak{M}^{(k)}_{n,m}$ *is an orbit of* $e_k = \left[\begin{array}{cc} I_k & 0 \\ 0 & 0 \end{array}\right]_{n\times m}$ *under the group* $G$ *of transformations*

$$x \to g_1 x g_2, \qquad g_1 \in GL(n,\mathbb{R}), \quad g_2 \in GL(m,\mathbb{R}),$$

*and*

$$\dim \mathfrak{M}^{(k)}_{n,m} = k(n+m-k). \tag{3.38}$$

(iii) *The manifold* $\overline{\mathfrak{M}}^{(k)}_{n,m}$ *is a collection of all matrices* $x \in \mathfrak{M}_{n,m}$ *of the form*

$$x = \gamma \left[\begin{array}{c} \omega \\ 0 \end{array}\right], \quad \gamma \in O(n), \quad \omega \in \mathfrak{M}_{k,m}, \tag{3.39}$$

*or*

$$x = v\omega, \qquad v \in V_{n,k}, \quad \omega \in \mathfrak{M}_{k,m}. \tag{3.40}$$

PROOF. (i) Let us consider (3.36). Since $\text{rank}(v\omega) \le k$, then $(\nu_k, \varphi) = 0$ for all functions $\varphi \in C_c(\mathfrak{M}_{n,m})$ supported away from $\overline{\mathfrak{M}}^{(k)}_{n,m}$, i.e. $\text{supp}\,\nu_k = \overline{\mathfrak{M}}^{(k)}_{n,m}$.

(ii) We have to show that each $x \in \mathfrak{M}^{(k)}_{n,m}$ is represented as $x = g_1 e_k g_2$ for some $g_1 \in GL(n,\mathbb{R})$ and $g_2 \in GL(m,\mathbb{R})$. We write $x = us$, where $u \in V_{n,m}$ and $s = (x'x)^{1/2}$ is a positive semi-definite $m \times m$ matrix of rank $k$ (see Appendix C, **11**). By taking into account that $s = g_2' \left[\begin{array}{cc} I_k & 0 \\ 0 & 0 \end{array}\right]_{m\times m} g_2$ for some $g_2 \in GL(m,\mathbb{R})$, and $u = \gamma \left[\begin{array}{c} I_m \\ 0 \end{array}\right]$ for some $\gamma \in O(n)$, we obtain $x = g_1 e_k g_2$ with

$$g_1 = \gamma \left[\begin{array}{cc} g_2' & 0 \\ 0 & I_{n-m} \end{array}\right] \in GL(n,\mathbb{R}).$$

Therefore, the manifold $\mathfrak{M}^{(k)}_{n,m}$ is a homogeneous space of the group $G$. This allows us to find the dimension of $\mathfrak{M}^{(k)}_{n,m}$ by the formula

$$\dim \mathfrak{M}^{(k)}_{n,m} = \dim G - \dim G_1,$$

where $G_1$ is the subgroup of $G$ leaving $e_k$ stable. In order to calculate the dimension of $G_1$, we write the condition $g_1 e_k g_2 = e_k$ in the form

$$\left[\begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array}\right] \left[\begin{array}{cc} I_k & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{cc} b_1 & b_2 \\ b_3 & b_4 \end{array}\right] = \left[\begin{array}{cc} I_k & 0 \\ 0 & 0 \end{array}\right],$$

where $a_1 \in GL(k,\mathbb{R})$ and $b_1 \in GL(k,\mathbb{R})$. This gives $a_1 b_1 = I_k$, $a_3 = 0$, $b_2 = 0$. Hence,

$$\dim G_1 = n^2 - k(n-k) + m(m-k),$$

and (3.38) follows.

(iii) It is clear that each matrix of the form (3.39) or (3.40) has rank $\le k$. Conversely, if $\text{rank}(x) \le k$ then, as above, $x = us = \gamma \left[\begin{array}{c} s \\ 0 \end{array}\right]$ where $\gamma \in O(n)$ and $s = (x'x)^{1/2}$ is a positive semi-definite $m \times m$ matrix of rank $\le k$. The latter can be written as

$$s = g\lambda g', \qquad g \in O(m), \qquad \lambda = \text{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0),$$

and therefore,

$$x = \gamma \begin{bmatrix} g\lambda \\ 0 \end{bmatrix} g' = \gamma \begin{bmatrix} g & 0 \\ 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} \lambda \\ 0 \end{bmatrix} g' = \gamma_1 \begin{bmatrix} \omega \\ 0 \end{bmatrix}$$

where $\gamma_1 = \gamma \begin{bmatrix} g & 0 \\ 0 & I_{n-m} \end{bmatrix}$ and $\omega \in \mathfrak{M}_{k,m}$. The representation (3.40) follows from (3.39). □

COROLLARY 3.12. *The integral (3.35) can be written as*

$$(3.41) \qquad (\zeta_k, f) = \int\limits_{\mathrm{rank}(x) \le k} f(x) d\nu_k(x) = \int\limits_{\mathrm{rank}(x)=k} f(x) d\nu_k(x).$$

PROOF. The first equality follows from Lemma 3.11 (i). The second one is clear from the observation that if $\mathrm{rank}(x) \le k-1$ then, by (3.40), $x = v\omega$, $v \in V_{n,k-1}$, $\omega \in \mathfrak{M}_{k-1,m}$. The set of all such pairs $(v, \omega)$ has measure 0 in $V_{n,k} \times \mathfrak{M}_{k,m}$. □

REMARK 3.13. We cannot obtain simple explicit representation of the conjugate normalized zeta distributions $(\zeta_k^-, f)$, $k = 0, 1, \ldots, m-1$. At the first glance, it would be natural to use the formula

$$(\zeta_\alpha^-, f) = c_\alpha \, (\zeta_{\alpha+1}^+, \mathcal{D}f), \qquad c_\alpha = (-1)^m \frac{\Gamma(\alpha+1-m)}{\Gamma(\alpha+1)},$$

see (3.28), in which $(\zeta_{\alpha+1}^+, \mathcal{D}f)$ can be evaluated for $\alpha = k$ by Theorem 3.6 or Corollary 3.9. Unfortunately, we cannot do this because $c_\alpha = \infty$ for such $\alpha$. On the other hand, Lemma 3.2 says that $(\zeta_k^-, f)$ is well defined by (3.13), namely,

$$(3.42) \qquad (\zeta_k^-, f) = 2^{-m} \, \pi^{-m(m-1)/4} \, (\omega_k, \Phi),$$

where

$$\begin{aligned} \omega_k(t_{1,1}, \ldots, t_{m,m}) &= \prod_{i=1}^m \frac{|t_{i,i}|^{\alpha-i} \, \mathrm{sgn}\,(t_{i,i})}{\Gamma((\alpha-i)/2+1)} \bigg|_{\alpha=k}, \\ \Phi(t_{1,1}, \ldots, t_{m,m}) &= \int\limits_{\mathbb{R}^{m(m-1)/2}} dt_* \int\limits_{O(m)} f(vt) \, \mathrm{sgn} \det(v) \, dv. \end{aligned}$$

In particular, for $k = 0$,

$$(3.43) \qquad \omega_0(t_{1,1}, \ldots, t_{m,m}) = \prod_{i=1}^m \frac{|t_{i,i}|^\lambda \, \mathrm{sgn}\,(t_{i,i})}{\Gamma(\lambda/2+1)} \bigg|_{\lambda=-i},$$

where the generalized functions

$$\frac{|s|^\lambda \, \mathrm{sgn}\,(s)}{\Gamma(\lambda/2+1)} \bigg|_{\lambda=-i}, \qquad i = 1, 2, \ldots, m,$$

are defined as follows. For $i$ odd:

$$\left( \frac{|s|^\lambda \, \mathrm{sgn}\,(s)}{\Gamma(\lambda/2+1)} \bigg|_{\lambda=-i}, \varphi \right) = \frac{1}{\Gamma(1-i/2)} \, (s^{-i}, \varphi)$$

$$= \int\limits_0^\infty s^{-i} \left\{ \varphi(s) - \varphi(-s) - 2 \left[ s\varphi'(0) + \frac{s^3}{3!} \, \varphi'''(0) + \ldots + \frac{s^{i-2}}{(i-2)!} \, \varphi^{(i-2)}(0) \right] \right\} ds.$$

For $i$ even:

$$\left( \frac{|s|^\lambda \operatorname{sgn}(s)}{\Gamma(\lambda/2+1)} \bigg|_{\lambda=-i}, \varphi \right) = \frac{(-1)^{i/2}\, \varphi^{(i-1)}(0)\,(i/2-1)!}{(i-1)!};$$

see [**GSh1**, Chapter 1, Section 3.5] . Note that the Fourier transform of the distribution $\zeta_0^-$ has the form

$$(\mathcal{F}\zeta_0^-)(y) = \frac{(-i)^m\, \pi^{m^2/2}}{\Gamma_m((m+1)/2)} \operatorname{sgn}\det(y). \tag{3.44}$$

This follows immediately from (3.24) and the Parseval formula (1.5).

In the following the expression $(\zeta_0^-, f)$ will be understood in the sense of regularization according to (3.42), (3.43).

### 3.3. Riesz potentials and the generalized Hilbert transform

The functional equation (3.22) for the zeta distribution can be written in the form

$$\frac{1}{\gamma_{n,m}(\alpha)} \mathcal{Z}(f, \alpha-n) = (2\pi)^{-nm} \mathcal{Z}(\mathcal{F}f, -\alpha)\ , \tag{3.45}$$

$$\gamma_{n,m}(\alpha) = \frac{2^{\alpha m}\, \pi^{nm/2}\, \Gamma_m(\alpha/2)}{\Gamma_m((n-\alpha)/2)}, \qquad \alpha \neq n-m+1,\, n-m+2, \dots. \tag{3.46}$$

We recall that $m \geq 2$. The excluded values $\alpha = n-m+1,\, n-m+2, \dots$ are poles of the gamma function $\Gamma_m((n-\alpha)/2)$. Note that for $n = m$, all $\alpha = 1, 2, \dots$ are excluded. The normalization in (3.45) gives rise to the *Riesz distribution* $h_\alpha$ defined by

$$\begin{aligned} (h_\alpha, f) &= 2^{-\alpha m}\, \pi^{-nm/2}\, \Gamma_m\left(\frac{n-\alpha}{2}\right) (\zeta_\alpha, f) \\ &= \text{a.c.}\ \frac{1}{\gamma_{n,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,m}} |x|_m^{\alpha-n} f(x) dx, \end{aligned} \tag{3.47}$$

where $f \in \mathcal{S}(\mathfrak{M}_{n,m})$ and "a.c." abbreviates analytic continuation in the $\alpha$-variable. For $Re\, \alpha > m-1$, the distribution $h_\alpha$ is regular and agrees with the usual function $h_\alpha(x) = |x|_m^{\alpha-n}/\gamma_{n,m}(\alpha)$. The *Riesz potential* of a function $f \in \mathcal{S}(\mathfrak{M}_{n,m})$ is defined by

$$(I^\alpha f)(x) = (h_\alpha, f_x), \qquad f_x(\cdot) = f(x-\cdot). \tag{3.48}$$

For $Re\, \alpha > m-1$, one can represent $I^\alpha f$ in the classical form by the absolutely convergent integral

$$(I^\alpha f)(x) = \frac{1}{\gamma_{n,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,m}} f(x-y) |y|_m^{\alpha-n} dy. \tag{3.49}$$

We also introduce *the generalized Hilbert transform*

$$(\mathcal{H}f)(x) = (\zeta_0^-, f_x) = (\zeta_0^- * f)(x). \tag{3.50}$$

By (3.44), this can be regarded as a pseudo-differential operator with the symbol

$$(\mathcal{F}\zeta_0^-)(y) = \frac{(-i)^m\, \pi^{m^2/2}}{\Gamma_m((m+1)/2)} \operatorname{sgn}\det(y). \tag{3.51}$$

Clearly, $\mathcal{H}$ extends as a linear bounded operator on $L^2(\mathfrak{M}_{n,m})$. For $m = 1$, it coincides (up to a constant multiple) with the usual Hilbert transform on the real line.

The following properties of Riesz potentials and Riesz distributions are inherited from those for the normalized zeta integrals. We denote

$$\gamma_1 = 2^{-km}\,\pi^{-km/2}\,\Gamma_m\left(\frac{n-k}{2}\right)/\Gamma_m\left(\frac{n}{2}\right), \tag{3.52}$$

$$\gamma_2 = 2^{-k(m+1)}\,\pi^{-k(m+n)/2}\,\Gamma_k\left(\frac{n-m}{2}\right). \tag{3.53}$$

THEOREM 3.14. *Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $n \geq m$. Suppose that $\alpha = k$ is a positive integer. If $k \neq n-m+1,\ n-m+2, \ldots,$ then*

$$(I^k f)(x) = \gamma_1 \int\limits_{\mathfrak{M}_{k,m}} d\omega \int\limits_{O(n)} f\left(x - \gamma\begin{bmatrix}\omega\\ 0\end{bmatrix}\right) d\gamma, \tag{3.54}$$

$$= \gamma_2 \int\limits_{V_{n,k}} dv \int\limits_{\mathfrak{M}_{k,m}} f(x - v\omega)d\omega. \tag{3.55}$$

*Furthermore,*

$$(I^0 f)(x) = f(x). \tag{3.56}$$

This statement is an immediate consequence of Theorem 3.6.

LEMMA 3.15. *Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $n \geq m$, $\alpha \in \mathbb{C}$, $\alpha \neq n-m+1,\ n-m+2, \ldots$ .*
(i) *The Fourier transform of the Riesz distribution $h_\alpha$ is evaluated by the formula $(\mathcal{F}h_\alpha)(y) = |y|_m^{-\alpha}$, the precise meaning of which is*

$$(h_\alpha, f) = (2\pi)^{-nm}(|y|_m^{-\alpha}, (\mathcal{F}f)(y)) = (2\pi)^{-nm}\mathcal{Z}(\mathcal{F}f, -\alpha). \tag{3.57}$$

(ii) *If $k = 0, 1, \ldots,$ and $\Delta$ is the Cayley-Laplace operator, then*

$$h_\alpha = (-1)^{mk}\Delta^k h_{\alpha+2k}, \quad \textit{i.e.} \quad (h_\alpha, f) = (-1)^{mk}(h_{\alpha+2k}, \Delta^k f). \tag{3.58}$$

(iii) *If $n = m$, $k = 1, 2, \ldots$ , and $\mathcal{D}$ is the Cayley operator, then*

$$h_\alpha = c\,\mathcal{D}^{2k-1}\zeta^-_{\alpha+2k-1}, \quad \textit{i.e.} \quad (h_\alpha, f) = c\,(-1)^m(\zeta^-_{\alpha+2k-1}, \mathcal{D}^{2k-1}f), \tag{3.59}$$

$$c = \frac{(-1)^{m(k+1)}\,\Gamma_m(1 + (m - \alpha - 2k)/2)}{2^{(\alpha+2k-1)m}\,\pi^{m^2/2}}.$$

PROOF. (i) follows immediately from the definition (3.47) and the functional equation (3.45). To prove (3.58), for sufficiently large $\alpha$, according to (1.49), we have

$$\begin{aligned}
\Delta^k h_{\alpha+2k}(x) &= \frac{1}{\gamma_{n,m}(\alpha+2k)}\Delta^k |x|_m^{\alpha+2k-n}\\
&= \frac{B_k(\alpha)}{\gamma_{n,m}(\alpha+2k)}|x|_m^{\alpha-n}\\
&= c h_\alpha(x),
\end{aligned}$$

where by (1.50) and (1.9),

$$c = \frac{B_k(\alpha)\,\gamma_{n,m}(\alpha)}{\gamma_{n,m}(\alpha+2k)} = \frac{B_k(\alpha)\,\Gamma_m(\alpha/2)\,\Gamma_m((n-\alpha)/2-k)}{4^{mk}\,\Gamma_m(\alpha/2+k)\,\Gamma_m((n-\alpha)/2)} = (-1)^{mk}.$$

For all admissible $\alpha \in \mathbb{C}$, (3.58) follows by analytic continuation. Let us prove (iii). Owing to (3.58), $h_\alpha = (-1)^{mk}\,\mathcal{D}^{2k-1}\,\mathcal{D}h_{\alpha+2k}$. Since, by (3.47 ) and (3.28),

$$h_{\alpha+2k} = \frac{\Gamma_m((m-\alpha-2k)/2)}{2^{(\alpha+2k)m}\,\pi^{m^2/2}}\,\zeta^+_{\alpha+2k}, \quad \text{and} \quad \mathcal{D}\zeta^+_{\alpha+2k} = \frac{\Gamma(\alpha+2k)}{\Gamma(\alpha+2k-m)}\,\zeta^-_{\alpha+2k-1}\,,$$

then (3.59) follows after simple calculation using (1.9). □

Lemma 3.15 implies the following.

THEOREM 3.16. *Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $n \ge m$.*
(i) *If $k = 0, 1, 2, \dots,$ then*

$$(I^{-2k}f)(x) = (-1)^{mk}(\Delta^k f)(x). \tag{3.60}$$

(ii) *If $k = 1, 2, \dots,$ and $n > m$, then*

$$(I^{1-2k}f)(x) = (-1)^{mk}(I^1\Delta^k f)(x) = c_1 \int\limits_{S^{n-1}} dv \int\limits_{\mathbb{R}^m} (\Delta^k f)(x - vy')\,dy, \tag{3.61}$$

$$c_1 = \frac{(-1)^{mk}\,\Gamma((n-m)/2)}{2^{m+1}\,\pi^{(m+n)/2}}.$$

(iii) *If $k = 1, 2, \dots,$ and $n = m$, then*

$$(I^{1-2k}f)(x) = c_2\,(\mathcal{H}\mathcal{D}^{2k-1}f)(x), \qquad c_2 = \frac{(-1)^{m(k+1)}\,\Gamma_m((m+1)/2)}{\pi^{m^2/2}}, \tag{3.62}$$

*$\mathcal{H}$ being the generalized Hilbert transform (3.50).*

PROOF. The equality (3.60) is a consequence of (3.48), (3.57), and (3.58). Namely,

$$\begin{aligned}
(I^{-2k}f)(x) \quad &= \quad (h_{-2k}, f_x) \overset{(3.58)}{=} (-1)^{mk}(h_0,\ \Delta^k f_x) \\
&\overset{(3.57)}{=} \quad (-1)^{mk}(2\pi)^{-nm}\mathcal{Z}(\mathcal{F}(\Delta^k f_x),\ 0) \\
&= \quad (-1)^{mk}(\Delta^k f_x)(0) = (-1)^{mk}(\Delta^k f)(x).
\end{aligned}$$

Similarly, by (3.48), $(I^{1-2k}f)(x) = (h_{1-2k}, f_x)$. If $n > m$, then, by (3.58),

$$(h_{1-2k}, f_x) = (-1)^{mk}(h_1,\ \Delta^k f_x) = (-1)^{mk}(h_1,\ (\Delta^k f)_x) = (-1)^{mk}(I^1\Delta^k f)(x),$$

and it remains to apply (3.55) (with $k = 1$). If $n = m$, then we apply (3.59 ) with $\alpha = 1 - 2k$ and get

$$(h_{1-2k}, f_x) = (-1)^m c_2\,(\zeta_0^-, \mathcal{D}^{2k-1}f_x) = c_2\,(\zeta_0^-, (\mathcal{D}^{2k-1}f)_x) = c_2\,(\mathcal{H}\mathcal{D}^{2k-1}f)(x).$$

□

Integral representations (3.49) and (3.54) (or (3.55)) can serve as definitions of $I^\alpha f$ for functions $f$ belonging to Lebesgue spaces and $\alpha$ belonging to the associated Wallach-like set

$$\mathbf{W}_{n,m} = \{0, 1, 2, \dots, k_0\} \cup \{\alpha : Re\,\alpha > m-1;\ \alpha \ne n-m+1,\ n-m+2, \dots\}, \tag{3.63}$$

$$k_0 = \min(m-1, n-m).$$

THEOREM 3.17. [**Ru7**]. *Let $f \in L^p(\mathfrak{M}_{n,m})$, $n \geq m$. If $\alpha \in \mathbf{W}_{n,m}$, then the Riesz potential $(I^\alpha f)(x)$ absolutely converges for almost all $x \in \mathfrak{M}_{n,m}$ provided*

$$1 \leq p < \frac{n}{Re\, \alpha + m - 1}. \tag{3.64}$$

For $m = 1$, the condition (3.64) is well known [**St1**] and best possible. We do not know whether (3.64) can be improved if $m > 1$.

**3.3.1. Inversion of Riesz potentials.** Let us discuss the following problem. Given a Riesz potential $g = I^\alpha f$, how do we recover its density $f$? In the rank-one case, a variety of pointwise inversion formulas for Riesz potentials is available in a large scale of function spaces [**Ru1**], [**Sa**]. However, in the higher rank case we encounter essential difficulties. Below we show how the unknown function $f$ can be recovered in the framework of the theory of distributions.

First we specify the space of test functions. The Fourier transform formula

$$(h_\alpha, f) = (2\pi)^{-nm}(|y|_m^{-\alpha}, (\mathcal{F}f)(y))$$

reveals that the Schwartz class $\mathcal{S} \equiv \mathcal{S}(\mathfrak{M}_{n,m})$ is not good enough because it is not invariant under multiplication by $|y|_m^{-\alpha}$. To get around this difficulty, we follow the idea of V.I. Semyanistyi [**Se**] suggested for $m = 1$. Let $\Psi \equiv \Psi(\mathfrak{M}_{n,m})$ be the subspace of functions $\psi(y) \in \mathcal{S}$ vanishing on the set of all matrices of rank $< m$, i.e.,

$$\{y : y \in \mathfrak{M}_{n,m},\ \mathrm{rank}(y) < m\} = \{y : y \in \mathfrak{M}_{n,m},\ |y'y| = 0\}, \tag{3.65}$$

with all derivatives. The coincidence of both sets in (3.65) is clear because $\mathrm{rank}(y) = \mathrm{rank}(y'y)$, see, e.g., [**FZ**, p. 5]. The set $\Psi$ is a closed linear subspace of $\mathcal{S}$. Therefore, it can be regarded as a linear topological space with the induced topology of $\mathcal{S}$. Let $\Phi \equiv \Phi(\mathfrak{M}_{n,m})$ be the Fourier image of $\Psi$. Since the Fourier transform $\mathcal{F}$ is an automorphism of $\mathcal{S}$ (i.e., a topological isomorphism of $\mathcal{S}$ onto itself), then $\Phi$ is a closed linear subspace of $\mathcal{S}$. Having been equipped with the induced topology of $\mathcal{S}$, the space $\Phi$ becomes a linear topological space isomorphic to $\Psi$ under the Fourier transform. We denote by $\Phi' \equiv \Phi'(\mathfrak{M}_{n,m})$ the space of all linear continuous functionals (generalized functions) on $\Phi$. Since for any complex $\alpha$, multiplication by $|y|_m^{-\alpha}$ is an automorphism of $\Psi$, then, according to the general theory [**GSh2**], $I^\alpha$, as a convolution with $h_\alpha$, is an automorphism of $\Phi$, and we have

$$\mathcal{F}[I^\alpha f](y) = |y|_m^{-\alpha}(\mathcal{F}f)(y)$$

for all $\Phi'$-distributions $f$. In the rank-one case, the spaces $\Phi$, $\Psi$, their duals and generalizations were studied by P.I. Lizorkin, S.G. Samko and others in view of applications to the theory of function spaces and fractional calculus; see [**Sa**], [**SKM**], [**Ru1**] and references therein.

THEOREM 3.18. *Let $\alpha \in \mathbf{W}_{n,m}$ and let $f$ be a locally integrable function such that $g(x) = (I^\alpha f)(x)$ is well defined as an absolutely convergent integral for almost all $x \in \mathfrak{M}_{n,m}$. Then $f$ can be recovered from $g$ in the sense of $\Phi'$-distributions by the formula*

$$(f, \phi) = (g, I^{-\alpha}\phi), \qquad \phi \in \Phi, \tag{3.66}$$

*where*

$$(I^{-\alpha}\phi)(x) = (\mathcal{F}^{-1}|y|_m^{\alpha}\mathcal{F}\phi)(x). \tag{3.67}$$

*In particular, if* $\alpha = 2k$ *is even, then*

$$(f, \phi) = (-1)^{mk}(g, \Delta^k \phi), \qquad \phi \in \Phi, \tag{3.68}$$

$\Delta$ *being the Cayley-Laplace operator (1.44).*

CHAPTER 4

# Radon transforms

## 4.1. Matrix planes

DEFINITION 4.1. Let $k, n$, and $m$ be positive integers, $0 < k < n$, $V_{n,n-k}$ be the Stiefel manifold of orthonormal $(n-k)$-frames in $\mathbb{R}^n$, $\mathfrak{M}_{n-k,m}$ be the space of $(n-k) \times m$ real matrices. For $\xi \in V_{n,n-k}$ and $t \in \mathfrak{M}_{n-k,m}$, the set

$$\tau \equiv \tau(\xi, t) = \{x : x \in \mathfrak{M}_{n,m};\ \xi' x = t\} \tag{4.1}$$

will be called a *matrix k-plane* in $\mathfrak{M}_{n,m}$. For $k = n - m$, the plane $\tau$ will be called a *matrix hyperplane*. We denote by $\mathfrak{T}$ the manifold of all matrix $k$- planes.

The parameterization $\tau = \tau(\xi, t)$ by the points $(\xi, t)$ of a "matrix cylinder" $V_{n,n-k} \times \mathfrak{M}_{n-k,m}$ is not one-to-one because for any orthogonal transformation $\theta \in O(n-k)$, the pairs $(\xi, t)$ and $(\xi\theta', \theta t)$ define the same plane $\tau$. We identify functions $\varphi(\tau)$ on $\mathfrak{T}$ with the corresponding functions $\varphi(\xi, t)$ on $V_{n,n-k} \times \mathfrak{M}_{n-k,m}$ satisfying $\varphi(\xi\theta', \theta t) = \varphi(\xi, t)$ for all $\theta \in O(n-k)$, and supply $\mathfrak{T}$ with the measure $d\tau$ so that

$$\int\limits_{\mathfrak{T}} \varphi(\tau)\, d\tau = \int\limits_{V_{n,n-k} \times \mathfrak{M}_{n-k,m}} \varphi(\xi, t)\, d\xi dt. \tag{4.2}$$

The plane $\tau$ is, in fact, a usual $km$-dimensional plane in $\mathbb{R}^{nm}$. To see this, we write $x = (x_{i,j}) \in \mathfrak{M}_{n,m}$ and $t = (t_{i,j}) \in \mathfrak{M}_{n-k,m}$ as column vectors

$$\bar{x} = \begin{pmatrix} x_{1,1} \\ x_{1,2} \\ \dots \\ x_{n,m} \end{pmatrix} \in \mathbb{R}^{nm}, \qquad \bar{t} = \begin{pmatrix} t_{1,1} \\ t_{1,2} \\ \dots \\ t_{n-k,m} \end{pmatrix} \in \mathbb{R}^{(n-k)m}, \tag{4.3}$$

and denote

$$\bar{\xi} = \mathrm{diag}(\xi, \dots, \xi) \in V_{nm,(n-k)m}. \tag{4.4}$$

Then (4.1) reads

$$\tau = \tau(\bar{\xi}, \bar{t}) = \{\bar{x} : \bar{x} \in \mathbb{R}^{nm};\ \bar{\xi}' \bar{x} = \bar{t}\}. \tag{4.5}$$

The $km$-dimensional planes (4.5) form a subset of measure zero in the affine Grassmann manifold of all $km$-dimensional planes in $\mathbb{R}^{nm}$.

The manifold $\mathfrak{T}$ can be regarded as a fibre bundle, the base of which is the ordinary Grassmann manifold $G_{n,k}$ of $k$-dimensional linear subspaces of $\mathbb{R}^n$, and whose fibres are homeomorphic to $\mathfrak{M}_{n-k,m}$. Indeed, let

$$\pi : \mathfrak{T} \to G_{n,k},$$

be the canonical projection which assigns to each matrix plane $\tau(\xi, t)$ the subspace

$$\eta = \eta(\xi) = \{y : y \in \mathbb{R}^n;\ \xi' y = 0\} \in G_{n,k}\ . \tag{4.6}$$

The fiber $H_\eta = \pi^{-1}(\eta)$ is the set of all matrix planes (4.1), where $t$ sweeps the space $\mathfrak{M}_{n-k,m}$. Regarding $\mathfrak{T}$ as a fibre bundle, one can utilize a parameterization which is alternative to (4.1) and one-to-one. Namely, let

$$x = [x_1 \dots x_m], \qquad t = [t_1 \dots t_m], \tag{4.7}$$

where $x_i \in \mathbb{R}^n$ and $t_i \in \mathbb{R}^{n-k}$ are column-vectors. For $\tau = \tau(\xi, t) \in \mathfrak{T}$, we have

$$\tau = \{x : x \in \mathfrak{M}_{n,m};\ \xi' x_i = t_i, \quad i = 1, \dots, m\}.$$

Each ordinary $k$-plane $\tau_i = \{x_i : x_i \in \mathbb{R}^n;\ \xi' x_i = t_i\}$ can be parameterized by the pair $(\eta, \lambda_i)$, where $\eta$ is the subspace (4.6), $\lambda_i \in \eta^\perp, \quad i = 1, \dots, m$, are columns of the matrix $\lambda = \xi t \in \mathfrak{M}_{n,m}$, and $\eta^\perp$ denotes the orthogonal complement of $\eta$ in $\mathbb{R}^n$. The corresponding parameterization

$$\tau = \tau(\eta, \lambda), \qquad \eta \in G_{n,k}, \quad \lambda = [\lambda_1 \dots \lambda_m], \quad \lambda_i \in \eta^\perp, \tag{4.8}$$

is one-to-one. Both parameterizations (4.1) and (4.8) will be useful in the following.

### 4.2. Definition and elementary properties of the Radon transform

The Radon transform $\hat{f}$ of a function $f(x)$ on $\mathfrak{M}_{n,m}$ assigns to $f$ a collection of integrals of $f$ over all matrix $k$-planes (4.1). Namely,

$$\hat{f}(\tau) = \int_{x \in \tau} f(x), \qquad \tau \in \mathfrak{T}.$$

In order to give this integral precise meaning, we note that the matrix plane $\tau = \tau(\xi, t)$, $\xi \in V_{n,n-k}$, $t \in \mathfrak{M}_{n-k,m}$, consists of "points"

$$x = g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix},$$

where $\omega \in \mathfrak{M}_{k,m}$, and $g_\xi \in SO(n)$ is a rotation satisfying

$$g_\xi \xi_0 = \xi, \qquad \xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k}. \tag{4.9}$$

This observation leads to the following

Definition 4.2. The Radon transform of a function $f(x)$ on $\mathfrak{M}_{n,m}$ is defined as a function on the "matrix cylinder" $V_{n,n-k} \times \mathfrak{M}_{n-k,m}$ by the formula

$$\hat{f}(\tau) \equiv \hat{f}(\xi, t) = \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix}\right) d\omega. \tag{4.10}$$

It is worth noting that the expression (4.10) is independent of the choice of the rotation $g_\xi : \xi_0 \to \xi$. Indeed, if $g_1$ and $g_2$ are two such rotations, then $g_1 = g_2 g$ where $g$ belongs to the isotropy subgroup of $\xi_0$. Hence $g$ has the form

$$g = \begin{bmatrix} \theta & 0 \\ 0 & I_{n-k} \end{bmatrix}, \qquad \theta \in SO(k).$$

Multiplying matrices and changing variable $\theta\omega \to \omega$, we get

$$\int\limits_{\mathfrak{M}_{k,m}} f\left(g_1 \begin{bmatrix} \omega \\ t \end{bmatrix}\right) d\omega = \int\limits_{\mathfrak{M}_{k,m}} f\left(g_2 \begin{bmatrix} \theta & 0 \\ 0 & I_{n-k} \end{bmatrix} \begin{bmatrix} \omega \\ t \end{bmatrix}\right) d\omega = \int\limits_{\mathfrak{M}_{k,m}} f\left(g_2 \begin{bmatrix} \omega \\ t \end{bmatrix}\right) d\omega.$$

Furthermore, since

$$g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix} = g_\xi \begin{bmatrix} \omega \\ 0 \end{bmatrix} + g_\xi \begin{bmatrix} 0 \\ t \end{bmatrix} = g_\xi \begin{bmatrix} \omega \\ 0 \end{bmatrix} + \xi t, \tag{4.11}$$

one can write

$$\hat{f}(\xi, t) = \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ 0 \end{bmatrix} + \xi t\right) d\omega \tag{4.12}$$

$$= \int\limits_{\{y \in \mathfrak{M}_{n,m} \, : \, \xi' y = 0\}} f(y + \xi t)\, dy. \tag{4.13}$$

If $m = 1$, then $\hat{f}(\xi, t)$ is the ordinary $k$-plane Radon transform that assigns to a function $f(x)$ on $\mathbb{R}^n$ the collection of integrals of $f$ over all $k$-dimensional planes. If $m = 1$ and $k = n - 1$, the definition (4.12) gives the classical hyperplane Radon transform [**H**], [**GGV**].

In terms of the one-to-one parameterization (4.8), where $\tau = \tau(\eta, \lambda)$, $\eta \in G_{n,k}$, $\lambda = [\lambda_1 \dots \lambda_m] \in \mathfrak{M}_{n,m}$, and $\lambda_i \in \eta^\perp$, the Radon transform (4.10) can be written as

$$\hat{f}(\tau) = \int\limits_{\eta} dy_1 \dots \int\limits_{\eta} f([y_1 + \lambda_1 \dots y_m + \lambda_m])\, dy_m. \tag{4.14}$$

For $m = 1$, this is the well known form of the $k$-plane transform in $\mathbb{R}^n$; cf. [**H**, p. 30, formula (56)].

The following sufficient conditions of the existence of the Radon transform $\hat{f}$ immediately follow from Definition 4.2. More subtle results will be presented in Section 4.7.2.

Lemma 4.3.
(i) *If $f \in L^1(\mathfrak{M}_{n,m})$, then the Radon transform $\hat{f}(\xi, t)$ exists for all $\xi \in V_{n,n-k}$ and almost all $t \in \mathfrak{M}_{n-k,m}$. Furthermore,*

$$\int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, t)\, dt = \int\limits_{\mathfrak{M}_{n,m}} f(x)\, dx, \qquad \forall\, \xi \in V_{n,n-k}. \tag{4.15}$$

(ii) *Let $||x|| = (\mathrm{tr}(x'x))^{1/2} = (x_{1,1}^2 + \ldots + x_{n,m}^2)^{1/2}$. If $f$ is a continuous function satisfying*

$$f(x) = O(||x||^{-a}), \qquad a > km, \tag{4.16}$$

*then $\hat{f}(\xi, t)$ exists for all $\xi \in V_{n,n-k}$ and all $t \in \mathfrak{M}_{n-k,m}$.*

PROOF. (i) is a consequence of the Fubini theorem:

$$\begin{aligned}\int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi,t)\,dt &= \int\limits_{\mathfrak{M}_{n-k,m}} dt \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix}\right) d\omega \\ &= \int\limits_{\mathfrak{M}_{n,m}} f(g_\xi x)\,dx = \int\limits_{\mathfrak{M}_{n,m}} f(x)\,dx.\end{aligned}$$

(ii) becomes obvious from (4.5) if we regard $\tau = \tau(\xi,t)$ as a $km$-dimensional plane in $\mathbb{R}^{nm}$. □

LEMMA 4.4. *Suppose that the Radon transform*

$$f(x) \longrightarrow \hat{f}(\xi,t), \qquad x \in \mathfrak{M}_{n,m}, \quad (\xi,t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m},$$

*exists (at least almost everywhere). Then*
(i) $\hat{f}(\xi,t)$ *is a "matrix-even" function, i.e.,*

$$\hat{f}(\xi\theta',\theta t) = \hat{f}(\xi,t), \qquad \forall\theta \in O(n-k). \tag{4.17}$$

(ii) *The Radon transform commutes with the group $M(n,m)$ of matrix motions. Specifically, if $g(x) = \gamma x\beta + b$ where $\gamma \in O(n)$, $\beta \in O(m)$, and $b \in \mathfrak{M}_{n,m}$, then*

$$(f \circ g)^\wedge(\xi,t) = \hat{f}(\gamma\xi, t\beta + \xi'\gamma' b). \tag{4.18}$$

PROOF. (i) Formula (4.17) is a matrix analog of the "evenness property" of the classical Radon transform (the case $m=1$, $k=n-1$): $\hat{f}(-\xi,-t) = \hat{f}(\xi,t)$ [**H**, p. 3]. By (4.10),

$$\hat{f}(\xi\theta',\theta t) = \int\limits_{\mathfrak{M}_{k,m}} f\left(g_{\xi\theta'} \begin{bmatrix} \omega \\ \theta t \end{bmatrix}\right) d\omega,$$

where one can choose $g_{\xi\theta'} = g_\xi \begin{bmatrix} I_k & 0 \\ 0 & \theta' \end{bmatrix}$. Hence

$$g_{\xi\theta'} \begin{bmatrix} \omega \\ \theta t \end{bmatrix} = g_\xi \begin{bmatrix} I_k & 0 \\ 0 & \theta' \end{bmatrix} \begin{bmatrix} \omega \\ \theta t \end{bmatrix} = g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix}$$

which gives (4.17).

(ii) By (4.10),

$$\begin{aligned}(f \circ g)^\wedge(\xi,t) &= \int\limits_{\mathfrak{M}_{k,m}} f\left(\gamma\left(g_\xi \begin{bmatrix} \omega\beta \\ t\beta \end{bmatrix}\right) + b\right) d\omega \qquad (\omega\beta \to \omega) \\ &= \int\limits_{\mathfrak{M}_{k,m}} f\left(g_{\gamma\xi}\left(\begin{bmatrix} \omega \\ t\beta \end{bmatrix} + g_{\gamma\xi}^{-1} b\right)\right) d\omega.\end{aligned}$$

We set $g_{\gamma\xi}^{-1} b = \begin{bmatrix} p \\ q \end{bmatrix}$, $p \in \mathfrak{M}_{k,m}$, $q \in \mathfrak{M}_{n-k,m}$, and change the variable $\omega + p \to \omega$. Then the last integral reads

$$\int\limits_{\mathfrak{M}_{k,m}} f\left(g_{\gamma\xi} \begin{bmatrix} \omega \\ t\beta + q \end{bmatrix}\right) d\omega.$$

By (4.9), $q = \xi_0' g_{\gamma\xi}^{-1} b = \xi_0' g_\xi' \gamma' b = (g_\xi \xi_0)' \gamma' b = \xi' \gamma' b$, and we are done.

□

COROLLARY 4.5. *For $x, y \in \mathfrak{M}_{n,m}$, let $f_x(y) = f(x+y)$. Then*

$$\hat{f}_x(\xi, t) = \hat{f}(\xi, \xi' x + t). \tag{4.19}$$

COROLLARY 4.6. *If the function $f : \mathfrak{M}_{n,m} \to \mathbb{C}$ is $O(n)$ left-invariant, i.e., $f(\gamma x) = f(x)$ for all $\gamma \in O(n)$, then*

$$\hat{f}(\gamma\xi\theta', \theta t) = \hat{f}(\xi, t), \qquad \forall\, \gamma \in O(n),\ \theta \in O(n-k). \tag{4.20}$$

## 4.3. Interrelation between the Radon transform and the Fourier transform

We recall that the Fourier transform of a function $f \in L^1(\mathfrak{M}_{n,m})$ is defined by

$$(\mathcal{F}f)(y) = \int\limits_{\mathfrak{M}_{n,m}} \exp(i\,\mathrm{tr}(y'x))\, f(x)\, dx, \qquad y \in \mathfrak{M}_{n,m}. \tag{4.21}$$

The following statement is a matrix generalization of the so-called *projection-slice theorem*. It links together the Fourier transform (4.21) and the Radon transform (4.10).

For $y = [y_1 \dots y_m] \in \mathfrak{M}_{n,m}$, let $\mathcal{L}(y) = \mathrm{lin}(y_1, \dots, y_m)$ be the linear hull of the vectors $y_1, \dots, y_m$, i.e., the smallest linear subspace containing $y_1, \dots, y_m$. Suppose that $\mathrm{rank}(y) = \ell$. Then $\dim \mathcal{L}(y) = \ell \le m$.

THEOREM 4.7. *Let $f \in L^1(\mathfrak{M}_{n,m})$, $1 \le k \le n-m$. If $y \in \mathfrak{M}_{n,m}$, and $\zeta$ is a $(n-k)$-dimensional plane containing $\mathcal{L}(y)$, then for any orthonormal frame $\xi \in V_{n,n-k}$ spanning $\zeta$, there exists $b \in \mathfrak{M}_{n-k,m}$ such that $y = \xi b$. In this case,*

$$(\mathcal{F}f)(y) = \int\limits_{\mathfrak{M}_{n-k,m}} \exp(i\,\mathrm{tr}(b't))\, \hat{f}(\xi, t)\, dt, \tag{4.22}$$

*or*

$$(\mathcal{F}f)(\xi b) = \mathcal{F}[\hat{f}(\xi, \cdot)](b), \quad \xi \in V_{n,n-k}, \quad b \in \mathfrak{M}_{n-k,m}. \tag{4.23}$$

PROOF. Since each vector $y_j$ $(j = 1, \dots, m)$ lies in $\zeta$, it decomposes as $y_j = \xi b_j$ for some $b_j \in \mathbb{R}^{n-k}$. Hence $y = \xi b$ where $b = [b_1 \dots b_m] \in \mathfrak{M}_{n-k,m}$. Thus it remains to prove (4.23). By (4.10),

$$\mathcal{F}[\hat{f}(\xi, \cdot)](b) = \int\limits_{\mathfrak{M}_{n-k,m}} \exp(i\,\mathrm{tr}(b't))\, dt \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix}\right) d\omega.$$

If $x = g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix}$, then, by (4.9),

$$\xi' x = \xi_0' g_\xi' g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix} = \xi_0' \begin{bmatrix} \omega \\ t \end{bmatrix} = t,$$

and the Fubini theorem yields

$$\mathcal{F}[\hat{f}(\xi, \cdot)](b) = \int\limits_{\mathfrak{M}_{n,m}} \exp(i\,\mathrm{tr}(b'\xi' x))\, f(x)\, dx = (\mathcal{F}f)(\xi b).$$

□

REMARK 4.8. It is clear that $\xi$ and $b$ in the basic equality (4.22) are not uniquely defined. If $\mathrm{rank}(y) = m$ one can choose some $\xi$ and $b$ as follows. By taking into account that $n-k \geq m$, we set

$$\xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k}, \quad \omega_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in V_{n-k,m}, \quad v_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in V_{n,m},$$

so that $\xi_0 \omega_0 = v_0$. Consider the polar decomposition

$$y = vr^{1/2}, \qquad v \in V_{n,n-k}, \quad r = y'y \in \mathcal{P}_m,$$

and let $g_v$ be a rotation with the property $g_v v_0 = v$. Then

$$y = vr^{1/2} = g_v v_0 r^{1/2} = g_v \xi_0 \omega_0 r^{1/2} = \xi b,$$

where

$$\xi = g_v \xi_0 \in V_{n,n-k}, \qquad b = \omega_0 r^{1/2} \in \mathfrak{M}_{n-k,m}. \tag{4.24}$$

THEOREM 4.9.
(i) *If $1 \leq k \leq n-m$, then the Radon transform $f \to \hat{f}$ is injective on the Schwartz space $\mathcal{S}(\mathfrak{M}_{n,m})$, and $f$ can be recovered by the formula*

$$\begin{aligned} f(x) \;&=\; \frac{2^{-m}}{(2\pi)^{nm}} \int\limits_{\mathcal{P}_m} |r|^{\frac{n-m-1}{2}} dr \\ &\times \int\limits_{V_{n,m}} \exp(-i\,\mathrm{tr}(x'vr^{1/2}))(F\hat{f}(g_v\xi_0, \cdot))(\xi_0' v_0 r^{1/2}) dv. \end{aligned} \tag{4.25}$$

(ii) *For $k > n-m$, the Radon transform is non-injective on $\mathcal{S}(\mathfrak{M}_{n,m})$.*

PROOF. By Theorem 4.7, given the Radon transform $\hat{f}$ of $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, the Fourier transform $(\mathcal{F}f)(y)$ can be recovered at each point $y \in \mathfrak{M}_{n,m}$ by the formula (4.22), so that if $\hat{f} \equiv 0$ then $\mathcal{F}f \equiv 0$. Since $\mathcal{F}$ is injective, then $f \equiv 0$, and we are done. Remark 4.8 allows us to reconstruct $f$ from $\hat{f}$, because (4.24) expresses $\xi$ and $b$ through $y \in \mathfrak{M}_{n,m}$ explicitly. This gives (4.25).

To prove (ii), let $\psi \not\equiv 0$ be a Schwartz function, the Fourier transform of which is supported in the set $\mathfrak{M}^{(m)}_{n,m}$ of matrices $x \in \mathfrak{M}_{n,m}$ of rank $m$. This is an open set in $\mathfrak{M}_{n,m}$. By (4.23),

$$\mathcal{F}[\hat{\psi}(\xi, \cdot)](b) = \hat{\psi}(\xi b) = 0 \quad \forall\, \xi \in V_{n,n-k}, \quad \forall\, b \in \mathfrak{M}_{n-k,m}, \tag{4.26}$$

because $\xi b \notin \mathfrak{M}^{(m)}_{n,m}$ (since $n-k < m$, then $\mathrm{rank}(\xi b) < m$). By injectivity of the Fourier transform in (4.26), we obtain $\hat{\psi}(\xi, t) = 0$ $\forall \xi, t$. Thus for $k > n-m$, the injectivity of the Radon transform fails. □

## 4.4. The dual Radon transform

DEFINITION 4.10. Let $\tau = \tau(\xi, t)$ be a matrix plane (4.1), $(\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$. We say that the plane $\tau \equiv \tau(\xi, t)$ contains a "point" $x \in \mathfrak{M}_{n,m}$ if $\xi' x = t$. The dual Radon transform $\check{\varphi}(x)$ assigns to a function $\varphi(\tau)$ on $\mathfrak{T}$ the integral of $\varphi$ over all matrix $k$-planes containing $x$. Namely,

$$\check{\varphi}(x) = \int\limits_{\tau \ni x} \varphi(\tau), \qquad x \in \mathfrak{M}_{n,m}.$$

The precise meaning of this integral is

$$\begin{aligned}
(4.27)\qquad \check{\varphi}(x) &= \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \varphi(\xi, \xi' x)\, d\xi \\
&= \int\limits_{SO(n)} \varphi(\gamma\xi_0, \xi_0'\gamma' x)\, d\gamma, \qquad \xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k}.
\end{aligned}$$

Clearly, $\check{\varphi}(x)$ exists for all $x \in \mathfrak{M}_{n,m}$ if $\varphi$ is a continuous function. Later we shall prove that $\check{\varphi}(x)$ is finite a.e. on $\mathfrak{M}_{n,m}$ for any locally integrable function $\varphi$.

REMARK 4.11. The definition (4.27) is independent of the parameterization $\tau = \tau(\xi, t)$ in the sense that for any other parameterization $\tau = \tau(\xi\theta', \theta t)$, $\theta \in O(n-k)$ (see Section 4.1), the equality (4.27) gives the same result:

$$\frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \varphi(\xi\theta', \theta\xi' x) d\xi = \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \varphi(\xi_1, \xi_1' x) d\xi_1 = \check{\varphi}(x).$$

LEMMA 4.12. *The duality relation*

$$\begin{aligned}
(4.28)\qquad \int\limits_{\mathfrak{M}_{n,m}} f(x)\check{\varphi}(x) dx &= \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi, t)\hat{f}(\xi, t) dt \\
\Bigg(\text{or} \int\limits_{\mathfrak{M}_{n,m}} f(x)\check{\varphi}(x) dx &= \frac{1}{\sigma_{n,n-k}} \int\limits_{\mathfrak{T}} \varphi(\tau)\hat{f}(\tau)\, d\tau\Bigg)
\end{aligned}$$

*is valid provided that either side of this equality is finite for $f$ and $\varphi$ replaced by $|f|$ and $|\varphi|$, respectively.*

PROOF. By (4.10), the right-hand side of (4.28) equals

$$(4.29)\qquad \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi, t)\, dt \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix}\right) d\omega.$$

Setting $x = g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix}$, we have

$$\xi' x = (g_\xi \xi_0)' g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix} = \xi_0' \begin{bmatrix} \omega \\ t \end{bmatrix} = t.$$

Hence, by the Fubini theorem, (4.29) reads

$$\frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n,m}} \varphi(\xi, \xi' x) f(x)\, dx = \int\limits_{\mathfrak{M}_{n,m}} \check{\varphi}(x) f(x)\, dx.$$

□

LEMMA 4.13. *The dual Radon transform commutes with the group $M(n,m)$ of matrix motions. Specifically, if $gx = \gamma x \beta + b$ where $\gamma \in O(n)$, $\beta \in O(m)$, and $b \in \mathfrak{M}_{n,m}$, then*

$$(4.30)\qquad (\varphi \circ g)^\vee(x) = \check{\varphi}(gx).$$

*More precisely, if* $\tau = \tau(\xi, t)$*, then*

$$(\varphi \circ g)(\xi, t) = \varphi(\gamma\xi, t\beta + \xi'\gamma' b)$$

*and*

$$(\varphi \circ g)^{\vee}(x) = \check{\varphi}(\gamma x \beta + b). \tag{4.31}$$

Proof. By (4.27),

$$\begin{aligned}(\varphi \circ g)^{\vee}(x) &= \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \varphi(\gamma\xi, \xi' x\beta + \xi'\gamma' b) d\xi \qquad (\gamma\xi \to \xi) \\ &= \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \varphi(\xi, \xi'(\gamma x \beta + b)) d\xi = \check{\varphi}(\gamma x \beta + b).\end{aligned}$$

□

Corollary 4.14. *If* $\gamma \in O(n)$ *and* $\varphi(\gamma\xi, t) = \varphi(\xi, t)$*, then* $\check{\varphi}(\gamma x) = \check{\varphi}(x)$*.*

## 4.5. Radon transforms of radial functions

In this section, we show that the Radon transform and the dual Radon transform of radial functions are represented by Gårding-Gindikin fractional integrals studied in Chapter 2. This phenomenon is well known in the rank-one case when diverse Radon transforms of radial functions are represented by the usual Riemann-Liouville fractional integrals. In the higher rank case, an exceptional role of the Gårding-Gindikin fractional integrals in the theory of the Radon transform on Grassmann manifolds was demonstrated in [**GR**]. We recall (see Section 1.7) that a function $f(x)$ on $\mathfrak{M}_{n,m}$ is radial if it is $O(n)$ left-invariant. Each such function has the form $f(x) = f_0(x'x)$. In the similar way one can define radial functions of matrix planes.

Definition 4.15. For $0 < k < n$ and $m \geq 1$, let $\mathfrak{T}$ be the manifold of matrix planes $\tau = \tau(\xi, t)$, $(\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$; see (4.1). A function $\varphi(\tau) \equiv \varphi(\xi, t)$ is called radial if it is $O(n)$ left-invariant in the $\xi$ variable, i.e., $\varphi(\gamma\xi, t) = \varphi(\xi, t)$ for all $\gamma \in O(n)$ and all (or almost all) $\xi$ and $t$.

Note that if $\varphi$ is a radial function, then

$$\varphi(\gamma\xi\theta', \theta t) = \varphi(\xi, t), \qquad \forall \gamma \in O(n),\ \theta \in O(n-k). \tag{4.32}$$

This equality is a result of parameterization which is not one-to-one; see Section 4.1.

Lemma 4.16. *Every function* $\varphi$ *on* $\mathfrak{T}$ *of the form* $\varphi(\xi, t) = \varphi_0(t't)$ *is radial. Conversely, if* $1 \leq k \leq n - m$ *and* $\varphi(\xi, t)$ *is radial, then there is a function* $\varphi_0(s)$ *on* $\overline{\mathcal{P}}_m$ *such that* $\varphi(\xi, t) = \varphi_0(t't)$ *for all* $\xi \in V_{n,n-k}$ *and all matrices* $t \in \mathfrak{M}_{n-k,m}$ *(if* $\varphi$ *is a "rough" function then "all" should be replaced by "almost all").*

Proof. The first statement is clear. By the polar decomposition (see Appendix C, **11**), for all matrices $t \in \mathfrak{M}_{n-k,m}$ we have

$$t = us^{1/2}, \quad u \in V_{n-k,m}, \quad s = t't \in \overline{\mathcal{P}}_m.$$

Let

$$\xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k}, \qquad u_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in V_{n-k,m}.$$

We choose $\gamma \in O(n)$ and $\theta \in O(n-k)$ so that $u = \theta' u_0$, $\gamma\xi\theta' = \xi_0$. Then

$$\begin{array}{rcl} \varphi(\xi, t) & = & \varphi(\xi, \theta' u_0 s^{1/2}) \\ & \overset{(4.32)}{=} & \varphi(\gamma\xi\theta', u_0 s^{1/2}) \\ & = & \varphi(\xi_0, u_0 s^{1/2}) \\ & \overset{\text{def}}{=} & \varphi_0(s). \end{array}$$

This is what we need. $\square$

THEOREM 4.17. *Let $f$ be a radial function on $\mathfrak{M}_{n,m}$ so that $f(x) = f_0(r)$, $r = x'x$. Let $I_-^{k/2} f_0$ be the Gårding-Gindikin fractional integral (2.34). Then*

$$\hat{f}(\xi, t) = \pi^{km/2} (I_-^{k/2} f_0)(s), \qquad s = t't \in \overline{\mathcal{P}}_m, \tag{4.33}$$

*provided that either side of this equality exists in the Lebesgue sense.*

PROOF. The statement follows immediately from (4.10):

$$\begin{aligned} \hat{f}(\xi, t) &= \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix}\right) d\omega = \int\limits_{\mathfrak{M}_{k,m}} f_0(\omega'\omega + t't)\, d\omega \\ &= \pi^{km/2} (I_-^{k/2} f_0)(s). \end{aligned} \tag{4.34}$$

$\square$

REMARK 4.18. The rank of $s$ in (4.33) does not exceed $m$. If $\text{rank}(s) = m$ then $s \in \mathcal{P}_m$. If $\text{rank}(s) < m$ (it always happens if $k > n - m$) then $s$ is a boundary point of the cone $\mathcal{P}_m$. The fact, that for radial $f$ the Radon transform $\hat{f}$ is also radial, follows immediately by Corollary 4.6, Definition 1.14, and Definition 4.15.

Let us pass to the dual Radon transform.

THEOREM 4.19. *For $(\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$, let $\varphi(\xi, t) = \varphi_0(t't)$. Suppose that $1 \le k \le n - m$ and denote*

$$\Phi_0(s) = |s|^\delta \varphi_0(s), \quad \delta = (n-k)/2 - d, \quad d = (m+1)/2, \quad c = \frac{\pi^{km/2} \sigma_{n-k,m}}{\sigma_{n,m}}.$$

*Let $I_+^{k/2} \Phi_0$ be the Gårding-Gindikin fractional integral (2.32) of $\Phi_0$. Then for any $x \in \mathfrak{M}_{n,m}$ of rank $m$,*

$$\check{\varphi}(x) = c |r|^{d-n/2} (I_+^{k/2} \Phi_0)(r), \quad r = x'x \in \mathcal{P}_m, \tag{4.35}$$

*provided that either side of this equality exists in the Lebesgue sense.*

PROOF. By (4.27),

$$\check{\varphi}(x) = \int\limits_{SO(n)} \varphi_0(x'\gamma\xi_0\xi_o'\gamma' x) d\gamma, \qquad \xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k}.$$

We write $x$ in the polar coordinates $x = vr^{1/2}$, $v \in V_{n,m}$, $r \in \mathcal{P}_m$, and get

$$\check{\varphi}(x) = \int\limits_{SO(n)} \varphi_0(r^{1/2} v' \gamma \xi_0 \xi_o' \gamma' v r^{1/2})\, d\gamma$$

$$= \frac{1}{\sigma_{n,m}} \int\limits_{V_{n,m}} \varphi_0(r^{1/2} v' \xi_0 \xi_o' v r^{1/2})\, dv. \tag{4.36}$$

Since $n - k \geq m$, one can transform (4.36) by making use of the bi-Stiefel decomposition from Lemma 1.13. Let us consider the cases $k < m$ and $k \geq m$ separately.

$1^0$. **The case $k < m$.** We set

$$v = \begin{bmatrix} a \\ u(I_m - a'a)^{1/2} \end{bmatrix}, \qquad a \in \mathfrak{M}_{k,m}, \quad u \in V_{n-k,m}.$$

Multiplying matrices and using (1.40), we obtain

$$\check{\varphi}(x) = c_1 \int\limits_{0<a'a<I_m} |I_m - a'a|^{\delta} \varphi_0(r^{1/2}(I_m - a'a) r^{1/2})\, da$$

$$= c_1 \int\limits_{0<bb'<I_m} |I_m - bb'|^{\beta} \varphi_0(r^{1/2}(I_m - bb') r^{1/2}) db,$$

where $c_1 = \sigma_{n-k,m}/\sigma_{n,m}$ . Then we pass to polar coordinates

$$b = vq^{1/2}, \quad v \in V_{m,k}, \quad q \in \mathcal{P}_k,$$

and use the equality $|I_m - bb'| = |I_k - b'b|$. This gives

$$\check{\varphi}(x) = 2^{-k} c_1 \int\limits_{V_{m,k}} dv \int\limits_0^{I_k} |q|^{(m-k-1)/2} |I_k - q|^{\delta} \varphi_0(r^{1/2}(I_m - vqv') r^{1/2}) dq$$

$$= 2^{-k} c_1 |r|^{-\delta} \int\limits_{V_{m,k}} dv \int\limits_0^{I_k} |q|^{(m-k-1)/2} \Phi_0(r^{1/2}(I_m - vqv') r^{1/2}) dq.$$

Hence, by (2.15),

$$\check{\varphi}(x) = \pi^{km/2} \frac{\sigma_{n-k,m}}{\sigma_{n,m}} |r|^{d-n/2} (I_+^{k/2} \Phi_0)(r),$$

and (4.35) follows.

$2^0$. **The case $k \geq m$.** Let us transform (4.36) by the formula (1.41). We have

$$\check{\varphi}(x) = c_2 \int\limits_0^{I_m} |s|^{\gamma} |I_m - s|^{\delta} \varphi_0(r^{1/2}(I_m - s) r^{1/2})\, ds$$

$$= c_2 \int\limits_0^{I_m} |I_m - s|^{\gamma} |s|^{\delta} \varphi_0(r^{1/2} s r^{1/2})\, ds,$$

$$c_2 = \frac{2^{-m} \sigma_{k,m}\, \sigma_{n-k,m}}{\sigma_{n,m}}, \qquad \gamma = k/2 - d.$$

By changing variable $r^{1/2}sr^{1/2} = s_0$ (see Lemma 1.1 (ii)) so that $|s| = |r|^{-1}|s_0|$, $ds_0 = |r|^d ds$, and

$$|I_m - s| = |r^{-1/2}(r - r^{1/2}sr^{1/2})r^{-1/2}| = |r|^{-1}|r - s_0|,$$

we obtain

$$\begin{aligned}\check{\varphi}(x) &= c_2|r|^{-\gamma-\delta-d}\int_0^r |r-s_0|^{\gamma}|s_0|^{\delta}\varphi_0(s_0)\,ds_0 \\ &= c_2|r|^{d-n/2}\int_0^r |r-s_0|^{k/2-d}\Phi_0(s_0)\,ds_0.\end{aligned}$$

By (1.37) this coincides with (4.35). □

REMARK 4.20. In Theorem 4.19, we have assumed $1 \le k \le n-m$, although, Definition 4.10 is also meaningful for $k > n-m$. In the last case the integral on the right-hand side of (4.35) requires regularization. Since the main object of our study here is the Radon transform $\hat{f}(\xi, t)$, and the condition $k \le n-m$ constitutes a natural framework of the inversion problem (see Theorem 4.9), we do not focus on the case $k > n-m$ and leave it for subsequent publications.

## 4.6. Examples

Formulas (4.33) and (4.35) allow us to compute the Radon transform and the dual Radon transform of some elementary functions. The following examples are useful in different occurrences. As above, we suppose

$$f \equiv f(x), \quad x \in \mathfrak{M}_{n,m}, \qquad \varphi \equiv \varphi(\xi, t), \quad (\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m},$$

and write $f \overset{\wedge}{\longrightarrow} \varphi$ ($\varphi \overset{\vee}{\longrightarrow} f$) to indicate that $\varphi$ is the Radon transform of $f$ ($f$ is the dual Radon transform of $\varphi$). Given a symmetric $m \times m$ matrix $s$ and $\alpha \in \mathbb{C}$, we denote

$$s_+^{\alpha} = \begin{cases} |s|^{\alpha} & \text{if} \quad s \in \mathcal{P}_m, \\ 0 & \text{if} \quad s \notin \mathcal{P}_m. \end{cases} \tag{4.37}$$

LEMMA 4.21. *Let*

$$\lambda_1 = \frac{\pi^{km/2}\Gamma_m((\lambda-k)/2)}{\Gamma_m(\lambda/2)}, \qquad Re\,\lambda > k+m-1. \tag{4.38}$$

*The following formulas hold.*

$$|x'x|^{-\lambda/2} \overset{\wedge}{\longrightarrow} \lambda_1\,|t't|^{(k-\lambda)/2}, \tag{4.39}$$

$$|I_m + x'x|^{-\lambda/2} \overset{\wedge}{\longrightarrow} \lambda_1\,|I_m + t't|^{(k-\lambda)/2}, \tag{4.40}$$

$$(a - x'x)_+^{(\lambda-k)/2-d} \overset{\wedge}{\longrightarrow} \lambda_1\,(a - t't)_+^{\lambda/2-d}, \qquad a \in \mathcal{P}_m. \tag{4.41}$$

PROOF. We denote

$$\begin{aligned} f_1(x) &= |x'x|^{-\lambda/2}, \\ f_2(x) &= |I_m + x'x|^{-\lambda/2}, \\ f_3(x) &= (a - x'x)_+^{(\lambda-k)/2-d}. \end{aligned}$$

Then $\hat{f}_i(\xi, t) = \varphi_i(t't)$ where by (4.34),

$$\begin{aligned} \varphi_1(s) &= \int\limits_{\mathfrak{M}_{k,m}} |\omega'\omega + s|^{-\lambda/2} d\omega, \\ \varphi_2(s) &= \int\limits_{\mathfrak{M}_{k,m}} |I_m + \omega'\omega + s|^{-\lambda/2} d\omega, \\ \varphi_3(s) &= \int\limits_{\mathfrak{M}_{k,m}} (a - \omega'\omega - s)_+^{(\lambda-k)/2-d} d\omega. \end{aligned}$$

Owing to (A.6), $\varphi_1(s) = \lambda_1 |s|^{(k-\lambda)/2}$ and $\varphi_2(s) = \lambda_1 |I_m + s|^{(k-\lambda)/2}$. For the third integral we have $\varphi_3(s) = 0$ if $a - s \notin \mathcal{P}_m$. Otherwise,

$$\varphi_3(s) = \int\limits_{\{\omega \in \mathfrak{M}_{k,m} \,:\, \omega'\omega < a - s\}} |a - \omega'\omega - s|^{(\lambda-k)/2-d} d\omega,$$

and the formula (A.7) gives $\varphi_3(s) = \lambda_1 |a - s|^{\lambda/2-d}$. □

LEMMA 4.22. *Let* $1 \le k \le n - m$,

$$\lambda_2 = \frac{\Gamma_m(n/2)\Gamma_m((\lambda-k)/2)}{\Gamma_m(\lambda/2)\Gamma_m((n-k)/2)}, \qquad Re\,\lambda > k + m - 1. \tag{4.42}$$

*Then*

$$|t't|^{(\lambda-n)/2} \overset{\vee}{\longrightarrow} \lambda_2 |x'x|^{(\lambda-n)/2}, \tag{4.43}$$

$$|t't|^{(\lambda-n)/2} |I_m + t't|^{-\lambda/2} \overset{\vee}{\longrightarrow} \lambda_2 |x'x|^{(\lambda-n)/2} |I_m + x'x|^{(k-\lambda)/2}, \tag{4.44}$$

$$|t't|^{(k-n)/2+d} (t't - a)_+^{(\lambda-k)/2-d} \overset{\vee}{\longrightarrow} \lambda_2 |x'x|^{d-n/2} (x'x - a)_+^{\lambda/2-d}. \tag{4.45}$$

PROOF. Let

$$\begin{aligned} \varphi_1(\xi, t) &= |t't|^{(\lambda-n)/2}, \\ \varphi_2(\xi, t) &= |t't|^{(\lambda-n)/2} |I_m + t't|^{-\lambda/2}, \\ \varphi_3(\xi, t) &= |t't|^{(k-n)/2+d} (t't - a)_+^{(\lambda-k)/2-d}. \end{aligned}$$

By (4.35),

$$\check{\varphi}_i(x) = c |r|^{d-n/2} (I_+^{k/2} \Phi_i)(r), \qquad r = x'x, \quad i = 1, 2, 3, \tag{4.46}$$

where $c = \pi^{km/2}\sigma_{n-k,m}/\sigma_{n,m}$,

$$\begin{aligned}\Phi_1(s) &= |s|^{(\lambda-k)/2-d},\\ \Phi_2(s) &= |s|^{(\lambda-k)/2-d}|I_m+s|^{-\lambda/2},\\ \Phi_3(s) &= (s-a)_+^{(\lambda-k)/2-d}.\end{aligned}$$

For $\check{\varphi}_1(x)$, owing to (4.46) and (2.32), we obtain

$$\check{\varphi}_1(x) = c|r|^{d-n/2}\pi^{-km/2}\int\limits_{\{\omega\in\mathfrak{M}_{k,m}:\,\omega'\omega<r\}} |r-\omega'\omega|^{(\lambda-k)/2-d}d\omega.$$

Hence (A.7) and (1.37) yield

$$\check{\varphi}_1(x) = \frac{\Gamma_m(n/2)\,\Gamma_m((\lambda-k)/2)}{\Gamma_m(\lambda/2)\,\Gamma_m((n-k)/2)}\;|r|^{(\lambda-n)/2}, \qquad r = x'x.$$

This coincides with (4.43).

For $\check{\varphi}_3(x)$, according to (4.46), we have

$$\check{\varphi}_3(x) = c|r|^{d-n/2}\pi^{-km/2}\int\limits_{\{\omega\in\mathfrak{M}_{k,m}:\,\omega'\omega<r\}} (r-\omega'\omega-a)_+^{(\lambda-k)/2-d}d\omega.$$

Hence $\check{\varphi}_3(x) = 0$ if $r-a \notin \mathcal{P}_m$, and

$$\check{\varphi}_3(x) = c|r|^{d-n/2}\pi^{-km/2}\int\limits_{\{\omega\in\mathfrak{M}_{k,m}:\,\omega'\omega<r-a\}} |r-a-\omega'\omega|^{(\lambda-k)/2-d}d\omega$$

if $r-a\in\mathcal{P}_m$. Applying (A.7), we obtain

$$\check{\varphi}_3(x) = \frac{\Gamma_m(n/2)\,\Gamma_m((\lambda-k)/2)}{\Gamma_m(\lambda/2)\,\Gamma_m((n-k)/2)}\;|r|^{d-n/2}\;|r-a|^{\lambda/2-d}.$$

This gives (4.45).

In order to prove (4.44), we consider the cases $k\ge m$ and $k<m$ separately.

$1^0$. **The case $k \ge m$.** By (4.46),

$$\begin{aligned}\check{\varphi}_2(x) &= c_1|r|^{d-n/2}\int\limits_0^r |s|^{(\lambda-k)/2-d}\;|r-s|^{k/2-d}\;|I_m+s|^{-\lambda/2}\;ds\\ &= c_1|r|^{d-n/2}\int\limits_{I_m}^{I_m+r} |r+I_m-s|^{k/2-d}\;|s-I_m|^{(\lambda-k)/2-d}\;|s|^{-\lambda/2}\;ds,\end{aligned}$$

$c_1 = 2^{-m}\sigma_{n-k,m}\sigma_{k,m}/\sigma_{n,m}$. Owing to (A.3), we obtain

$$\check{\varphi}_2(r) = c_1 B_m(k/2,(\lambda-k)/2)\,|r|^{(\lambda-n))/2}\;|I_m+r|^{(k-\lambda)/2},$$

and (4.44) follows.

$2^0$. **The case $k<m$.** By (4.46) and (2.19),

$$\check{\varphi}_2(x) = c\;|r|^{d-n/2}(I_+^{k/2}\Phi_2)(r) = c\;|r|^{(k-n)/2}(I_-^{k/2}G_2)(r^{-1})$$

where

$$G_2(s) = |s|^{-k/2-d}\Phi_2(s^{-1}) = |I_m+s|^{-\lambda/2}.$$

Hence, in view of (2.19),

$$\check{\varphi}_2(x) = c\,\pi^{-km/2}\,|r|^{(k-n)/2} \int\limits_{\mathfrak{M}_{k,m}} |I_m + r^{-1} + \omega'\omega|^{-\lambda/2} d\omega.$$

This integral can be evaluated by the formula (A.6), and we obtain

$$\begin{aligned}\check{\varphi}_2(x) &= \frac{c\,\Gamma_m((\lambda-k)/2)}{\Gamma_m(\lambda/2)}\,|r|^{(k-n)/2}\,|I_m + r^{-1}|^{(k-\lambda)/2} \\ &= \lambda_2\,|r|^{(\lambda-n)/2}\,|I_m + r|^{(k-\lambda)/2}, \qquad r = x'x.\end{aligned}$$

□

## 4.7. Integral identities. Existence of the Radon transform

**4.7.1. Integral identities.** It is important to have precise information about the behavior of the Radon transform at infinity and near certain manifolds. The same is desirable for the dual Radon transform. Examples in Section 4.6 combined with duality (4.28) give rise to integral identities which provide this information in integral terms. For $m = 1$, similar results were obtained in [**Ru4**]; see also [**Ru5**] for Radon transforms on affine Grassmann manifolds.

We present these results in two theorems. The constants $\lambda_1$ and $\lambda_2$ below are defined by (4.38) and (4.42), respectively, $d = (m+1)/2$, and all equalities hold provided that either side exists in the Lebesgue sense.

THEOREM 4.23. *If* $1 \le k \le n-m$, $Re\,\lambda > k+m-1$, *then*

$$\frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi,t)\,|t't|^{(\lambda-n)/2}\,dt = \lambda_2 \int\limits_{\mathfrak{M}_{n,m}} f(x)\,|x'x|^{(\lambda-n)/2}\,dx, \tag{4.47}$$

$$\frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi,t)\,|t't|^{(\lambda-n)/2}\,|I_m + t't|^{-\lambda/2}\,dt = \lambda_2 \int\limits_{\mathfrak{M}_{n,m}} f(x)\,|x'x|^{(\lambda-n)/2}\,|I_m + x'x|^{(k-\lambda)/2}\,dx, \tag{4.48}$$

$$\frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi,t)\,|t't|^{-\delta}\,(t't - a)_+^{(\lambda-k)/2-d}\,dt = \lambda_2 \int\limits_{\mathfrak{M}_{n,m}} f(x)\,|x'x|^{d-n/2}\,(x'x - a)_+^{\lambda/2-d}\,dx, \tag{4.49}$$

$$\delta = (n-k)/2 - d.$$

THEOREM 4.24. *If* $1 \le k \le n-m$, *Re* $\lambda > k+m-1$, *then*

$$\int\limits_{\mathfrak{M}_{n,m}} \check{\varphi}(x)\, |x'x|^{-\lambda/2}\, dx = \frac{\lambda_1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi,t)\, |t't|^{(k-\lambda)/2}\, dt, \tag{4.50}$$

$$\int\limits_{\mathfrak{M}_{n,m}} \check{\varphi}(x)\, |I_m + x'x|^{-\lambda/2}\, dx = \frac{\lambda_1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi,t)\, |I_m + t't|^{((k-\lambda)/2}\, dt, \tag{4.51}$$

$$\int\limits_{\mathfrak{M}_{n,m}} \check{\varphi}(x)\, (a - x'x)_+^{(\lambda-k)/2-d}\, dx = \frac{\lambda_1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi,t)\, (a - t't)_+^{\lambda/2-d}\, dt. \tag{4.52}$$

**4.7.2. Existence of the Radon transform.** In most of the formulas presented above we a priori assumed that the Radon transform is well defined. Now we arrive at one of the central questions: for which functions $f$ does the Radon transform $\hat{f}(\xi,t)$ exist? In other words, which functions are integrable over all (or almost all) matrix planes $\xi'x = t$ in $\mathfrak{M}_{n,m}$? The crux is how to specify the behavior of $f(x)$ at infinity. Below we study this problem if (a) $f$ is a continuous function, (b) $f$ is a locally integrable function, and (c) $f \in L^p$.

THEOREM 4.25. *Let $f(x)$ be a continuous function on $\mathfrak{M}_{n,m}$, satisfying*

$$f(x) = O(|I_m + x'x|^{-\lambda/2}). \tag{4.53}$$

*If $\lambda > k+m-1$ then the Radon transform $\hat{f}(\xi,t)$ is finite for all $(\xi,t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$. If $\lambda \le k+m-1$ then there is a function $f_\lambda(x)$ which obeys (4.53) and $\hat{f}_\lambda(\xi,t) \equiv \infty$.*

PROOF. It suffices to consider the function $f_\lambda(x) = |I_m + x'x|^{-\lambda/2}$. By (4.40), $\hat{f}_\lambda(\xi,t) = \lambda_1\, |I_m + t't|^{(k-\lambda)/2}$ provided $\lambda > k+m-1$. This proves the first statement of the theorem. Conversely, owing to (4.33), the Radon transform $\hat{f}_\lambda(\xi,t)$ is finite if and only if $(I_-^{k/2} f_\lambda)(t't) < \infty$ (we utilize the same notation $f_\lambda$ both for $|I_m + x'x|^{-\lambda/2}$ and $|I_m + r|^{-\lambda/2}$, $r = x'x$). If $\lambda \le k+m-1$, then $I_-^{k/2} f_\lambda \equiv \infty$; see Remark 2.20. Thus $\hat{f}_\lambda(\xi,t) \equiv \infty$ for the same $\lambda$. $\square$

THEOREM 4.26. *Let $f$ be a locally integrable function on $\mathfrak{M}_{n,m}$. If*

$$\int\limits_{\{y\in\mathfrak{M}_{n,m}\,:\,y'y>R\}} |y'y|^{(k-n)/2}|f(y)|\,dy<\infty \qquad \textit{for all}\ \ R\in\mathcal{P}_m, \tag{4.54}$$

*then $\hat f(\xi,t)$ is finite for almost all $(\xi,t)\in V_{n,n-k}\times\mathfrak{M}_{n-k,m}$. If (4.54) fails for some $R\in\mathcal{P}_m$, then $\hat f(\xi,t)$ may be identically infinite.*

PROOF. STEP 1. Suppose first that $f$ is a radial function so that $f(x)=f_0(r)$, $r=x'x$. Then (4.54) becomes

$$\int\limits_R^\infty |r|^{k/2-d}\,|f_0(r)|\,dr<\infty \qquad \text{for all}\ \ R\in\mathcal{P}_m. \tag{4.55}$$

By Theorem 4.17,

$$\hat f(\xi,t)=\pi^{km/2}(I_-^{k/2}f_0)(t't),$$

and therefore, $\hat f(\xi,t)<\infty$ if and only if $(I_-^{k/2}f_0)(t't)<\infty$. Now the result follows from Lemma 2.19 and Remark 2.20.

STEP 2. The general case reduces to the radial one. Indeed, let

$$F(x)=\int\limits_{SO(n)} f(\gamma x)d\gamma$$

be the "radialization" of $f$. If (4.55) holds, then, by STEP 1, the Radon transform of $F$ is a radial function of the form $\hat F(\xi,t)=\Phi_0(s)$, $s=t't$, which is finite for almost all $t$. On the other hand, owing to commutation (4.18), $\Phi_0(t't)$ is the mean value of $\hat f(\xi,t)$ over all $\xi\in V_{n,n-k}$, and therefore,

$$\frac{1}{\sigma_{n,n-k}}\int\limits_{V_{n,n-k}} \hat f(\xi,t)d\xi=\Phi_0(t't)<\infty$$

for almost all $t$. It follows that $\hat f(\xi,t)<\infty$ for almost all $(\xi,t)\in V_{n,n-k}\times\mathfrak{M}_{n-k,m}$. □

THEOREM 4.27. *Let $f\in L^p(\mathfrak{M}_{n,m})$. The Radon transform $\hat f(\xi,t)$ is finite for almost all $(\xi,t)\in V_{n,n-k}\times\mathfrak{M}_{n-k,m}$ if and only if*

$$1\le p<p_0=\frac{n+m-1}{k+m-1}. \tag{4.56}$$

PROOF. By Hölder's inequality,

$$\int\limits_{\{y\in\mathfrak{M}_{n,m}\,:\,y'y>R\}} |y'y|^{(k-n)/2}|f(y)|\,dy\le A\|f\|_p,$$

where

$$\begin{aligned} A^{p'} &= \int\limits_{\{y\in\mathfrak{M}_{n,m}\,:\,y'y>R\}} |y'y|^{p'(k-n)/2}\,dy \\ &= c\int\limits_{R}^{\infty} |r|^{p'(k-n)/2+n/2-d}dr \qquad (d=(m+1)/2,\ c=c(n,m)) \\ &= c\int\limits_{0}^{R^{-1}} |r|^{p'(n-k)/2-n/2-d}\,dr<\infty \end{aligned}$$

provided $p<p_0$. Thus, by Theorem 4.26, the Radon transform $\hat{f}(\xi,t)$ is finite for almost all $(\xi,t)\in V_{n,n-k}\times\mathfrak{M}_{n-k,m}$. Conversely, if $p>p_0$, then one can choose $\lambda$ satisfying

$$\frac{n+m-1}{p}<\lambda\le k+m-1.$$

For such $\lambda$, the function $f_\lambda(x)=|I_m+x'x|^{-\lambda/2}$ belongs to $L^p(\mathfrak{M}_{n,m})$ (use the formula (A.6)), and $\hat{f}_\lambda(\xi,t)\equiv\infty$; see the proof of Theorem 4.25. In order to cover the case $p=p_0$, we need a more subtle counter-example. One can show that for $p\ge p_0$, the function

$$F(x)=|2I_m+x'x|^{-(n+m-1)/2p}(\log|2I_m+x'x|)^{-1} \tag{4.57}$$

belongs to $L^p(\mathfrak{M}_{n,m})$, and $\hat{F}(\xi,t)\equiv\infty$. The proof of this statement is technical, and presented in Appendix B. □

Remark 4.28. (i) Theorems 4.25 and 4.27 agree with each other in the sense that all functions satisfying (4.53) belong to $L^p(\mathfrak{M}_{n,m})$ where

$$\frac{n+m-1}{\lambda}<p<p_0.$$

The last statement holds by the formula (A.6), according to which the function $|I_m+x'x|^{-\lambda p/2}$ is integrable for such $p$.

(ii) The equality (4.48) implies part (ii) of Theorem 0.5 and provides alternative proof of the "if part" of Theorem 4.27 when $1\le k\le n-m$. Indeed, choosing $\lambda=n$, we have

$$\begin{aligned} &\frac{1}{\sigma_{n,n-k}}\int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} |\hat{f}(\xi,t)|\,|I_m+t't|^{-n/2}\,dt \\ &= \lambda_2\int\limits_{\mathfrak{M}_{n,m}} |f(x)|\,|I_m+x'x|^{(k-n)/2}\,dx. \end{aligned} \tag{4.58}$$

Hence, if $|I_m+x'x|^{\alpha/2}f(x)\in L^1(\mathfrak{M}_{n,m})$ for some $\alpha\ge k-n$, then $\hat{f}(\xi,t)$ is finite for almost all $(\xi,t)\in V_{n,n-k}\times\mathfrak{M}_{n-k,m}$. Moreover, by Hölder's inequality, the right-hand side of (4.58) does not exceed $A\|f\|_p$, where

$$A^{p'}=\int\limits_{\mathfrak{M}_{n,m}} |I_m+x'x|^{p'(k-n)/2}dx. \tag{4.59}$$

By (A.6), the last integral is finite when $p'(k-n) > k+m-1$, i.e., for $p < p_0$. Thus the left-hand side of (4.58) is finite too, and therefore, the Radon transform $\hat{f}(\xi, t)$ is well defined for almost all $(\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$.

**4.7.3. Existence of the dual Radon transform.** The special case $\lambda = k + 2d$ in (4.52) deserves particular mentioning. In this case

$$\int\limits_{x'x<a} \check{\varphi}(x)dx = \frac{\lambda_1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi, t)\,(a - t't)_+^{k/2}\,dt, \tag{4.60}$$

and therefore,

$$\int\limits_{x'x<a} |\check{\varphi}(x)|\,dx \le c \int\limits_{V_{n,n-k}} d\xi \int\limits_{t't<a} |\varphi(\xi, t)|\,dt, \quad c = \frac{\lambda_1\,|a|^{k/2}}{\sigma_{n,n-k}}. \tag{4.61}$$

Since $a \in \mathcal{P}_m$ is arbitrary, this implies the following.

THEOREM 4.29. *If $\varphi(\xi, t)$ is a locally integrable function on the set $V_{n,n-k} \times \mathfrak{M}_{n-k,m}$, $1 \le k \le n-m$, then the dual Radon transform $\check{\varphi}(x)$ is finite for almost all $x \in \mathfrak{M}_{n,m}$.*

CHAPTER 5

# Analytic families associated to the Radon transform

## 5.1. Matrix distances and shifted Radon transforms

In this section, we introduce important mean value operators which can be called *the shifted Radon transforms.* Here we adopt the terminology of F. Rouvière [**Rou**] that admirably suits our case. In contrast to the rank-one case $m = 1$, where averaging parameters are positive numbers, in the higher rank case $m > 1$ these parameters are matrix-valued.

Some geometric preliminaries are in order. We shall need a natural analog of the euclidean distance for the space $\mathfrak{M}_{n,m}$ of rectangular $n \times m$ matrices. Unlike $m = 1$, when the distance is represented by a positive number, in the higher rank case our "distance" will be matrix-valued and represented by a positive semi-definite $m \times m$ matrix.

DEFINITION 5.1. A *matrix distance* between two points $x$ and $y$ in $\mathfrak{M}_{n,m}$ is defined by

$$d(x, y) = [(x - y)'(x - y)]^{1/2}. \tag{5.1}$$

Given a point $x \in \mathfrak{M}_{n,m}$ and a matrix $k$-plane $\tau = \tau(\xi, t) \in \mathfrak{T}$ (see (4.1)), a matrix distance between $x$ and $\tau$ is defined by

$$d(x, \tau) = [(\xi' x - t)'(\xi' x - t)]^{1/2}. \tag{5.2}$$

Abusing notation, we write (cf. (3.1))

$$|x - y|_m = \det(d(x, y)), \qquad |x - \tau|_m = \det(d(x, \tau)). \tag{5.3}$$

Let us comment on this definition. We first note that for $m = 1$, (5.1) is the usual euclidean distance between points in $\mathbb{R}^n$. If $y = 0$, then $d(x, 0) = (x'x)^{1/2}$. This agrees with the polar decomposition $x = vr^{1/2}$, $r = x'x$, $v \in V_{n,m}$. If $\text{rank}(x) = m$, then $d(x, 0) = r^{1/2}$ is a positive definite matrix. If $\text{rank}(x) < m - 1$ then $d(x, 0) \in \partial\mathcal{P}_m$ and $\det(d(x, 0)) = 0$.

Let us explain (5.2). For $\xi \in V_{n,n-k}$, let $\{\xi\}$ denote the $(n - k)$-dimensional subspace spanned by $\xi$, and $\text{Pr}_{\{\xi\}}$ the orthogonal projection onto $\{\xi\}$ which is a linear map with $n \times n$ matrix $\xi\xi'$. Then, as in the euclidean case, it is natural to define the distance between the point $x \in \mathfrak{M}_{n,m}$ and the plane $\tau = \tau(\xi, t) \in \mathfrak{T}$ as that between two points, namely, $\text{Pr}_{\{\xi\}} x = \xi\xi' x$ and $\xi t$. By (5.1), we have

$$d(x, \tau) = d(\xi\xi' x, \xi t) = [(\xi\xi' x - \xi t)'(\xi\xi' x - \xi t)]^{1/2} = [(\xi' x - t)'(\xi' x - t)]^{1/2}.$$

This can be regarded as a matrix distance between $x$ and the projection $\text{Pr}_\tau x$ of $x$ onto the plane $\tau$.

LEMMA 5.2.
(i) *The group* $M(n,m)$ *of matrix motions,*

$$\mathfrak{M}_{n,m} \ni x \longrightarrow \gamma x\beta + b, \qquad \gamma \in O(n), \qquad \beta \in O(m), \qquad b \in \mathfrak{M}_{n,m},$$

*acts on the manifolds* $\mathfrak{M}_{n,m}$ *and* $\mathfrak{T}$ *transitively.*
(ii) *The determinants* $|x-y|_m$ *and* $|x-\tau|_m$ *are invariant under the action of* $M(n,m)$*. Namely,*

$$|gx - gy|_m = |x-y|_m, \qquad |gx - g\tau|_m = |x-\tau|_m, \qquad g \in M(n,m).$$

(iii) *The distances* $d(x,y)$ *and* $d(x,\tau)$ *are invariant under the subgroup* $M(n)$ *of* $M(n,m)$*, acting by the rule* $x \to \gamma x + b$*,* $\gamma \in O(n)$*,* $b \in \mathfrak{M}_{n,m}$ *(i.e.* $\beta = 0$*).*

PROOF. The statements follow immediately from Definition 5.1, by taking into account that

$$g: \ \tau(\xi,t) \longrightarrow \tau(\gamma\xi, t\beta + \xi'\gamma' b). \tag{5.4}$$

□

DEFINITION 5.3. Let $r \in \overline{\mathcal{P}}_m$. The *shifted Radon transform* of a function $f(x)$ on $\mathfrak{M}_{n,m}$ is defined by

$$\hat{f}_r(\xi,t) = \frac{1}{\sigma_{n-k,m}} \int\limits_{V_{n-k,m}} dv \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ t + vr^{1/2} \end{bmatrix}\right) d\omega \tag{5.5}$$

$$= \frac{1}{\sigma_{n-k,m}} \int\limits_{V_{n-k,m}} \hat{f}(\xi, t + vr^{1/2}) dv, \tag{5.6}$$

where $(\xi,t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$, $g_\xi \in SO(n)$ is a rotation satisfying (4.9).

If $r = 0$ then $\hat{f}_r(\xi,t)$ coincides with the Radon transform $\hat{f}(\xi,t)$; cf. (4.10). The integral (5.5) averages $f(x)$ over all $x$ satisfying $\xi' x = t + vr^{1/2}$. This means that $\hat{f}_r(\xi,t)$ is the integral of $f(x)$ over all $x$ at matrix distance $r^{1/2}$ from the plane $\tau = \tau(\xi,t)$ (see Definition 4.1). One can write

$$\hat{f}_r(\tau) = \hat{f}_r(\xi,t) = \int\limits_{d(x,\tau) = r^{1/2}} f(x). \tag{5.7}$$

DEFINITION 5.4. Let $1 \le k \le n-m$, $r \in \overline{\mathcal{P}}_m$, $\mathfrak{T}$ be the manifold of all matrix $k$-planes

$$\tau \equiv \tau(\xi,t) = \{x \in \mathfrak{M}_{n,m} : \xi' x = t\}, \qquad (\xi,t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}.$$

Given a point $z \in \mathfrak{M}_{n-k,m}$ at distance $r^{1/2}$ from the origin (i.e. $z'z = r$), the *shifted dual Radon transform* of a function $\varphi(\tau) \equiv \varphi(\xi,t)$ on $\mathfrak{T}$ is defined by

$$\check{\varphi}_r(x) = \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \varphi(\xi, \xi' x + z) d\xi. \tag{5.8}$$

Let us comment on this definition. We first note that (5.8) is independent of the choice of $z$ with $z'z = r$. Indeed, by passing to polar coordinates

$$z = \theta u_0 r^{1/2}, \quad \theta \in O(n-k), \quad u_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in V_{n-k,m},$$

we have

$$\begin{aligned}
\check{\varphi}_r(x) &= \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \varphi(\xi, \xi' x + \theta u_0 r^{1/2})\, d\xi \qquad (\text{set } \xi = \eta\theta') \\
&= \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \varphi(\eta\theta', \theta(\eta' x + u_0 r^{1/2})\, d\eta \quad (\text{use} \quad \varphi(\xi\theta', \theta t) = \varphi(\xi, t)) \\
&= \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \varphi(\eta, \eta' x + u_0 r^{1/2})\, d\eta. \qquad (5.9)
\end{aligned}$$

If $r = 0$ then $\check{\varphi}_r(x)$ is the usual dual Radon transform (4.27). Owing to (5.2), the matrix distance $d(x, \tau)$ between $x$ and $\tau = \tau(\xi, \xi' x + z)$ is $r^{1/2}$. Hence, $\check{\varphi}_r(x)$ may be regarded as the mean value of $\varphi(\xi, t)$ over all matrix planes at distance $r^{1/2}$ from $x$:

$$\check{\varphi}_r(x) = \int\limits_{d(x,\tau) = r^{1/2}} \varphi(\tau) \ . \tag{5.10}$$

## 5.2. Intertwining operators

Given a sufficiently good function $w(r)$ on $\mathbb{R}_+$, we define the following intertwining operator

$$\begin{aligned}
(Wf)(\tau) \equiv (Wf)(\xi, t) &= \int\limits_{\mathfrak{M}_{n,m}} f(x)\, w(|\xi' x - t|_m)\, dx \\
&= \int\limits_{\mathfrak{M}_{n,m}} f(x)\, w(|x - \tau|_m)\, dx,
\end{aligned} \tag{5.11}$$

which assigns to a sufficiently good function $f(x)$ on $\mathfrak{M}_{n,m}$ a function $(Wf)(\tau)$ on the manifold $\mathfrak{T}$ of matrix $k$-planes $\tau = \tau(\xi, t)$, $(\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$. The dual operator, which maps a function $\varphi(\tau)$ on $\mathfrak{T}$ to the corresponding function $(W^*\varphi)(x)$ on $\mathfrak{M}_{n,m}$, is defined by

$$\begin{aligned}
(W^*\varphi)(x) &= \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi, t)\, w(|\xi' x - t|_m)\, dt \\
&= \frac{1}{\sigma_{n,n-k}} \int\limits_{\mathfrak{T}} \varphi(\tau)\, w(|x - \tau|_m)\, d\tau.
\end{aligned} \tag{5.12}$$

We recall that $|x - \tau|_m$ denotes for the determinant of the matrix distance between $x$ and $\tau$; see (5.3). For $m = 1$, operators (5.11) and (5.12) were introduced in [**Ru4**]. The following statement follows immediately from Lemma 5.2.

LEMMA 5.5. *Operators $W$ and $W^*$ commute with the group $M(n, m)$ of matrix motions.*

LEMMA 5.6. *If $\tau = \tau(\xi, t)$, $(\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$, then*

$$(Wf)(\tau) \equiv (Wf)(\xi, t) = \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, z)\, w(|t-z|_m)\, dz, \tag{5.13}$$

*provided that either side of (5.13) is finite for $f$ replaced by $|f|$.*

PROOF. Let $g_\xi \in SO(n)$ be a rotation satisfying

$$g_\xi \xi_0 = \xi, \qquad \xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k}.$$

The change of variable $x = g_\xi y$ in (5.11) gives

$$(Wf)(\xi, t) = \int\limits_{\mathfrak{M}_{n,m}} f(g_\xi y)\, w(|\xi_0' y - t|_m)\, dy, \qquad \xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k}.$$

By setting

$$y = \begin{bmatrix} \omega \\ z \end{bmatrix}, \qquad \omega \in \mathfrak{M}_{k,m}, \qquad z \in \mathfrak{M}_{n-k,m},$$

we have

$$\begin{aligned} (Wf)(\xi, t) &= \int\limits_{\mathfrak{M}_{n-k,m}} w(|t-z|_m)\, dz \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ z \end{bmatrix}\right) d\omega \\ &= \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, z)\, w(|t-z|_m)\, dz. \end{aligned}$$

$\square$

LEMMA 5.7. *Let $1 \le k \le n-m$, $\delta = (n-k)/2 - d$, $d = (m+1)/2$,*

$$\tau = \tau(\xi, t), \qquad (\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}, \qquad u_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in V_{n-k,m}.$$

*Then*

$$(Wf)(\xi, t) = 2^{-m} \sigma_{n-k,m} \int\limits_{\mathcal{P}_m} |r|^\delta\, w(|r|^{1/2})\, \hat{f}_r(\xi, t)\, dr, \tag{5.14}$$

$$(W^*\varphi)(x) = 2^{-m} \sigma_{n-k,m} \int\limits_{\mathcal{P}_m} |r|^\delta\, w(|r|^{1/2})\, \check{\varphi}_r(x)\, dr. \tag{5.15}$$

PROOF. From (5.13), by passing to the polar coordinates (see Lemma 1.11), we obtain

$$\begin{aligned} (Wf)(\xi, t) &= \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, t+z)\, w(|z|_m)\, dz \\ &= 2^{-m} \int\limits_{\mathcal{P}_m} |r|^\delta\, w(|r|^{1/2})\, dr \int\limits_{V_{n-k,m}} \hat{f}(\xi, t + vr^{1/2})\, dv. \end{aligned}$$

By (5.6), this coincides with (5.14). Furthermore, (5.12) yields

$$\begin{aligned}(W^*\varphi)(x) &= \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi, t+\xi' x)\, w(|t|_m)\, dt \\ &= \frac{2^{-m}}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathcal{P}_m} |r|^{\delta}\, w(|r|^{1/2})\, dr \int\limits_{V_{n-k,m}} \varphi(\xi, ur^{1/2}+\xi' x)\, du \\ &= \frac{2^{-m}\sigma_{n-k,m}}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathcal{P}_m} |r|^{\delta}\, w(|r|^{1/2})\, dr \int\limits_{O(n-k)} \varphi(\xi, \theta u_0 r^{1/2}+\xi' x)\, d\theta.\end{aligned}$$

Now we change the order of integration and replace $\xi$ by $\xi\theta'$. Since $\varphi(\xi\theta', \theta t) = \varphi(\xi, t)$, then

$$(W^*\varphi)(x) = \frac{2^{-m}\sigma_{n-k,m}}{\sigma_{n,n-k}} \int\limits_{\mathcal{P}_m} |r|^{\delta}\, w(|r|^{1/2})\, dr \int\limits_{V_{n,n-k}} \varphi(\xi, u_0 r^{1/2}+\xi' x)\, d\xi.$$

By (5.9), this gives (5.15). $\square$

### 5.3. The generalized Semyanistyi fractional integrals

To avoid possible confusion, we shall discriminate between operators acting on $\mathfrak{M}_{n,m}$ and the similar operators on $\mathfrak{M}_{n-k,m}$. As before, the notation $I^\alpha$, $\Delta$, $\mathcal{D}$, and $\mathcal{H}$ will be used for the Riesz potential, the Cayley-Laplace operator, the Cayley operator, and the generalized Hilbert transform on $\mathfrak{M}_{n,m}$. We write $\tilde{I}^\alpha$, $\tilde{\Delta}$, $\tilde{\mathcal{D}}$, and $\tilde{\mathcal{H}}$ for the similar operators on $\mathfrak{M}_{n-k,m}$. These will be applied to functions $\hat{f}(\xi, t)$ and $\varphi(\xi, t)$ in the $t$-variable. We assume $1 \le k \le n-m$, and denote by $\mathcal{S}(\mathfrak{T})$ the space of functions $\varphi(\xi, t)$ which are infinitely differentiable in the $\xi$-variable and belong to the Schwartz space $\mathfrak{M}_{n-k,m}$ in the $t$-variable uniformly in $\xi \in V_{n,n-k}$.

Consider the following operators

$$P^\alpha f = \tilde{I}^\alpha \hat{f}, \qquad \overset{*}{P}{}^\alpha \varphi = (\tilde{I}^\alpha \varphi)^\vee, \tag{5.16}$$

where $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $\varphi \in \mathcal{S}(\mathfrak{T})$, and

$$\alpha \in \mathbb{C}, \quad \alpha \ne n-k-m+1, n-k-m+2, \dots\ . \tag{5.17}$$

The right-hand sides of equalities in (5.16) absolutely converge for $Re\,\alpha > m-1$ and are understood in the sense of analytic continuation for other values of $\alpha$. Below we prove a series of lemmas giving explicit representation of operators (5.16) for different values of $\alpha$.

LEMMA 5.8. *Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $\varphi \in \mathcal{S}(\mathfrak{T})$. If*

$$Re\,\alpha > m-1, \qquad \alpha \ne n-k-m+1,\ n-k-m+2, \dots\ ,$$

*then $P^\alpha$ and $\overset{*}{P}{}^\alpha$ are intertwining operators of the form (5.11) and (5.12), respectively. Namely,*

$$(P^\alpha f)(\xi,t) = \frac{1}{\gamma_{n-k,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,m}} f(x)\,|\xi'x-t|_m^{\alpha+k-n}\,dx, \tag{5.18}$$

$$(\overset{*}{P}{}^\alpha\varphi)(x) = \frac{1}{\gamma_{n-k,m}(\alpha)} \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi,t)\,|\xi'x-t|_m^{\alpha+k-n}\,dt, \tag{5.19}$$

*where $d_*\xi = \sigma_{n,n-k}^{-1}\,d\xi$ is the normalized measure on $V_{n,n-k}$ and $\gamma_{n-k,m}(\alpha)$ is the normalized constant for the Riesz potential on $\mathfrak{M}_{n-k,m}$; cf. (3.46).*

Proof. The formula (5.19) follows immediately from the definitions (5.16), (4.27), and (3.49). Furthermore, by (5.16), (4.10), and (3.49), we have

$$\begin{aligned}(P^\alpha f)(\xi,t) &= \frac{1}{\gamma_{n-k,m}(\alpha)} \int\limits_{\mathfrak{M}_{n-k,m}} |y|_m^{\alpha+k-n} dy \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ t-y \end{bmatrix}\right) d\omega \\ &= \frac{1}{\gamma_{n-k,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,m}} f(x)\,|\xi'x-t|_m^{\alpha+k-n}\,dx.\end{aligned}$$

This proves (5.18). □

Lemma 5.9. *Let $f\in \mathcal{S}(\mathfrak{M}_{n,m})$, $\varphi\in\mathcal{S}(\mathfrak{T})$, $1\le k\le n-m$. If $\ell$ is a positive integer so that $\ell\le n-k-m$ (cf. (5.17)), then*

$$(P^\ell f)(\xi,t) = c_\ell \int\limits_{\mathfrak{M}_{k+\ell,m}} dz \int\limits_{O(n-k)} f\left(\xi t - g_\xi \begin{bmatrix} I_k & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} z \\ 0 \end{bmatrix}\right) d\gamma, \tag{5.20}$$

$$(\overset{*}{P}{}^\ell\varphi)(x) = c_\ell \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{\ell,m}} dz \int\limits_{O(n-k)} \varphi\left(\xi,\ \xi'x-\gamma\begin{bmatrix} z \\ 0 \end{bmatrix}\right) d\gamma, \tag{5.21}$$

*where*

$$c_\ell = 2^{-\ell m}\,\pi^{-\ell m/2}\,\Gamma_m\Big(\frac{n-k-\ell}{2}\Big)/\Gamma_m\Big(\frac{n-k}{2}\Big). \tag{5.22}$$

*Moreover,*

$$(P^0 f)(\xi,t) = \hat f(\xi,t), \qquad (\overset{*}{P}{}^0\varphi)(x) = \check\varphi(x). \tag{5.23}$$

Proof. By (5.16), analytic continuation of $P^\alpha f$ and $\overset{*}{P}{}^\alpha\varphi$ reduces to that of the Riesz potential on $\mathfrak{M}_{n-k,m}$. One can readily see that $\hat f(\xi,t)$, defined by (4.10), is a Schwartz function in the $t$-variable. Thus,

$$P^\ell f = \tilde I^\ell \hat f, \qquad \overset{*}{P}{}^\ell\varphi = (\tilde I^\ell\varphi)^\vee. \tag{5.24}$$

Now (5.21) follows from (3.54), and (5.23) is a consequence of (3.56). Furthermore, by (3.54) and (4.10),

$$
\begin{aligned}
(P^{\ell} f)(\xi,t) &= c_\ell \int\limits_{\mathfrak{M}_{\ell,m}} dz \int\limits_{O(n-k)} \hat{f}\left(\xi, t+\gamma \begin{bmatrix} z \\ 0 \end{bmatrix}\right) d\gamma \\
&= c_\ell \int\limits_{\mathfrak{M}_{\ell,m}} dz \int\limits_{O(n-k)} d\gamma \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ t+\gamma \begin{bmatrix} z \\ 0 \end{bmatrix} \end{bmatrix}\right) d\omega.
\end{aligned}
$$

Since

$$
g_\xi \begin{bmatrix} \omega \\ t+\gamma \begin{bmatrix} z \\ 0 \end{bmatrix} \end{bmatrix} = \xi t + g_\xi \begin{bmatrix} I_k & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} \omega \\ z \\ 0 \end{bmatrix},
$$

then (5.20) follows if we change the notation $\begin{bmatrix} \omega \\ z \end{bmatrix} \to z$. $\square$

REMARK 5.10. We call $P^\alpha f$ and $\overset{*}{P}{}^\alpha \varphi$ *the generalized Semyanistyi fractional integrals*; see Remark 5.14 for comments. Formulas (5.18)–(5.23) can serve as definitions of $P^\alpha f$ and $\overset{*}{P}{}^\alpha \varphi$ if $f$ and $\varphi$ are arbitrary locally integrable functions so that the corresponding integrals converge.

LEMMA 5.11. *Let* $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $\varphi \in \mathcal{S}(\mathfrak{T})$. *If* $\ell = 1, 2, \dots$ , *then*

$$
(P^{-2\ell} f)(\xi,t) = (-1)^{m\ell} \tilde{\Delta}^\ell \hat{f}(\xi,t), \tag{5.25}
$$

*and*

$$
(\overset{*}{P}{}^{-2\ell}\varphi)(x) = (-1)^{m\ell}[\tilde{\Delta}^\ell \varphi]^\vee(x) = (-1)^{m\ell} \int\limits_{V_{n,n-k}} \tilde{\Delta}^\ell \varphi(\xi,t)\Big|_{t=\xi' x} \, d_*\xi. \tag{5.26}
$$

PROOF. The statement follows from (5.16) and (3.60). $\square$

LEMMA 5.12. *Let* $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $\varphi \in \mathcal{S}(\mathfrak{T})$, $\ell = 1, 2, \dots$ . *We denote*

$$
c_1 = \frac{(-1)^{m\ell}\,\Gamma((n-k-m)/2)}{2^{m+1}\,\pi^{(m+n-k)/2}}, \qquad c_2 = \frac{(-1)^{m(\ell+1)}\,\Gamma_m((m+1)/2)}{\pi^{m^2/2}}. \tag{5.27}
$$

(i) *If* $1 \le k \le n-m$ *and* $F_\xi(t) = \tilde{\Delta}^\ell \hat{f}(\xi,t)$ *then*

$$
\begin{aligned}
(P^{1-2\ell} f)(\xi,t) &= (-1)^{m\ell}(\tilde{I}^1 F_\xi)(t) \\
&= c_1 \int\limits_{S^{n-k-1}} dv \int\limits_{\mathbb{R}^m} F_\xi(t - vy')\, dy
\end{aligned} \tag{5.28}
$$

*and*

$$
\begin{aligned}
(\overset{*}{P}{}^{1-2\ell}\varphi)(x) &= (-1)^{m\ell}[\tilde{I}^1 \tilde{\Delta}^\ell \varphi]^\vee(x) \\
&= c_1 \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{S^{n-k-1}} dv \int\limits_{\mathbb{R}^m} \tilde{\Delta}^\ell \varphi(\xi,t)\Big|_{t=\xi' x - vy'} \, dy.
\end{aligned} \tag{5.29}
$$

(ii) *If $k = n - m$, then*

$$(P^{1-2\ell} f)(\xi, t) = c_2\, (\tilde{\mathcal{H}}\tilde{\mathcal{D}}^{2\ell-1} \hat{f}(\xi, \cdot))(t) \tag{5.30}$$

*and*

$$\begin{aligned} (\overset{*}{P}{}^{1-2\ell}\varphi)(x) &= c_2\, (\tilde{\mathcal{H}}\tilde{\mathcal{D}}^{2\ell-1}\varphi(\xi,\cdot))^{\vee}(x) \\ &= c_2 \int\limits_{V_{n,n-k}} (\tilde{\mathcal{H}}\tilde{\mathcal{D}}^{2\ell-1}\varphi(\xi,\cdot))(\xi' x)\, d_*\xi, \end{aligned} \tag{5.31}$$

$\tilde{\mathcal{H}}$ *being the generalized Hilbert transform; cf. (3.50).*

Proof. (i) follows from (5.16) and (3.61); (ii) is a consequence of (5.16) and (3.62). □

The following statement is the main results of this chapter.

Theorem 5.13. *Let $1 \le k \le n-m$, $\alpha \in \mathbb{C}$; $\alpha \ne n-k-m+1, n-k-m+2, \dots$ .*
(i) *If $f \in \mathcal{S}(\mathcal{S}_m)$ then*

$$(\overset{*}{P}{}^{\alpha}\hat{f})(x) = c_{n,k,m}(I^{\alpha+k} f)(x) \tag{5.32}$$

*(the generalized Fuglede formula), where*

$$c_{n,k,m} = 2^{km}\pi^{km/2}\Gamma_m\left(\frac{n}{2}\right)/\Gamma_m\left(\frac{n-k}{2}\right). \tag{5.33}$$

(ii) *Let $\ell_0 = \min\{m-1, n-k-m\}$,*

$$\mathfrak{A} = \{0, 1, 2, \dots, \ell_0\} \cup \{\alpha : Re\,\alpha > m-1;\ \alpha \ne n-k-m+1, n-k-m+2, \dots\}.$$

*Suppose that $f \in L^1_{loc}(\mathfrak{M}_{n,m})$ and $\alpha \in \mathfrak{A}$. Then equality (5.32) holds provided that the Riesz potential $(I^{\alpha+k} f)(x)$ is finite for $f$ replaced by $|f|$ (e.g., for $f \in L^p$, $1 \le p < n/(Re\,\alpha + k + m - 1)$).*

Proof. (i) Let $f \in \mathcal{S}(\mathcal{S}_m)$. We make use of the equality (4.47) with $\lambda = \alpha + k$ and $Re\,\alpha > m-1$. This gives

$$\begin{aligned} &\frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, t)\, |t|_m^{\alpha+k-n}\, dt \\ &= \frac{\Gamma_m(n/2)\,\Gamma_m(\alpha/2)}{\Gamma_m((\alpha+k)/2)\,\Gamma_m((n-k)/2)} \int\limits_{\mathfrak{M}_{n,m}} f(y)\, |y|_m^{\alpha+k-n}\, dy. \end{aligned} \tag{5.34}$$

Replacing $f(y)$ by the shifted function $f_x(y) = f(x+y)$ and taking into account (4.19), we get

$$\begin{aligned} &\frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, \xi' x + t)\, |t|_m^{\alpha+k-n}\, dt \\ &= \frac{\Gamma_m(n/2)\,\Gamma_m(\alpha/2)}{\Gamma_m((\alpha+k)/2)\,\Gamma_m((n-k)/2)} \int\limits_{\mathfrak{M}_{n,m}} f(x+y)\, |y|_m^{\alpha+k-n}\, dy, \end{aligned} \tag{5.35}$$

cf. (5.19) and (3.49). Hence, (5.32) follows when $Re\,\alpha > m-1$ with the constant

$$\begin{aligned} c_{n,k,m} &= \frac{\Gamma_m(n/2)\,\Gamma_m(\alpha/2)\,\gamma_{n,m}(\alpha+k)}{\Gamma_m((\alpha+k)/2)\,\Gamma_m((n-k)/2)\,\gamma_{n-k,m}(\alpha)} \\ &\overset{(3.46)}{=} 2^{km}\pi^{km/2}\Gamma_m\left(\frac{n}{2}\right)/\Gamma_m\left(\frac{n-k}{2}\right). \end{aligned}$$

By analytic continuation, it is true for all $\alpha \in \mathbb{C}$, $\alpha \neq n-k-m+1, n-k-m+2, \dots$.

(ii) Suppose $f \in L^1_{loc}(\mathfrak{M}_{n,m})$. For $Re\,\alpha > m-1$, (5.32) follows from (5.35) by taking into account that (5.35) was derived from (4.47), and the latter is also true for locally integrable functions. For $\alpha = \ell$, $\ell = 1, 2, \dots \ell_0$, we have

$$\begin{aligned} (\overset{*}{P}{}^{\ell}\hat{f})(x) &= \frac{c_\ell}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{\ell,m}} dz \int\limits_{O(n-k)} d\gamma \int\limits_{\mathfrak{M}_{k,m}} f\left(x - g_\xi \begin{bmatrix} \omega \\ \gamma \begin{bmatrix} z \\ 0 \end{bmatrix} \end{bmatrix}\right) d\omega \\ &= c_\ell \int\limits_{O(n)} d\beta \int\limits_{\mathfrak{M}_{\ell,m}} dz \int\limits_{O(n-k)} d\gamma \int\limits_{\mathfrak{M}_{k,m}} f\left(x - \beta \begin{bmatrix} \omega \\ \gamma \begin{bmatrix} z \\ 0 \end{bmatrix} \end{bmatrix}\right) d\omega, \end{aligned}$$

where $c_\ell$ is the constant (5.22). We write

$$\begin{bmatrix} \omega \\ \gamma \begin{bmatrix} z \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} \omega \\ z \\ 0 \end{bmatrix}.$$

Then the change of variables $\beta \begin{bmatrix} I_k & 0 \\ 0 & \gamma \end{bmatrix} \to \beta$ gives

$$\begin{aligned} (\overset{*}{P}{}^{\ell}\hat{f})(x) &= c_\ell \int\limits_{\mathfrak{M}_{\ell,m}} dz \int\limits_{\mathfrak{M}_{k,m}} d\omega \int\limits_{O(n)} f\left(x - \beta \begin{bmatrix} \omega \\ z \\ 0 \end{bmatrix}\right) d\beta \\ &= c_\ell \int\limits_{\mathfrak{M}_{\ell+k,m}} dy \int\limits_{O(n)} f\left(x - \beta \begin{bmatrix} y \\ 0 \end{bmatrix}\right) d\beta \\ &= c_{n,k,m}(I^{\ell+k} f)(x), \end{aligned}$$

where

$$\begin{aligned} c_{n,k,m} &= c_\ell\, 2^{(\ell+k)m}\, \pi^{(\ell+k)m/2}\, \Gamma_m\Big(\frac{n}{2}\Big)/\Gamma_m\Big(\frac{n-\ell-k}{2}\Big) \\ &= 2^{km}\pi^{km/2}\Gamma_m\left(\frac{n}{2}\right)/\Gamma_m\left(\frac{n-k}{2}\right). \end{aligned}$$

$\square$

REMARK 5.14. Some comments are in order. The idea to study Radon transforms as members of the corresponding analytic families goes back to Semyanistyi [**Se**] who considered the hyperplane Radon transform on $\mathbb{R}^n$ (the case $m = 0,\ k = n-1$). This approach was extended by Rubin to Radon transforms of different kinds, see, e.g., [**Ru2**]. In the rank-one case $m = 1$ the formula (5.32) is due to Fuglede [**Fu**] for $\alpha = 0$ and to Rubin [**Ru2**], [**Ru4**] for any $\alpha$. In the higher rank case $m > 1$, it was established for $\alpha = 0$ in [**OR**] (for sufficiently good functions) and justified for $f \in L^p$ in [**Ru7**].

The formula (5.32) has the same nature as the classical decomposition of distributions in plane waves. To see this, one should formally set $x = 0$ in (5.32) and regard this equality in the framework of the theory of distributions. The idea of decomposition of a function in integral of plane waves amounts to Radon [**R**] and John [**Jo**], and has proved to be very fruitful in PDE. For the distribution $|x|^\lambda$ on $\mathbb{R}^n$ an account of this theory is presented in [**GSh1**, Section 3.10]. This method was generalized by Petrov [**P2**] for the matrix case for $k = n - m$, and briefly outlined by Shibasov [**Sh2**] for $k < n - m$. Our way of thinking differs from that in [**Sh2**], [**P2**], and [**GSh1**], and the result is more general because we allow $f$ to be a "rough" function.

For the sake of completeness, we also present the following statement which follows from Theorem 5.13 and the semigroup property of Riesz potentials. Concerning this property, see [**Kh**] and [**Ru7**].

THEOREM 5.15. *Let*

$$Re\,\alpha > m-1, \qquad Re\,\beta > m-1, \qquad Re\,(\alpha+\beta) < n-k-m+1.$$

*If the integral* $I^{\alpha+\beta+k} f$ *absolutely converges then*

$$\overset{*}{P}{}^{\alpha} P^{\beta} f = c_{n,k,m} I^{\alpha+\beta+k} f, \tag{5.36}$$

$c_{n,k,m}$ *being the constant (5.33).*

PROOF. By (5.32) and (5.16), we have

$$c_{n,k,m} I^{\alpha+\beta+k} f = \overset{*}{P}{}^{\alpha+\beta} \hat{f} = (\tilde{I}^{\alpha+\beta} \hat{f})^\vee = (\tilde{I}^{\alpha} \tilde{I}^{\beta} \hat{f})^\vee = \overset{*}{P}{}^{\alpha} P^{\beta} f.$$

□

CHAPTER 6

# Inversion of the Radon transform

## 6.1. The radial case

Theorems 4.17, 4.27, and the inversion formula (2.51) for the Gårding-Gindikin fractional integrals imply the following result for the Radon transform of radial functions.

THEOREM 6.1. *Suppose that* $f(x) \equiv f_0(r)$, $x \in \mathfrak{M}_{n,m}$, $r = x'x$, *and let*

$$f \in L^p(\mathfrak{M}_{n,m}), \qquad 1 \le p < \frac{n+m-1}{k+m-1}. \tag{6.1}$$

*Then the Radon transform* $\hat{f}(\xi, t)$ *is well defined by (4.10) for almost all* $(\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$ *and represents a radial function, namely,*

$$\hat{f}(\xi, t) = \pi^{km/2}(I_-^{k/2} f_0)(s) = \varphi_0(s), \qquad s = t't \in \overline{\mathcal{P}}_m. \tag{6.2}$$

*If* $1 \le k \le n-m$, *then* $f_0$ *can be recovered from* $\varphi_0$ *by the formula*

$$f_0(r) = \pi^{-km/2}(D_-^{k/2}\varphi_0)(r), \tag{6.3}$$

*where* $D_-^{k/2}$ *is defined in the sense of* $\mathcal{D}'(\mathcal{P}_m)$*-distributions by (2.52).*

REMARK 6.2. The condition (6.1) can be replaced by the weaker one. Indeed, owing to Theorem 2.19, it suffices to assume that

$$\int_R^\infty |r|^{(k-m-1)/2}\, |f_0(r)|\, dr < \infty \qquad \text{for all } \; R \in \mathcal{P}_m. \tag{6.4}$$

Furthermore, we know that by Theorem 4.9, the assumption $k \le n-m$ is necessary for injectivity of the Radon transform. One might expect that once we restrict to radial functions, then this assumption can be reduced. However, it is not so, because the function $\psi$ in the proof of Theorem 4.9 can be chosen to be radial. Note that if $k > n-m$, then the exterior variable $s$ in (6.2) ranges on the boundary of the cone, and we "lose the dimension".

In the same manner, Theorem 4.19 and Lemma 2.23 allow us to obtain an inversion formula for the dual Radon transform.

THEOREM 6.3. *Let* $\varphi(\xi, t) \equiv \varphi_0(s)$, $(\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$, $s = t't$. *We assume* $1 \le k \le n-m$ *and denote*

$$\Phi_0(s) = |s|^\delta \varphi_0(s), \qquad \delta = (n-k)/2 - d, \qquad d = (m+1)/2.$$

*If* $\Phi_0(s) \in L^1_{loc}(\mathcal{P}_m)$ *then the dual Radon transform* $\check{\varphi}(x)$ *is well defined by (4.27) for almost all* $x \in \mathfrak{M}_{n,m}$ *and represents the radial function*

$$\check{\varphi}(x) = c|r|^{d-n/2}(I_+^{k/2}\Phi_0)(r) = f_0(r), \quad r = x'x \in \overline{\mathcal{P}}_m, \tag{6.5}$$

$c = \pi^{km/2}\sigma_{n-k,m}/\sigma_{n,m}$. *The function* $\varphi_0$ *can be recovered from* $f_0$ *by the formula*

$$\varphi_0(s) = c^{-1}|s|^{-\delta}(D_+^j I_+^{j-k/2} F_0)(s), \qquad F_0(r) = |r|^{n/2-d} f_0(r), \tag{6.6}$$

*for any integer* $j \geq k/2$. *If* $k$ *is even, then*

$$\varphi_0(s) = c^{-1}|s|^{-\delta}(D_+^{k/2} F_0)(s). \tag{6.7}$$

*The differential operator* $D_+$ *in (6.6) and (6.7) is understood in the sense of* $\mathcal{D}'(\mathcal{P}_m)$*-distributions (cf. (2.48) and (2.49)).*

## 6.2. The method of mean value operators

As it was mentioned in Introduction, after the 1917 paper by Radon [**R**], the following two inversion methods have been customarily used for reconstruction of functions from their integrals over affine planes. These are *the method of mean value operators*, and *the method of Riesz potentials.* The third classical method presented in [**GSh1**] is *the method of plane waves.* Below we focus on the first method and extend it to functions of matrix argument.

Definition 6.4. Given a function $f(x)$ on $\mathfrak{M}_{n,m}$, we define

$$(M_r f)(x) = \frac{1}{\sigma_{n,m}} \int\limits_{V_{n,m}} f(vr^{1/2} + x)dv, \qquad r \in \mathcal{P}_m. \tag{6.8}$$

This is a matrix generalization of the usual spherical mean on $\mathbb{R}^n$.

We say that $r \in \mathcal{P}_m$ tends to zero when $\mathrm{tr}(r) \to 0$.

Lemma 6.5. *If* $f \in L^p(\mathfrak{M}_{n,m})$, $1 \leq p < \infty$, *then*

$$\lim_{r \to 0} (M_r f)(x) = f(x) \tag{6.9}$$

*in the* $L^p$*-norm. If* $f \in C_0(\mathfrak{M}_{n,m})$, *i.e.,* $f(x) \to 0$ *as* $||x|| = (\mathrm{tr}(x'x))^{1/2} \to \infty$, *this limit is uniform on* $\mathfrak{M}_{n,m}$.

Proof. By the generalized Minkowski inequality,

$$\|M_r f - f\|_p \leq \frac{1}{\sigma_{n,m}} \int\limits_{V_{n,m}} \|f(vr^{1/2} + \cdot) - f(\cdot)\|_p \, dv.$$

Since the integrand does not exceed $2\|f\|_p$, by the Lebesgue theorem on dominated convergence, one can pass to the limit under the sign of integration. Let $y = vr^{1/2} \in \mathfrak{M}_{n,m}$. We represent the corresponding $nm$-vector $\bar{y} = (y_{1,1}, \ldots, y_{n,m})$ in polar coordinates

$$\bar{y} = \theta\rho, \qquad \rho = \|y\| = (\mathrm{tr}(y'y))^{1/2}, \qquad \theta \in S^{nm-1},$$

so that

$$\lim_{r \to 0} \|f(vr^{1/2} + \cdot) - f(\cdot)\|_p^p = \lim_{\rho \to 0} \int\limits_{\mathbb{R}^{nm}} |f(\theta\rho + \bar{x}) - f(\bar{x})|^p d\bar{x} = 0.$$

This gives (6.9) in the $L^p$-norm. The proof of the second part of the statement is the same. $\square$

The following lemma which combines the Radon transform and the shifted dual Radon transform (5.7) is the core of the method. It reduces the inversion problem for the Radon transform to the case of radial functions. Thus "dimension of the problem" becomes essentially smaller.

LEMMA 6.6. *For fixed $x \in \mathfrak{M}_{n,m}$, let $F_x(r) = (M_r f)(x)$, $r \in \mathcal{P}_m$. Then*

$$(\hat{f})^{\vee}_s(x) = \pi^{km/2}(I_-^{k/2} F_x)(s), \qquad s \in \mathcal{P}_m, \tag{6.10}$$

*provided that either side of this equality exists in the Lebesgue sense.*

PROOF. We denote $\varphi(\xi, t) = \hat{f}(\xi, t)$, $f_x(y) = f(x+y)$. Let $z \in \mathfrak{M}_{n-k,m}$ be a matrix at distance $s^{1/2}$ from the origin. By (5.8) and (4.19),

$$\begin{aligned}(\hat{f})^{\vee}_s(x) &= \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \hat{f}_x(\xi, z) d\xi \\ &= \int\limits_{SO(n)} \hat{f}_x(\gamma\xi, z) d\gamma.\end{aligned}$$

Interchanging the order of the Radon transform and integration over $SO(n)$, and using (4.18), we get

$$\check{\varphi}_s(x) = \int\limits_{\mathfrak{M}_{k,m}} d\omega \int\limits_{SO(n)} f_x\left(\gamma \left[\begin{array}{c} \omega \\ z \end{array}\right]\right) d\gamma.$$

Hence, $z \to \check{\varphi}_s(x)$ is the Radon transform of the radial function

$$\tilde{f}_x(y) = \int\limits_{SO(n)} f_x(\gamma y) d\gamma = \int\limits_{SO(n)} f(x + \gamma y) d\gamma, \qquad y \in \mathfrak{M}_{n,m}.$$

In other words,

$$\check{\varphi}_s(x) = (\tilde{f}_x)^{\wedge}(\xi, z), \tag{6.11}$$

where

$$\tilde{f}_x(y) = \frac{1}{\sigma_{n,m}} \int\limits_{V_{n,m}} f(x + vr^{1/2}) dv = F_x(r), \qquad r = y'y.$$

Owing to (6.11), the desired equality (6.10) follows from the representation (4.33) for the Radon transform of a radial function. □

COROLLARY 6.7. *Let $f \in L^p(\mathfrak{M}_{n,m})$, $1 \le p < (n+m-1)/(k+m-1)$. Then (6.10) holds for almost all $x \in \mathfrak{M}_{n,m}$. In particular, (6.10) holds for any continuous function $f$ satisfying*

$$f(x) = O(|I_m + x'x|^{-\lambda/2}), \qquad \lambda > k+m-1.$$

PROOF. It suffices to show that the right side of (6.10) is finite for almost all $x \in \mathfrak{M}_{n,m}$ whenever $f \in L^p(\mathfrak{M}_{n,m})$, $1 \le p < (n+m-1)/(k+m-1)$. By Theorem 2.19, the integral $(I_-^{k/2} F_x)(s)$ is well defined provided

$$I = \int\limits_R^{\infty} |r|^{k/2-d} |F_x(r)| \, dr < \infty \qquad \text{for all} \quad R \in \mathcal{P}_m. \tag{6.12}$$

The proof of (6.12) is simple. Indeed, taking into account (6.8) and using Lemma 1.11, we obtain

$$\begin{aligned} I &\le \frac{1}{\sigma_{n,m}} \int\limits_R^\infty |r|^{k/2-d}\, dr \int\limits_{V_{n,m}} |f(vr^{1/2}+x)|\, dv \\ &= \frac{2^m}{\sigma_{n,m}} \int\limits_{\{y\in\mathfrak{M}_{n,m}\,:\, y'y>R\}} |y'y|^{(k-n)/2} |f(x+y)|\, dy. \end{aligned}$$

By Hölder's inequality, $I \le A\|f\|_p$, where the constant $A$ is the same as in the proof of Theorem 4.27. If $p<(n+m-1)/(k+m-1)$ then $A<\infty$, and we are done. □

Now we can prove the main result.

THEOREM 6.8. *Let* $1\le k\le n-m$,

$$f\in L^p(\mathfrak{M}_{n,m}), \qquad 1\le p< \frac{n+m-1}{k+m-1}. \tag{6.13}$$

*Then the Radon transform* $\varphi(\xi,t)=\hat f(\xi,t)$ *is well defined by (4.10) for almost all* $(\xi,t)\in V_{n,n-k}\times\mathfrak{M}_{n-k,m}$ *and can be inverted by the formula*

$$f(x)=\pi^{-km/2} \overset{(L^p)}{\lim_{r\to 0}} (D_-^{k/2}\Phi_x)(r), \qquad \Phi_x(s)=\check\varphi_s(x), \tag{6.14}$$

*where* $D_-^{k/2}$ *is defined in the sense of* $\mathcal{D}'(\mathcal{P}_m)$*-distributions by (2.52). If* $f$ *is a continuous function satisfying*

$$f(x)=O(|I_m+x'x|^{-\lambda/2}), \qquad \lambda>k+m-1, \tag{6.15}$$

*then the limit in (6.14) can be treated in the sup-norm.*

PROOF. By Lemmas 6.6 and 2.24, one can recover $F_x(r)=(M_rf)(x)$ and get

$$(M_rf)(x)=\pi^{-km/2}(D_-^{k/2}\Phi_x)(r), \qquad r\in\mathcal{P}_m.$$

Now the result follows by Lemma 6.5. □

## 6.3. The method of Riesz potentials

The second traditional inversion method for the Radon transform reduces the problem to inversion of the Riesz potentials. This method relies on the formula (5.32) where $\alpha$ is in our disposal. The case $\alpha=0$ corresponds to the Fuglede formula

$$(\hat f)^\vee(x)=c_{n,k,m}(I^kf)(x), \tag{6.16}$$

which together with Theorem 3.17 implies the following.

THEOREM 6.9. *Let* $f\in L^p(\mathfrak{M}_{n,m})$, $1\le p<n/(k+m-1)$. *Then the Radon transform* $\varphi=\hat f$ *is well defined, and* $f$ *can be recovered from* $\varphi$ *in the sense of* $\Phi'$*-distributions by the formula*

$$c_{n,k,m}(f,\phi)=(\check\varphi, I^{-k}\phi), \tag{6.17}$$

$$\phi\in\Phi, \qquad c_{n,k,m}=2^{km}\pi^{km/2}\Gamma_m\left(\frac{n}{2}\right)/\Gamma_m\left(\frac{n-k}{2}\right),$$

*where the operator $I^{-k}$ is defined by*

$$(I^{-k}\phi)(x) = (\mathcal{F}^{-1}|y|_m^k \mathcal{F}\phi)(x).$$

*In particular, for $k$ even,*

$$c_{n,k,m}(f,\phi) = (-1)^{mk/2}(\check{\varphi}, \Delta^{k/2}\phi), \tag{6.18}$$

*$\Delta$ being the Cayley-Laplace operator (1.44).*

REMARK 6.10. For $k$ odd, the Radon transform can be inverted under more restrictive assumptions as follows. Let $k < n-m$. We choose $\alpha = 1$ in (5.32) and get

$$\tilde{I}^1 \hat{f} = c_{n,k,m} I^{k+1} f. \tag{6.19}$$

If $f \in L^p(\mathfrak{M}_{n,m})$, $1 \le p < n/(k+m)$, then $f$ can be recovered from $\varphi = \hat{f}$ by the formula

$$c_{n,k,m}(f,\phi) = (-1)^{m(k+1)/2}(\tilde{I}^1\varphi, \Delta^{(k+1)/2}\phi), \qquad \phi \in \Phi. \tag{6.20}$$

## 6.4. Decomposition in plane waves

This method was developed in [**P2**] for the case $k = n-m$, and outlined in [**Sh2**] for $1 \le k \le n-m$. It is based on decomposition of distributions in (matrix) plane waves. As we have already noted in Remark 5.14, our formula (5.32) has the same nature. Together with (5.26), (5.29), and (5.31), it enables us to invert the Radon transform of functions $f \in \mathcal{S}(\mathfrak{M}_{n,m})$.

THEOREM 6.11. *Let $1 \le k \le n-m$, $f \in \mathcal{S}(\mathcal{S}_m)$. The Radon transform $\varphi(\xi,t) = \hat{f}(\xi,t)$ can be inverted by the following formulas.*
(i) *For $k$ even,*

$$f(x) = (-1)^{mk/2} c_{n,k,m}^{-1} \int\limits_{V_{n,n-k}} \tilde{\Delta}^{k/2}\varphi(\xi,t)\Big|_{t=\xi' x} \, d_*\xi\,, \tag{6.21}$$

$$c_{n,k,m} = 2^{km}\pi^{km/2}\Gamma_m\left(\frac{n}{2}\right)/\Gamma_m\left(\frac{n-k}{2}\right)$$

*(we recall that $d_*\xi$ stands for the invariant measure on $V_{n,n-k}$ of total mass 1).*
(ii) *For $k$ odd and $k < n-m$,*

$$f(x) = c_1 \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{S^{n-k-1}} dv \int\limits_{\mathbb{R}^m} \tilde{\Delta}^{(k+1)/2}\varphi(\xi,t)\Big|_{t=\xi' x - vy'} \, dy, \tag{6.22}$$

$$c_1 = (-1)^{m(k+1)/2} 2^{-km-m-1} \pi^{(k-n)/2 - m(k/2+1)} \Gamma_{m+1}\left(\frac{n-k}{2}\right)/\Gamma_m\left(\frac{n}{2}\right).$$

(iii) *For $k$ odd and $k = n-m$,*

$$f(x) = c_2 \int\limits_{V_{n,n-k}} (\tilde{\mathcal{H}}\tilde{\mathcal{D}}^k\varphi(\xi,\cdot))(\xi' x)\, d_*\xi, \tag{6.23}$$

$$c_2 = (-1)^{m(k+3)/2} 2^{-km} \pi^{-m(k+m)/2} \Gamma_m\left(\frac{m}{2}\right)\Gamma_m\left(\frac{m+1}{2}\right)/\Gamma_m\left(\frac{n}{2}\right).$$

PROOF. We write (5.32) with $\alpha = -k$ so that

$$f(x) = c_{n,k,m}^{-1} (\overset{*}{P}{}^{-k} \varphi)(x), \tag{6.24}$$

where $\overset{*}{P}{}^{-k}$ is the operator (5.16). Now it remains to apply formulas (5.26), (5.29), and (5.31).

□

Formulas (6.21)–(6.23) differ from those in [**P2**] and [**Sh2**].

APPENDIX A

# Table of integrals

Let $k, m \in \mathbb{N}$; $\alpha, \beta, \gamma \in \mathbb{C}$; $d = (m+1)/2$. We recall that $\mathcal{S}_m, \mathcal{P}_m$, and $\overline{\mathcal{P}}_m$ denote the set of all $m \times m$ real symmetric matrices, the cone of positive definite matrices in $\mathcal{S}_m$, and the closed cone of positive semi-definite matrices in $\mathcal{S}_m$, respectively. The following formulas hold.

$$\int_a^b |r-a|^{\alpha-d}|b-r|^{\beta-d}dr = B_m(\alpha,\beta)|b-a|^{\alpha+\beta-d}, \tag{A.1}$$

$$a \in \mathcal{S}_m, \quad b > a, \quad Re\,\alpha > d-1, \quad Re\,\beta > d-1;$$

$$\int_c^b |b-r|^{\alpha-d}|r-c|^{\beta-d}\frac{dr}{|r|^{\alpha+\beta}} = \frac{B_m(\alpha,\beta)}{|a|^{\alpha}|b|^{\beta}}|b-a|^{\alpha+\beta-d}, \tag{A.2}$$

$$a \in \mathcal{S}_m, \quad b > a, \quad c = b^{1/2}a^{1/2}b^{-1}a^{1/2}b^{1/2}, \quad Re\,\alpha > d-1, \quad Re\,\beta > d-1;$$

$$\int_{I_m}^b |b-r|^{\alpha-d}|r-I_m|^{\beta-d}\frac{dr}{|r|^{\alpha+\beta}} = \frac{B_m(\alpha,\beta)}{|b|^{\beta}}|b-I_m|^{\alpha+\beta-d}, \tag{A.3}$$

$$b > I_m, \quad Re\,\alpha > d-1, \quad Re\,\beta > d-1;$$

$$\int_s^\infty |r|^{-\gamma}|r-s|^{\alpha-d}dr = |s|^{\alpha-\gamma}B_m(\alpha,\gamma-\alpha), \tag{A.4}$$

$$s \in \mathcal{P}_m, \quad Re\,\alpha > d-1, \quad Re\,(\gamma-\alpha) > d-1;$$

$$\int_s^\infty |I_m+r|^{-\gamma}|r-s|^{\alpha-d}dr = |I_m+s|^{\alpha-\gamma}B_m(\alpha,\gamma-\alpha), \tag{A.5}$$

$$s \in \overline{\mathcal{P}}_m, \quad Re\,\alpha > d-1, \quad Re\,(\gamma-\alpha) > d-1;$$

$$\int_{\mathfrak{M}_{k,m}} |b+y'y|^{-\lambda/2}dy = \frac{\pi^{km/2}\Gamma_m((\lambda-k)/2)}{\Gamma_m(\lambda/2)}|b|^{(k-\lambda)/2}, \tag{A.6}$$

$$b \in \mathcal{P}_m, \quad Re\,\lambda > k+m-1;$$

$$\text{(A.7)} \qquad \int\limits_{\{y\in\mathfrak{M}_{k,m}:\, y'y<b\}} |b-y'y|^{(\lambda-k)/2-d}dy = \frac{\pi^{km/2}\Gamma_m((\lambda-k)/2)}{\Gamma_m(\lambda/2)}|b|^{\lambda/2-d},$$

$$b\in\mathcal{P}_m, \quad Re\,\lambda > k+m-1.$$

PROOF

**(A.1).** We have

$$I \equiv \int\limits_a^b |r-a|^{\alpha-d}|b-r|^{\beta-d}dr = \int\limits_0^c |s|^{\alpha-d}|c-s|^{\beta-d}\,ds,$$

$c=b-a$. Let $s=c^{1/2}tc^{1/2}$. By Lemma 1.1 (ii),

$$I = |c|^{\alpha+\beta-d}\int\limits_0^{I_m}|t|^{\alpha-d}|I_m-t|^{\beta-d}\,dt = |c|^{\alpha+\beta-d}B_m(\alpha,\beta).$$

**(A.2), (A.3).** Let us write (A.1) from the right to the left, and set

$$r=a^{1/2}b^{1/2}\tau b^{1/2}a^{1/2}, \qquad dr=(|a||b|)^d d\tau.$$

This gives

$$B_m(\alpha,\beta)|b-a|^{\alpha+\beta-d} = (|a||b|)^{\alpha+\beta-d}\int\limits_{b^{-1}}^{c^{-1}}|\tau-b^{-1}|^{\alpha-d}|c^{-1}-\tau|^{\beta-d}d\tau.$$

Then we set $\tau=s^{-1}$, $d\tau=|s|^{-2d}ds$, and get

$$B_m(\alpha,\beta)|b-a|^{\alpha+\beta-d} = |a|^{\alpha}|b|^{\beta}\int\limits_c^b|b-s|^{\alpha-d}|s-c|^{\beta-d}\frac{ds}{|s|^{\alpha+\beta}}$$

which was required. The equality (A.3) follows from (A.2).

**(A.4), (A.5).** By setting $r=q^{-1}$, $dr=|q|^{-m-1}dq$, one can write the left-hand side of (A.4) as

$$|s|^{\alpha-d}\int\limits_0^{s^{-1}}|q|^{\gamma-\alpha-d}|s^{-1}-q|^{\alpha-d}dq \overset{(A.1)}{=} |s|^{\alpha-\gamma}B_m(\alpha,\gamma-\alpha),$$

and we are done. The equality (A.5) follows from (A.4) if we replace $s$ and $r$ by $I_m+s$ and $I_m+r$, respectively.

**(A.6), (A.7).** By changing variable $y\to yb^{1/2}$, we obtain

$$\int\limits_{\mathfrak{M}_{k,m}}|b+y'y|^{-\lambda/2}dy = |b|^{(k-\lambda)/2}J_1,$$

$$\int\limits_{\{y\in\mathfrak{M}_{k,m}:\, y'y<b\}}|b-y'y|^{(\lambda-k)/2-d}dy = |b|^{\lambda/2-d}J_2,$$

where

$$\begin{aligned} J_1 &= \int\limits_{\mathfrak{M}_{k,m}} |I_m + y'y|^{-\lambda/2} dy, \\ J_2 &= \int\limits_{\{y \in \mathfrak{M}_{k,m}\colon\, y'y < I_m\}} |I_m - y'y|^{(\lambda-k)/2-d} dy. \end{aligned}$$

Let us show that

$$J_1 = J_2 = \frac{\pi^{km/2}\Gamma_m((\lambda-k)/2)}{\Gamma_m(\lambda/2)}.$$

**The case $k \geq m$.** We write both integrals in the polar coordinates according to Lemma 1.11. For $J_1$ we have

$$\begin{aligned} J_1 &= 2^{-m}\sigma_{k,m} \int\limits_{\mathcal{P}_m} |r|^{k/2-d} |I_m + r|^{-\lambda/2} dr \\ &= 2^{-m}\sigma_{k,m} B_m\left(\frac{k}{2}, \frac{\lambda-k}{2}\right) \end{aligned}$$

(the second equality holds by (A.5) with $s = 0,\ \alpha = k/2,\ \gamma = \lambda/2$). Similarly,

$$\begin{aligned} J_2 &= 2^{-m}\sigma_{k,m} \int\limits_0^{I_m} |r|^{k/2-d} |I_m - r|^{(\lambda-k)/2-d} dr \\ &= 2^{-m}\sigma_{k,m} B_m\left(\frac{k}{2}, \frac{\lambda-k}{2}\right). \end{aligned}$$

Now the result follows by (1.13) and (1.37).

**The case $k < m$.** We replace $y$ by $y'$ and pass to the polar coordinates. This yields

$$\begin{aligned} J_1 &= \int\limits_{\mathfrak{M}_{m,k}} |I_m + yy'|^{-\lambda/2} dy \\ &= 2^{-k} \int\limits_{V_{m,k}} dv \int\limits_{\mathcal{P}_k} |I_m + vqv'|^{-\lambda/2} |q|^{(m-k-1)/2} dq, \\ &\quad (|I_m + vqv'| = |I_k + q|) \\ &= 2^{-k}\sigma_{m,k} \int\limits_{\mathcal{P}_k} |q|^{(m-k-1)/2} |I_k + q|^{-\lambda/2} dq. \end{aligned}$$

By (A.5) (with $s = 0,\ m = k,\ \gamma = \lambda/2$) and (1.10),

$$\begin{aligned} J_1 &= 2^{-k}\sigma_{m,k} B_k\left(\frac{m}{2}, \frac{\lambda-m}{2}\right) \\ &= \frac{\pi^{km/2}\Gamma_k((\lambda-m)/2)}{\Gamma_k(\lambda/2)} \\ &= \frac{\pi^{km/2}\Gamma_m((\lambda-k)/2)}{\Gamma_m(\lambda/2)}. \end{aligned}$$

Similarly,

$$\begin{aligned}
J_2 &= \int\limits_{\{y\in\mathfrak{M}_{m,k}:\ yy'<I_m\}} |I_m - yy'|^{(\lambda-k)/2-d} dy \\
&= 2^{-k} \int\limits_{V_{m,k}} dv \int\limits_{\{q\in\mathcal{P}_k:\ vqv'<I_m\}} |I_m - vqv'|^{(\lambda-k)/2-d} |q|^{(m-k-1)/2} dq \\
&= 2^{-k}\sigma_{m,k} \int\limits_0^{I_k} |I_k - q|^{(\lambda-m-k-1)/2} |q|^{(m-k-1)/2} dq \\
&= 2^{-k}\sigma_{m,k} B_k\left(\frac{m}{2}, \frac{\lambda-m}{2}\right),
\end{aligned}$$

and we get the same.

APPENDIX B

# A counterexample to Theorem 4.27

Let us prove that the function

(B.1) $$F(x) = F_0(x'x) = |2I_m + x'x|^{-(n+m-1)/2p}(\log|2I_m + x'x|)^{-1}$$

belongs to $L^p(\mathfrak{M}_{n,m})$, but

(B.2) $$\hat{F}(\xi, t) \equiv \infty \quad \text{for} \quad p \geq p_0 = (n+m-1)/(k+m-1).$$

We shall put " $\simeq$ ", " $\lesssim$ " , and " $\gtrsim$ " instead of "=", "$\leq$", and "$\geq$", respectively, if the corresponding relation holds up to a constant multiple. To prove (B.2), owing to (4.34), we have

$$\hat{F}(\xi, t) = \int\limits_{\mathfrak{M}_{k,m}} |2I_m + \omega'\omega + s|^{-(n+m-1)/2p}(\log|2I_m + \omega'\omega + s|)^{-1} d\omega,$$

where $s = t't$. This gives ( set $\omega = y(2I_m + s)^{1/2}$, $\quad d\omega = |2I_m + s|^{k/2} dy$)

$$\begin{aligned} \hat{F}(\xi, t) &= |2I_m + s|^{k/2-(n+m-1)/2p} I_1(z), \qquad z = \log|2I_m + s|, \\ I_1(z) &= \int\limits_{\mathfrak{M}_{k,m}} |I_m + y'y|^{-(n+m-1)/2p}(z + \log|I_m + y'y|)^{-1} dy. \end{aligned}$$

For $k \geq m$, by Lemma 1.11, we obtain

$$I_1(z) \simeq \int\limits_{\mathcal{P}_m} |r|^{k/2}|I_m + r|^{-(n+m-1)/2p}(z + \log|I_m + r|)^{-1} d_*r \ ,$$

$d_*r = |r|^{-(m+1)/2} dr$. Let us pass to polar coordinates on $\mathcal{P}_m$:

(B.3) $$r = \gamma' a \gamma, \quad \gamma \in O(m), \quad a = \mathrm{diag}(a_1, \dots, a_m), \quad a_j > 0.$$

Then

(B.4) $$d_*r = c_m\, v_m(a) \Big( \prod_{j=1}^m a_j^{-(m+1)/2}\, da_j \Big) d\gamma,$$

where

$$v_k(a) = \prod_{1 \leq i < j \leq k} |a_i - a_j|, \quad k = 2, \dots, m, \quad c_m^{-1} = \pi^{-(m^2+m)/4} \prod_{j=1}^m j\ \Gamma(j/2),$$

[**T**, p. 23, 43]. By (B.3) and (B.4),

$$\begin{aligned}I_1(z) &\simeq \int_0^\infty \dots \int_0^\infty v_m(a)\left[z+\sum_{j=1}^m \log(1+a_j)\right]^{-1}\\ &\times \left[\prod_{j=1}^m a_j^{(k-m-1)/2}(1+a_j)^{-(n+m-1)/2p}da_j\right].\end{aligned}$$

The transformation $a_j+1=b_j$ yields

$$I_1(z) \gtrsim \int_2^3 b_1^\lambda db_1 \int_4^5 b_2^\lambda db_2 \dots \int_{2m}^\infty v_m(b)\left[z+\sum_{j=1}^m \log b_j\right]^{-1} b_m^\lambda db_m, \tag{B.5}$$

where $\lambda=(k-m-1)/2-(n+m-1)/2p$. Note that in (B.5),

$$v_m(b)=\prod_{1\le i<j\le m}|b_j-b_i| = \prod_{i=1}^{m-1}|b_m-b_i| \prod_{1\le i<j\le m-1}|b_j-b_i| \gtrsim (b_m-2m)^{m-1}$$

and $\log b_1<\dots<\log b_m$. Hence, for $p\ge p_0$,

$$I_1(z)\gtrsim \int_{2m}^\infty b_m^\lambda (b_m-2m)^{m-1}\left[z+m\log b_m\right]^{-1}db_m=\infty.$$

If $k<m$ then

$$\begin{aligned}I_1(z) &= \int_{\mathfrak{M}_{m,k}} |I_m+\omega\omega'|^{-(n+m-1)/2p}(z+\log|I_m+\omega\omega'|)^{-1}d\omega\\ &\simeq \int_{V_{m,k}} dv \int_{\mathcal{P}_k} |q|^{(m-k-1)/2}|I_m+vqv'|^{-(n+m-1)/2p}(z+\log|I_m+vqv'|)^{-1}dq.\end{aligned}$$

Using the equality $|I_m+vqv'|=|I_k+q|$ [**Mu**, p. 575], and setting $q=\beta' b\beta$, $\beta\in O(k)$, $b=\mathrm{diag}(b_1,\dots,b_k)$, $b_j>0$, we have

$$\begin{aligned}I_1(z) &\simeq \int_{\mathcal{P}_k} |q|^{(m-k-1)/2}|I_k+q|^{-(n+m-1)/2p}(z+\log|I_k+q|)^{-1}dq\\ &\simeq \int_0^\infty \dots \int_0^\infty \Big[\prod_{j=1}^k b_j^{(m-k-1)/2}(1+b_j)^{-(n+m-1)/2p}\Big]\\ &\times \left[z+\sum_{j=1}^k \log(1+b_j)\right]^{-1} v_k(b)\,db_1\dots db_k.\end{aligned}$$

Proceeding as above, for $\nu=(m-k-1)/2-(n+m-1)/2p$, $p\ge p_0$, we obtain

$$I_1(z)\gtrsim \int_{2k}^\infty b_k^\nu (b_k-2k)^{k-1}\left[z+k\log b_k\right]^{-1}db_k=\infty.$$

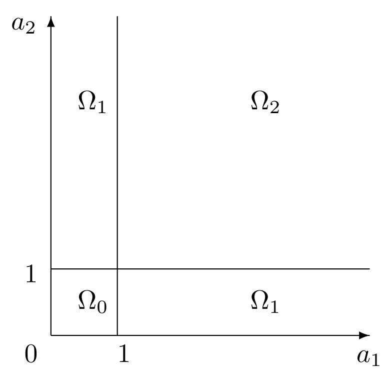
Let $I_2 = \|F\|_p^p$. To complete the proof it remains to show that $I_2 < \infty$. As above, we have

$$\begin{aligned} I_2 &\simeq \int\limits_{\mathcal{P}_m} |r|^{n/2} |F_0(r)|^p d_* r \\ &= \int\limits_{\mathcal{P}_m} |r|^{n/2} |2I_m + r|^{-(n+m-1)/2} (\log|2I_m + r|)^{-p} d_* r \\ &\simeq \int\limits_{\mathbb{R}_+^m} \Big[\prod_{j=1}^m a_j^{n/2-(m+1)/2} (2+a_j)^{-(n+m-1)/2}\Big] \\ &\times \left[\sum_{j=1}^m \log(2+a_j)\right]^{-p} v_m(a)\, da_1 \dots da_m, \end{aligned}$$

where $\mathbb{R}_+^m$ is the set of points $a = (a_1, \dots, a_m)$ with positive coordinates. Let us split $\mathbb{R}_+^m$ into $m+1$ pieces $\Omega_0, \dots, \Omega_m$, where

$$\begin{aligned} \Omega_0 &= \{a : \ 0 < a_j < 1 \quad \text{for all} \quad j = 1, \dots, m\}, \\ \Omega_m &= \{a : \ a_j > 1 \quad \text{for all} \quad j = 1, \dots, m\}, \end{aligned}$$

and $\Omega_\ell$ $(\ell = 1, 2, \dots, m-1)$ is the set of points $a \in \mathbb{R}_+^m$ having $\ell$ coordinates $> 1$ and $m - \ell$ coordinates $\leq 1$. Then $I_2 \simeq \sum\limits_{\ell=0}^m A_\ell$, where $A_\ell = \int_{\Omega_\ell}(...)$.




(The case $m = 2$)

For $p > 1$, we have

$$\begin{aligned} A_0 &= \int_0^1 \dots \int_0^1 \Big[\prod_{j=1}^m a_j^{n/2-(m+1)/2} (2+a_j)^{-(n+m-1)/2}\Big] \\ &\times \left[\sum_{j=1}^m \log(2+a_j)\right]^{-p} v_m(a)\, da_1 \dots da_m \\ &\lesssim \prod_{j=1}^m \int_0^1 a_j^{n/2-(m+1)/2} da_j < \infty. \end{aligned}$$

In order to estimate $A_m$, we first note that the integrand is a symmetric function of $a_1, \dots, a_m$. Hence, $A_m = (m!)B_m$, where $B_m$ is the integral over the set $\{a : a \in \Omega_m;\ a_1 > a_2 > \dots > a_m\}$. Using obvious inequalities

$$a_j^{n/2-(m+1)/2}(2+a_j)^{-(n+m-1)/2} < a_j^{-m}, \tag{B.6}$$

we have

$$\begin{aligned} B_m \quad &\lesssim \quad \int_1^\infty \frac{da_1}{a_1^m \log^p(2+a_1)} \int_1^{a_1} a_2^{-m} da_2 \dots \int_1^{a_{m-2}} a_{m-1}^{-m} \prod_{1\le i<j\le m-1} |a_i - a_j|\, da_{m-1} \\ &\times \int_1^{a_{m-1}} a_m^{-m} \prod_{i=1}^{m-1} |a_i - a_m|\, da_m. \end{aligned}$$

Since

$$\begin{aligned} \int_1^{a_{m-1}} a_m^{-m} \prod_{i=1}^{m-1} |a_i - a_m|\, da_m \quad &\le \quad \Big(\prod_{i=1}^{m-1} a_i\Big) \int_1^{a_{m-1}} \frac{da_m}{a_m^m} \\ &= \quad \frac{1 - a_{m-1}^{1-m}}{m-1} \prod_{i=1}^{m-1} a_i \lesssim \prod_{i=1}^{m-1} a_i, \end{aligned}$$

it follows that

$$B_m \lesssim \int_1^\infty \frac{da_1}{a_1^{m-1} \log^p(2+a_1)} \int_1^{a_1} a_2^{1-m} da_2 \dots \int_1^{a_{m-2}} a_{m-1}^{1-m} \prod_{1\le i<j\le m-1} |a_i - a_j|\, da_{m-1}.$$

Repeating this process, we get

$$B_m \lesssim \int_1^\infty \frac{d\,a_1}{a_1 \log^p(2+a_1)} < \infty.$$

Let us estimate $A_\ell$, $\ell = 1, \dots, m-1$. Owing to the symmetry, it suffices to consider the integral

$$\begin{aligned} B_\ell \quad &= \quad \int_1^\infty da_1 \int_1^{a_1} da_2 \dots \int_1^{a_{\ell-1}} \Big[\prod_{j=1}^{\ell} a_j^{n/2-(m+1)/2}(2+a_j)^{-(n+m-1)/2}\Big] \\ &\times \quad J(a_1, \dots, a_\ell)\, da_\ell, \end{aligned}$$

where

$$\begin{aligned} J(a_1, \dots, a_\ell) \quad &= \quad \int_0^1 \dots \int_0^1 \Big[\prod_{j=\ell+1}^{m} a_j^{n/2-(m+1)/2}(2+a_j)^{-(n+m-1)/2}\Big] \\ &\times \quad \left[\sum_{j=1}^m \log(2+a_j)\right]^{-p} v_m(a)\, da_{\ell+1} \dots da_m. \end{aligned}$$

We use the inequality

$$v_m(a) = \prod_{1\le i<j\le m} |a_i - a_j| \le \prod_{i=1}^{\ell} \prod_{j=i+1}^{m} |a_i - a_j| \le \prod_{1\le i<j\le \ell} |a_i - a_j| \prod_{i=1}^{\ell} (a_i + 1)^{m-\ell}.$$

This gives

$$J(a_1, \dots, a_\ell) \le [\log(2 + a_1)]^{-p} \prod_{1\le i<j\le \ell} |a_i - a_j| \prod_{i=1}^{\ell} (a_i + 1)^{m-\ell},$$

and, owing to (B.6),

$$\begin{aligned} B_\ell \quad &\lesssim \quad \int_1^{\infty} \frac{da_1}{a_1^{\ell} \log^p(2 + a_1)} \int_1^{a_1} a_2^{-\ell} da_2 \dots \int_1^{a_{\ell-2}} a_{\ell-1}^{-\ell} \prod_{1\le i<j\le \ell-1} |a_i - a_j| \, da_{\ell-1} \\ &\times \int_1^{a_{\ell-1}} a_\ell^{-\ell} \prod_{i=1}^{\ell-1} |a_i - a_\ell| \, da_\ell. \end{aligned}$$

Proceeding as above (see the argument for $B_m$), we obtain

$$B_\ell \lesssim \int_1^{\infty} \frac{d\, a_1}{a_1 \log^p(2 + a_1)} < \infty.$$

Thus, $I_2 < \infty$, and we are done.

APPENDIX C

# Some facts from algebra

We recall some well-known facts repeatedly used throughout the monograph. We do this for convenience of the reader by taking into account that the literature on this subject is rather sparse. More results from matrix algebra can be found, i.e., in [**Mu**] and [**FZ**].

Notation:

- $\mathfrak{M}_{n,m}$ is the space of real matrices $x = (x_{i,j})$ having $n$ rows and $m$ columns.
- $x'$ denotes the transpose of $x$.
- $\text{rank}(x) = \text{rank of } x$.
- $I_m$ is the identity $m \times m$ matrix.
- $\text{tr}(a)$ is the trace of the square matrix $a$.
- $|a|$ is the absolute value of the determinant $\det(a)$.
- $O(m)$ is the group of orthogonal $m \times m$ matrices.
- $V_{n,m} = \{v \in \mathfrak{M}_{n,m} : v'v = I_m\}$ is the Stiefel manifold.

Some useful facts:

**1.** If $x$ is $n \times m$ and $y$ is $m \times n$, then $|I_n + xy| = |I_m + yx|$.
**2.** If $a$ is a square matrix and $|a| \neq 0$ then $(a^{-1})' = (a')^{-1}$.
**3.** $\text{rank}(x) = \text{rank}(x') = \text{rank}(xx') = \text{rank}(x'x)$.
**4.** $\text{rank}(xy) \le \min(\text{rank}(x), \text{rank}(y))$.
**5.** $\text{rank}(x + y) \le \text{rank}(x) + \text{rank}(y)$.
**6.** $\text{rank}(axb) = \text{rank}(x)$ if $a$ and $b$ are nonsingular square matrices.
**7.** $\text{tr}(a) = \text{tr}(a')$.
**8.** $\text{tr}(a + b) = \text{tr}(a) + \text{tr}(b)$.
**9.** If $x$ is $n \times m$ and $y$ is $m \times n$ then $\text{tr}(xy) = \text{tr}(yx)$.
**10.** Let $\mathcal{S}_m$ be the space of $m \times m$ real symmetric matrices $r = (r_{i,j})$, $r_{i,j} = r_{j,i}$. A matrix $r \in \mathcal{S}_m$ is called positive definite (positive semi-definite) if $u'ru > 0$ ($u'ru \ge 0$) for all vectors $u \neq 0$ in $\mathbb{R}^m$; this is commonly expressed as $r > 0$ ($r \ge 0$). Given $r_1$ and $r_2$ in $\mathcal{S}_m$, the inequality $r_1 > r_2$ means $r_1 - r_2 > 0$.
**(i)** If $r > 0$ then $r^{-1} > 0$.
**(ii)** $x'x \ge 0$ for any matrix $x$.
**(iii)** If $r \ge 0$ then $r$ is nonsingular if and only if $r > 0$.
**(iv)** If $r > 0,\ s > 0$, and $r - s > 0$ then $s^{-1} - r^{-1} > 0$ and $|r| > |s|$.
**(v)** A symmetric matrix is positive definite (positive semi-definite) if and only if all its eigenvalues are positive (non-negative).

**(vi)** If $r \in \mathcal{S}_m$ then there exists an orthogonal matrix $\gamma \in O(m)$ such that $\gamma' r \gamma = \Lambda$ where $\Lambda =$diag$(\lambda_1, \ldots, \lambda_m)$. Each $\lambda_j$ is real and equal to the $j$th eigenvalue of $r$.

**(vii)** If $r$ is a positive semi-definite $m \times m$ matrix then there exists a positive semi-definite $m \times m$ matrix, written as $r^{1/2}$, such that $r = r^{1/2} r^{1/2}$. If $r = \gamma \Lambda \gamma'$, $\gamma \in O(m)$, then $r^{1/2} = \gamma \Lambda^{1/2} \gamma'$ where $\Lambda^{1/2} =$diag$(\lambda_1^{1/2}, \ldots, \lambda_m^{1/2})$.

**11.** If $x$ is $n \times k$ and $y$ is $m \times k$, $n \geq m$, then $x'x = y'y$ if and only if there exists $v \in V_{n,m}$ such that $x = vy$. In particular, if $x$ is $n \times m$, $n \geq m$, then there exists $v \in V_{n,m}$ such that $x = vy$, where $y = (x'x)^{1/2}$.

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# Subject index

# Index of symbols