

# PRODUCTS OF SPECIAL SETS OF REAL NUMBERS

BOAZ TSABAN AND TOMASZ WEISS

ABSTRACT. We describe a simple machinery which translates results on algebraic sums of sets of reals into the corresponding results on their cartesian product. Some consequences are:

- (1) The product of a meager/null-additive set and a strong measure zero/strongly meager set in the Cantor space has strong measure zero/is strongly meager, respectively. This extends a result of Scheepers from [12].
- (2) Using Scheepers' notation for selection principles:
  - (a) Borel's Conjecture for  $S_1(\Omega, \Omega)$  implies Borel's Conjecture.
  - (b)  $S_{fin}(\Omega, \Omega^{gp}) \cap S_1(\mathcal{O}, \mathcal{O}) = S_1(\Omega, \Omega^{gp})$ .

## 1. PRODUCTS IN THE CANTOR SPACE

The Cantor space  $\mathcal{C} = \{0, 1\}^{\mathbb{N}}$  is equipped with the product topology. For distinct  $x, y \in \mathcal{C}$ , write  $N(x, y) = \min\{n : x(n) \neq y(n)\}$ . Then the topology of  $\mathcal{C}$  is generated by the following metric:

$$d(x, y) = \begin{cases} 0 & x = y \\ \frac{1}{N(x, y)+1} & \text{otherwise} \end{cases}$$

(so that  $d(x, y) \leq 1$  for all  $x, y \in \mathcal{C}$ ). A canonical measure  $\mu$  is defined on  $\mathcal{C}$  by taking the product of the uniform probability measure on  $\{0, 1\}$ . Fix a natural number  $k$ , and consider the product space  $\mathcal{C}^k$ . Define the *product metric*  $d_k$  on  $\mathcal{C}^k$  by

$$d_k((x_0, \dots, x_{k-1}), (y_0, \dots, y_{k-1})) = \max\{d(x_0, y_0), \dots, d(x_{k-1}, y_{k-1})\}.$$

Then  $d_k$  generates the topology of  $\mathcal{C}^k$ . The measure on  $\mathcal{C}^k$  is the product measure  $\mu \times \dots \times \mu$  ( $k$  times).

$\mathcal{C}$ , with the operation  $\oplus$  defined by  $(x \oplus y)(n) = x(n) + y(n) \bmod 2$  is a topological group, and therefore so is  $\mathcal{C}^k$  for all  $k$ .

**Lemma 1.1.** *The function  $\Psi_k : \mathcal{C}^k \rightarrow \mathcal{C}$  defined by*

$$\Psi_k(x_0, \dots, x_{k-1})(mk + i) = x_i(m)$$

*for each  $m$  and each  $i < k$ , is a bi-Lipschitz measure preserving group isomorphism.*

---

1991 *Mathematics Subject Classification.* Primary: 26A03; Secondary: 37F20, 03E75.

*Key words and phrases.* Special sets of real numbers, products.

The first author is partially supported by the Golda Meir Fund and the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany).



We will say that  $\mathcal{P}$  is *0-productive* if we only require that (1) is satisfied, and *iso-productive* if we only require that (2) is satisfied.

As changing the order of coordinates is a bi-Lipschitz measure preserving group isomorphism, we have the following.

**Lemma 1.5.** *Assume that  $\mathcal{P} \subseteq \tilde{P}(\mathcal{C})$  is semiproductive. Then for each  $k, l$ , and  $X \in \mathcal{P} \cap \mathcal{C}^k$ , if 0 is the zero element of  $\mathcal{C}^l$ , then  $\{0\} \times X \in \mathcal{P}$ .*

**Theorem 1.6.** *Assume that  $\mathcal{I}$ ,  $\mathcal{J}$ , and  $\mathcal{K}$  are subsets of  $\tilde{P}(\mathcal{C})$  such that  $\mathcal{I}$  and  $\mathcal{J}$  are semiproductive, and  $\mathcal{K}$  is iso-productive. If*

*for each  $X \in \mathcal{I} \cap P(\mathcal{C})$  and  $Y \in \mathcal{J} \cap P(\mathcal{C})$ ,  $X \oplus Y \in \mathcal{K}$ ,*

*then*

*for each  $X \in \mathcal{I}$  and  $Y \in \mathcal{J}$ ,  $X \times Y \in \mathcal{K}$ .*

*Proof.* Assume that  $X \in \mathcal{I} \cap P(\mathcal{C}^k)$  and  $Y \in \mathcal{J} \cap P(\mathcal{C}^l)$ . Then  $\tilde{X} = \Psi_k[X] \in \mathcal{I} \cap P(\mathcal{C})$ , and  $\tilde{Y} = \Psi_l[Y] \in \mathcal{J} \cap P(\mathcal{C})$ .

$$\begin{aligned}\Psi_2[\tilde{X} \times \tilde{Y}] &= \Psi_2[(\tilde{X} \times \{0\}) \oplus (\{0\} \times \tilde{Y})] = \\ &= \Psi_2[\tilde{X} \times \{0\}] \oplus \Psi_2[\{0\} \times \tilde{Y}] = X' \oplus Y',\end{aligned}$$

Thus,  $X' \oplus Y' \in \mathcal{K}$ . As  $\Psi_2$  is bijective,  $\tilde{X} \times \tilde{Y} = \Psi_2^{-1}[X' \oplus Y'] \in \mathcal{K}$ . As  $\Psi_k \times \Psi_l$  is a bi-Lipschitz measure preserving group isomorphism and  $\Psi_k^{-1} \times \Psi_l^{-1} : \tilde{X} \times \tilde{Y} \rightarrow X \times Y$  is surjective,  $X \times Y \in \mathcal{K}$ .  $\square$

Almost all properties of special sets of reals which were considered in the literature are closed under taking bi-Lipschitz images and products with singletons, e.g., Hausdorff dimension, strong measure zero, and all properties in the special-sets version of the Cichoń Diagram [8] as well as in the Scheepers Diagram (see Section 4) and its extensions [13, 14]; see, e.g., [5, 11] for many more examples. The property of having measure zero is not closed under taking Lipschitz images:  $\mathcal{C} \times \{0\}$  has measure zero in  $\mathcal{C}^2$ , but its Lipschitz image  $\mathcal{C}$  does not. However this property is by definition closed under taking measure preserving images.

Assume that  $\mathcal{I} \subseteq \tilde{P}(\mathcal{C})$ . A subset  $X$  of  $\mathcal{C}^k$  is  $\mathcal{I}$ -additive if for each  $I \in \mathcal{I} \cap P(\mathcal{C}^k)$ ,  $X \oplus I \in \mathcal{I}$ . Clearly if  $X, Y \subseteq \mathcal{C}$  are  $\mathcal{I}$ -additive, then  $X \oplus Y$  is  $\mathcal{I}$ -additive. More generally, we will say that a subset  $X$  of  $\mathcal{C}^k$  is  $(\mathcal{I}, \mathcal{J})$ -additive if for each  $I \in \mathcal{I} \cap \mathcal{C}^k$ ,  $X \oplus I \in \mathcal{J}$ . Let  $\mathcal{I}^*$  and  $(\mathcal{I}, \mathcal{J})^*$  denote the classes of all  $\mathcal{I}$ -additive sets and  $(\mathcal{I}, \mathcal{J})$ -additive sets, respectively.

**Theorem 1.7.** *Assume that  $\mathcal{I}, \mathcal{J}, \mathcal{K} \subseteq \tilde{P}(\mathcal{C})$  are iso-productive, and  $(\mathcal{J}, \mathcal{K})^*$ ,  $(\mathcal{I}, \mathcal{J})^*$  are 0-productive. Then for each  $X \in (\mathcal{J}, \mathcal{K})^*$  and  $Y \in (\mathcal{I}, \mathcal{J})^*$ ,  $X \times Y \in (\mathcal{I}, \mathcal{K})^*$ . In particular, if  $\mathcal{I}$  is iso-productive and  $\mathcal{I}^*$  is 0-productive, then  $\mathcal{I}^*$  is closed under taking finite products.*

*Proof.* Assume that  $X \in (\mathcal{J}, \mathcal{K})^* \cap P(\mathcal{C})$ ,  $Y \in (\mathcal{I}, \mathcal{J})^* \cap P(\mathcal{C})$ , and  $I \in \mathcal{I}$ . Then  $Y \oplus I \in \mathcal{J}$  and therefore  $X \oplus (Y \oplus I) \in \mathcal{K}$ . Thus  $X \oplus Y \in (\mathcal{I}, \mathcal{K})^*$ .

**Lemma 1.8.** *Assume that  $\mathcal{I}, \mathcal{J} \subseteq \tilde{P}(\mathcal{C})$  are iso-productive. Then  $(\mathcal{I}, \mathcal{J})^*$  is iso-productive.*

*Proof.* Assume that  $X \in (\mathcal{I}, \mathcal{J})^* \cap P(\mathcal{C}^k)$  and  $\Phi : \mathcal{C}^k \rightarrow \mathcal{C}^l$  is a bi-Lipschitz measure preserving group isomorphism. Then for each  $I \in \mathcal{I} \cap P(\mathcal{C}^l)$ ,

$$\Phi[X] \oplus I = \Phi[X \oplus \Phi^{-1}[I]].$$

As  $\Phi^{-1}[I] \in \mathcal{I}$ ,  $X \oplus \Phi^{-1}[I] \in \mathcal{J}$  and therefore  $\Phi[X] \oplus I \in \mathcal{J}$ , that is,  $\Phi[X] \in (\mathcal{I}, \mathcal{J})^*$ .  $\square$

Thus  $(\mathcal{J}, \mathcal{K})^*$  and  $(\mathcal{I}, \mathcal{J})^*$  are semiproductive, and the theorem follows from Theorem 1.6.  $\square$

Following is a useful criterion for the 0-productivity required in Theorem 1.7.

**Lemma 1.9.** *Assume that  $\mathcal{I}, \mathcal{J} \subseteq \tilde{P}(\mathcal{C})$  are iso-productive, and for the element  $0 \in \mathcal{C}$  and each  $X \in (\mathcal{I}, \mathcal{J})^* \cap P(\mathcal{C})$ ,  $\Psi_2[X \times \{0\}] \in (\mathcal{I}, \mathcal{J})^*$ . Then  $(\mathcal{I}, \mathcal{J})^*$  is 0-productive. In particular, if  $\mathcal{I}$  is iso-productive and for each  $X \in \mathcal{I}^* \cap P(\mathcal{C})$ ,  $\Psi_2[X \times \{0\}] \in \mathcal{I}^*$ , then  $\mathcal{I}^*$  is 0-productive.*

*Proof.* Assume that  $X \in (\mathcal{I}, \mathcal{J})^* \cap P(\mathcal{C}^k)$  and fix  $l$ . By Lemma 1.8,  $\tilde{X} = \Psi_k[X] \in (\mathcal{I}, \mathcal{J})^* \cap P(\mathcal{C})$ . Thus  $\Psi_2 \circ (\Psi_k \times \Psi_l)[X \times \{0\}] = \Psi_2[\tilde{X} \times \{0\}] \in (\mathcal{I}, \mathcal{J})^*$ . Applying Lemma 1.8 again, we get that  $X \times \{0\} \in (\mathcal{I}, \mathcal{J})^*$ .  $\square$

We now give some applications. Let  $X$  be a metric space. Following Borel, we say that  $X$  has *strong measure zero* if for each sequence of positive reals  $\{\epsilon_n\}_{n \in \mathbb{N}}$ , there exists a cover  $\{I_n\}_{n \in \mathbb{N}}$  of  $X$  such that  $\text{diam}(I_n) < \epsilon_n$  for all  $n$ .  $X$  has *Hausdorff dimension zero* if for each positive reals  $\epsilon, \delta$  there exists a cover  $\{I_n\}_{n \in \mathbb{N}}$  of  $X$  with  $\sum_n \text{diam}(I_n)^\delta < \epsilon$ . Clearly strong measure zero implies Hausdorff dimension zero.  $X$  has the *Hurewicz property* if for each sequence  $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$  of open covers of  $X$  there exist finite subsets  $\mathcal{F}_n \subseteq \mathcal{U}_n$ ,  $n \in \mathbb{N}$ , such that  $X \subseteq \bigcup_n \bigcap_{m > n} \mathcal{U}_m$ . Let SMZ (respectively,  $\mathcal{H}$ ) denote the collections of metric spaces having strong measure zero (respectively, the Hurewicz property).

In [12], Scheepers proves the following Theorem 1.10. In [7], the corresponding theorem is proved with  $X \oplus Y$  instead of  $X \times Y$  (for a direct, combinatorial proof of this result, see Theorem A.3 in the appendix). It is easy to see that the collections  $\mathcal{H} \cap \tilde{P}(\mathcal{C})$  and SMZ  $\cap \tilde{P}(\mathcal{C})$  (and, therefore,  $\mathcal{H} \cap \text{SMZ} \cap \tilde{P}(\mathcal{C})$ ) are semiproductive. By Theorem 1.6, the first result actually follows from the second. (For yet another proof of Theorem 1.10, see Theorem A.2 in the appendix.)

**Theorem 1.10** (Scheepers [12]). *Assume that  $X \in \mathcal{H} \cap \text{SMZ} \cap \tilde{P}(\mathcal{C})$  and  $Y \in \text{SMZ} \cap \tilde{P}(\mathcal{C})$ . Then  $X \times Y \in \text{SMZ}$ .*

We now treat the classes of meager-additive and null-additive sets. Let  $\mathcal{M}$  and  $\mathcal{N}$  denote the meager (i.e., first category) and null (i.e., measure zero) sets, respectively.

**Theorem 1.11.** *The classes  $\mathcal{M}^* \cap \tilde{P}(\mathcal{C})$  and  $\mathcal{N}^* \cap \tilde{P}(\mathcal{C})$  are semiproductive, and are closed under taking finite products.*

*Proof.* Clearly, the classes  $\mathcal{M} \cap \tilde{P}(\mathcal{C})$  and  $\mathcal{N} \cap \tilde{P}(\mathcal{C})$  are semiproductive. By Theorem 1.7, it is enough to show that the classes  $\mathcal{M}^* \cap \tilde{P}(\mathcal{C})$  and  $\mathcal{N}^* \cap \tilde{P}(\mathcal{C})$  are 0-productive. We first treat  $\mathcal{M}^*$ .







We do not know whether the analogue of Theorem 1.11 in the Euclidean space holds.

**Problem 2.6.** *Are the classes  $\mathcal{M}^* \cap \tilde{P}(\mathbb{R})$  and  $\mathcal{N}^* \cap \tilde{P}(\mathbb{R})$  0-productive?*

If  $\mathcal{M}^* \cap \tilde{P}(\mathbb{R})$  is 0-productive, then it is closed under taking finite products. (The same assertion holds for  $\mathcal{N}^* \cap \tilde{P}(\mathbb{R})$ .)

### 3. THE EUCLIDEAN SPACE THROUGH THE LOOKING GLASS

The results in the Section 2 are not easy to use, as one should verify first that the additive results given in the literature for  $\mathbb{R}$  actually hold in  $\mathbb{R}^k$  for all  $k$ . We suggest here another approach, which covers some of the cases of interest.

**Definition 3.1.** The function  $T : \{0, 1\}^{\mathbb{N}} \rightarrow [0, 1]$  is defined by

$$x \mapsto \sum_{i \in \mathbb{N}} \frac{x(i)}{2^{i+1}} = 0.x(0)x(1)x(2)\dots,$$

where the last term is in base 2. Let  $C$  denote the collection of all eventually constant elements of  $\{0, 1\}^{\mathbb{N}}$ , and  $\mathbb{Q}_2 = T[C]$  denote the 2-adic rational numbers in  $[0, 1]$ .

**Lemma 3.2** (folklore).

- (1)  $T$  is a uniformly continuous surjection.
- (2)  $C$  is countable.
- (3)  $T : \{0, 1\}^{\mathbb{N}} \setminus C \rightarrow [0, 1] \setminus \mathbb{Q}_2$  is a homeomorphism which preserves measure in both directions.

*Proof.* (1)  $T$  is continuous on its compact domain, and clearly it is onto.

(2) is obvious, and the only nontrivial part of (3) is that  $T^{-1}$  is continuous on  $[0, 1] \setminus \mathbb{Q}_2$  (it is not uniformly continuous: Take  $x_n = 0.10^n\overline{01}$  and  $y_n = 0.01^n\overline{01}$ , then  $x_n - y_n \rightarrow 0$ , but  $d(T^{-1}(x_n), T^{-1}(y_n)) = 1$  for all  $n$ ). Write  $\tilde{x}$  for  $T^{-1}(y)$ . If  $x_n \rightarrow x$  are elements of  $[0, 1] \setminus \mathbb{Q}_2$  then from some  $n$  onwards, the  $\tilde{x}_n(0) = \tilde{x}(0)$ : Assume that this is not the case. Then by moving to a subsequence we may assume that for all  $n$ ,  $\tilde{x}(0) \neq \tilde{x}_n(0)$ . Assume that  $\tilde{x}(0) = 0$  and  $\tilde{x}_n(0) = 1$  for all  $n$  (the other case is similar). Let  $k = \min\{m > 0 : x(m) = 0\}$  (recall that  $\tilde{x}$  is not eventually constant). Then  $x = 0.01^{k-1}0\dots$ , thus

$$x_n - x \geq 0.1 - 0.01^{k-1}0\overline{1} = 0.1 - 0.01^k = 0.0^k 1 = \frac{1}{2^{k+1}}$$

for all  $n$ , a contradiction.

An inductive argument shows that for each  $k$ ,  $\tilde{x}_n(k) = \tilde{x}(k)$  for all large enough  $n$ .  $\square$

In principle, we can use the function  $T$  to translate questions about products in  $\mathbb{R}$  into questions about products in  $\mathcal{C}$ , apply the results of Section 1, and translate back to  $\mathbb{R}$ . The problem is that in the first test-case that comes to mind, namely, Scheepers' Theorem 2.5, we must deal with strong measure zero sets. By (1) of Lemma 3.2, if  $Y$  has strong measure zero then so does  $T[Y]$ . The other direction does not follow from

Lemma 3.2, since a homeomorphic image of a strong measure zero set need not have strong measure zero [10].

**Proposition 3.3.** *For each  $k$ ,  $T^k : \mathcal{C}^k \rightarrow [0, 1]^k$  preserves strong measure zero sets in both directions.*

*Proof.* Since  $T$  is uniformly continuous,  $T^k$  is uniformly continuous. Thus, if  $X \subseteq \mathcal{C}^k$  has strong measure zero, then so does  $T^k[X]$ .

We now prove the other direction. The following is an easy exercise.

**Lemma 3.4** (folklore). *Assume that there exists  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that the metric space  $(X, d)$  has the following property: For each sequence  $\{\epsilon_n\}_{n \in \mathbb{N}}$  of positive reals, there exist a cover  $\{I_m^n : n \in \mathbb{N}, m \leq f(n)\}$  of  $X$  satisfying  $\text{diam}(I_m^n) < \epsilon_n$  for each  $n$  and  $m$ . Then  $(X, d)$  has strong measure zero.*

To use Lemma 3.4, we make the following observation.

**Lemma 3.5.** *Assume that  $I \subseteq [0, 1]$  and  $\text{diam}(I) < \epsilon$ . Then there exist  $A_0, A_1 \subseteq \mathcal{C}$ , both with diameter  $< \epsilon$ , such that  $T^{-1}[I] \subseteq A_0 \cup A_1$ .*

*Proof.* Let  $n$  be the maximal such that  $2^{n-1}\epsilon \leq 1$ . Then there exists  $k < 2^n - 1$  such that

$$I \subseteq \left[ \frac{k}{2^n}, \frac{k+2}{2^n} \right] = \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right] \cup \left[ \frac{k+1}{2^n}, \frac{k+2}{2^n} \right].$$

There exist sequences  $s_0, s_1 \in \{0, 1\}^n$  such that for  $i = 0, 1$ , if  $A_i = \{x \in \mathcal{C} : s_i \subseteq x\}$ , then  $T[A_i] = [(k+i)/2^n, (k+i+1)/2^n]$ .  $\square$

Assume that  $Y \subseteq [0, 1]^k$  has strong measure zero, and  $\{\epsilon_n\}_{n \in \mathbb{N}}$  is a sequence of positive reals. Let  $I_n \subseteq [0, 1]^k$  be such that  $\text{diam}(I_n) < \epsilon_n$  and  $Y \subseteq \bigcup_n I_n$ . Fix  $n$ . Then  $I_n$  is contained in the product of  $k$  intervals,  $I_1^n, \dots, I_k^n \subseteq [0, 1]$ , each with diameter  $< \epsilon_n$ . For each  $i = 1, \dots, k$ , use Lemma 3.5 to obtain sets  $A_0^{n,i}, A_1^{n,i}$  as in the lemma. Then

$$I_n \subseteq \bigcup_{s \in \{0,1\}^k} \Pi_{i=1}^k T[A_{s(i)}^{n,i}] = T^k \left[ \bigcup_{s \in \{0,1\}^k} \Pi_{i=1}^k A_{s(i)}^{n,i} \right],$$

so that  $T^{-k}[I_n]$  is covered by  $2^k$  sets of diameter  $< \epsilon_n$ . Since the sets  $I_n$  cover  $Y$ , we have by Lemma 3.4 that  $T^{-k}[Y]$  has strong measure zero.  $\square$

We are now ready to give an alternative proof of Theorem 2.5, which is more self-contained than the one given in Section 2. Observe that Scheepers [12] proof of this theorem is more direct and simple. The proof given here is only meant to demonstrate the applicability of our translation machinery, which is perhaps applicable to other results which appear in the literature for algebraic sums of sets of reals.

**Theorem 3.6.** *Assume that  $X \in \mathcal{H} \cap \text{SMZ} \cap \tilde{P}(\mathbb{R})$  and  $Y \in \text{SMZ} \cap \tilde{P}(\mathbb{R})$ . Then  $X \times Y \in \text{SMZ}$ .*

*Proof.* We first use the following.





Thus, the arguments in the last proof actually establish the following.

**Theorem 4.3.** *For a set of reals  $X$ , the following are equivalent:*

- (1)  $X$  satisfies  $S_{fin}(\Omega, \Omega^{gp})$  and has strong measure-zero,
- (2)  $X$  satisfies  $S_{fin}(\Omega, \Omega^{gp})$  and  $S_1(\mathcal{O}, \mathcal{O})$ ,
- (3)  $X$  satisfies  $S_{fin}(\Omega, \Omega^{gp})$  and is meager-additive,
- (4)  $X$  satisfies  $S_{fin}(\Omega, \Omega^{gp})$  and  $S_1(\Omega, \Omega)$ ,
- (5)  $X$  satisfies  $S_1(\Omega, \Omega^{gp})$ .

*Proof.* Clearly,  $5 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$ , and  $3 \Rightarrow 1$ .

( $1 \Rightarrow 5$ ) Assume that (1) holds. In [4] it is proved that  $S_{fin}(\Omega, \Omega^{gp})$  is equivalent to satisfying the Hurewicz property  $U_{fin}(\mathcal{O}, \Gamma)$  in all finite powers. By Theorem 2.5, all finite powers of  $X$  satisfy  $U_{fin}(\mathcal{O}, \Gamma)$  and have strong measure zero. By [4],  $X$  satisfies  $S_1(\Omega, \Omega^{gp})$ .

( $1 \Rightarrow 3$ ) In [7] it is proved that every strong measure zero set of reals with the Hurewicz property is meager additive.  $\square$

The theorem also holds when  $X \subseteq \mathcal{C}$ . In this case, the quoted assertion in the last proof can be proved directly – see Theorem A.3 in the appendix.

*Acknowledgements.* We thank Marcin Kysiak for his useful suggestion regarding Theorem 2.5. (After finishing the writing of this paper, we have learned that Theorem 2.5 was already proved by Scheepers. We have given proper credit inside the paper.) We also thank Tomek Bartoszynski for the permission to publish his combinatorial proof of Theorem A.2.

## REFERENCES

- [1] T. Bartoszyński and H. Judah, *Set Theory: On the structure of the real line*, A. K. Peters, Massachusetts: 1995.
- [2] T. Bartoszynski and B. Tsaban, *Heredity topological diagonalizations and the Menger-Hurewicz Conjectures*, Proceedings of the American Mathematical Society, to appear. <http://arxiv.org/abs/math.L0/0208224>
- [3] W. Just, A. W. Miller, M. Scheepers, and P. Szeptycki, *Combinatorics of open covers II*, Topology and Its Applications, **73** (1996), 241–266.
- [4] Lj. D. R. Kočinac and M. Scheepers, *Combinatorics of open covers (VII): groupability*, Fundamenta Mathematicae **179** (2003), 131–155.
- [5] A.W. Miller, *Special subsets of the real line*, in: **Handbook of Set Theoretic Topology** (eds. K. Kunen and J.E. Vaughan), 201–233, North Holland, Amsterdam: 1984.
- [6] A. W. Miller, *The  $\gamma$ -Borel conjecture*, preprint.
- [7] A. Nowik, M. Scheepers, and T. Weiss, *The algebraic sum of sets of real numbers with strong measure zero sets*, The Journal of Symbolic Logic **63** (1998), 301–324.
- [8] J. Pawlikowski and I. Recław, *Parametrized Cichon's diagram and small sets*, Fundamenta Mathematicae **147** (1995), 135–155.
- [9] I. Recław, *Products of perfectly meagre sets*, Proceedings of the American Mathematical Society **112** (1991), 1029–1031.
- [10] F. Rothberger, *Sur des familles indénombrables de suites de nombres naturels, et les problèmes concernant la propriété C*, Proceedings of the Cambridge Philosophical Society **37** (1941), 109–126.

- [11] M. Scheepers, *Combinatorics of open covers I: Ramsey Theory*, Topology and its Applications **69** (1996), 31–62.
- [12] M. Scheepers, *Finite powers of strong measure zero sets*, The Journal of Symbolic Logic **64** (1999), 1295–1306.
- [13] M. Scheepers and B. Tsaban, *The combinatorics of Borel covers*, Topology and its Applications **121** (2002), 357–382.  
<http://arxiv.org/abs/math.GN/0302322>
- [14] B. Tsaban, *Selection principles and the minimal tower problem*, Note di Matematica, to appear.  
<http://arxiv.org/abs/math.LO/0105045>
- [15] B. Tsaban, *Problem of the month*, SPM Bulletin **7** (January 2004), 3.  
<http://arxiv.org/abs/math.GN/0401155>
- [16] B. Tsaban, *The Hurewicz covering property and slaloms in the Baire space*, submitted.  
<http://arxiv.org/abs/math.GN/0301085>
- [17] T. Weiss and B. Tsaban, *Topological diagonalizations and Hausdorff dimension*, Note di Matematica, to appear.  
<http://arxiv.org/abs/math.LO/0212009>

#### APPENDIX A. DIRECT PROOFS OF QUOTED THEOREMS

Following is Bartoszynski's (unpublished) combinatorial proof of theorem 1.10. The proof uses the following characterization of strong measure zero.

**Lemma A.1** ([1, Lemma 8.1.13]). *For  $X \subseteq \mathcal{C}$ , the following are equivalent:*

- (1)  *$X$  has strong measure zero,*
- (2) *For each  $f \in \mathbb{N}^{\mathbb{N}}$  there exists a function  $g$  such that  $g(n) \in \{0, 1\}^{f(n)}$  for all  $n$ , and for each  $x \in X$  there exist infinitely many  $n$  such that  $x \upharpoonright f(n) = g(n)$ ,*
- (3) *For each increasing sequence  $\{m_n\}_{n \in \mathbb{N}}$  there exists  $z \in \mathcal{C}$  such that for each  $x \in X$  there exist infinitely many  $n$  such that  $x \upharpoonright [m_n, m_{n+1}) = z \upharpoonright [m_n, m_{n+1})$ .*

Let  $\mathbb{N}^{\nearrow \mathbb{N}}$  denote the subspace of the Baire space  $\mathbb{N}^{\mathbb{N}}$  consisting of the increasing functions in  $\mathbb{N}^{\mathbb{N}}$ . In [16] it is proved that if  $X$  has the Hurewicz property and  $\Psi : X \rightarrow \mathbb{N}^{\nearrow \mathbb{N}}$  is continuous, then  $\Psi[X]$  admits some slalom  $h \in \mathbb{N}^{\nearrow \mathbb{N}}$ , that is, such that for each  $x \in X$  and all but finitely many  $n$ , there exists  $k$  such that  $h(n) \leq \Psi(x)(k) < h(n+1)$ . This fact will be used in the proof.

**Theorem A.2.** *Assume that  $X \subseteq \{0, 1\}^{\mathbb{N}}$  has the Hurewicz property and strong measure zero, and  $Y \subseteq \mathcal{C}$  has strong measure zero. Then  $X \times Y$  has strong measure zero.*

*Proof.* Fix  $f \in \mathbb{N}^{\nearrow \mathbb{N}}$  and let  $g$  be as in Lemma A.1 for  $X$  and  $f$ . Define a function  $\Psi : X \rightarrow \mathbb{N}^{\nearrow \mathbb{N}}$  so that for each  $x \in X$ ,  $\Psi(x)$  is the increasing enumeration of the set  $\{n : x \upharpoonright f(n) = g(n)\}$ . Then  $\Psi$  is continuous, thus there exists  $h \in \mathbb{N}^{\nearrow \mathbb{N}}$  such that for each  $x \in X$  and all but finitely many  $n$ , there exists  $k$  such that  $h(n) \leq \Psi(x)(k) < h(n+1)$ .

Consider a mapping  $\Phi$  defined on  $Y$  by

$$\Phi(y)(n) = \langle (g(k), y \upharpoonright f(k)) : h(n) \leq k < h(n+1) \rangle.$$

Then  $\Phi$  is uniformly continuous. Thus (essentially, by Lemma A.1) there exists a function  $r$  such that for all  $y \in Y$  there exist infinitely many  $n$  such that  $\Phi(y)(n) =$

$r(n)$ . From  $r$  we decode a function  $s$  such that  $s(n) \in \{0, 1\}^{f(n)} \times \{0, 1\}^{f(n)}$  by  $s(k) = r(n)(k)$  where  $n$  is such that  $h(n) \leq k < h(n+1)$ .

Then for all  $x \in X$  and  $y \in Y$  there exist infinitely many  $n$  such that  $(x \upharpoonright f(n), y \upharpoonright f(n)) = s(n)$ , which shows that  $X \times Y$  has strong measure zero.  $\square$

Using the bi-Lipschitz transformations  $\Psi_k$  of Section 1, we obtain the general Theorem 1.10 from Theorem A.2.

Bartoszynski's proof of Theorem A.2 can be modified to obtain a direct, combinatorial proof of one of the main theorems in [7] when restricted to the Cantor space.

**Theorem A.3** ([7]). *Assume that  $X \in \tilde{P}(\mathcal{C})$ ,  $X$  has the Hurewicz property  $U_{fin}(\mathcal{O}, \Gamma)$ , and strong measure zero. Then  $X$  is meager-additive.*

*Proof.* We prove the result for  $X \subseteq \mathcal{C}$  and use the machinery of Section 1 to deduce the general theorem.

Assume that  $\{m_n\}_{n \in \mathbb{N}}$  is an arbitrary increasing sequence. By Lemma 1.12, it suffices to find a sequence  $\{l_n\}_{n \in \mathbb{N}}$  and  $z \in \mathcal{C}$  such that for each  $x \in X$  and all but finitely many  $n$ ,  $l_n \leq m_k < m_{k+1} \leq l_{n+1}$  and  $x \upharpoonright [m_k, m_{k+1}] = z \upharpoonright [m_k, m_{k+1}]$  for some  $k$ .

By Lemma A.1, there exists  $z \in \mathcal{C}$  such that for each  $x \in X$  there exist infinitely many  $n$  such that  $x \upharpoonright [m_n, m_{n+1}] = z \upharpoonright [m_n, m_{n+1}]$ . Again, for each  $x \in X$  let  $\Psi(x)$  be the increasing enumeration of these  $n$ s, and use the fact that  $X$  has the Hurewicz property to find a slalom  $h \in \mathbb{N}^{\nearrow \mathbb{N}}$  for  $\Psi[X]$ .

Take  $l_n = m_{h(n)}$  for each  $n$ . Fix  $x \in X$ . Since  $h$  is a slalom for  $\Psi[X]$ , for all but finitely many  $n$  there exists  $k$  such that for  $i = \Psi(x)(k)$ ,  $h(n) \leq i < h(n+1)$ . Then

$$l_n = m_{h(n)} \leq m_i < m_{i+1} \leq m_{h(n+1)} = l_{n+1},$$

and by the definition of  $\Psi(x)$ ,  $x \upharpoonright [m_i, m_{i+1}] = z \upharpoonright [m_i, m_{i+1}]$ .  $\square$

EINSTEIN INSTITUTE OF MATHEMATICS, HEBREW UNIVERSITY OF JERUSALEM, GIVAT RAM,  
JERUSALEM 91904, ISRAEL

*E-mail address:* tsaban@math.huji.ac.il

*URL:* <http://www.cs.biu.ac.il/~tsaban>

INSTITUTE OF MATHEMATICS, AKADEMIA PODLASKA 08-119 SIEDLCE, POLAND

*E-mail address:* tomaszweiss@go2.pl