AN ANALOGUE OF THE FUGLEDE FORMULA IN INTEGRAL GEOMETRY ON MATRIX SPACES

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Abstract. The well known formula of B. Fuglede expresses the mean value of the Radon k-plane transform on $\mathbb{R}^n$ as a Riesz potential. We extend this formula to the space of $n \times m$ real matrices and show that the corresponding matrix k-plane transform $f \to \hat{f}$ is injective if and only if $n - k \geq m$. Different inversion formulas for this transform are obtained. We assume that $f \in L^p$ or f is a continuous function satisfying certain "minimal" conditions at infinity.

Preprint 2003/2004, No. 01

1. Introduction

In 1958, B. Fuglede [Fu] proved the following remarkable formula

$$c(\hat{f})^\vee(x) = (I^k f)(x), \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $\hat{f} \equiv \hat{f}(\tau)$ is the integral of $f(x)$ over a k-dimensional plane τ, $0 < k < n$; $(\hat{f})^\vee(x)$ is the mean value of $\hat{f}(\tau)$ over all k-planes through x, $(I^k f)(x)$ denotes the Riesz potential of order k, and $c = c(n,k)$ is a constant. For $k = n-1$, when τ is a hyperplane, this formula was implicitly exhibited by J. Radon [R] who was indebted to W. Blaschke for this idea. A consequence of (1.1) is the inversion formula

$$f = c(-\Delta)^{k/2}(\hat{f})^\vee \tag{1.2}$$

in which Δ denotes the Laplace operator.

2000 *Mathematics Subject Classification.* Primary 44A12; Secondary 47G10.

Key words and phrases. Matrix Radon transform, Fourier transform, Riesz potential, inversion formulas.

The work was supported in part by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany).

Our aim is to extend (1.1) and (1.2) to the case when x is an $n \times m$ real matrix. We note that Riesz potentials of functions of matrix argument and their generalizations arise in different contexts in harmonic analysis, integral geometry, and PDE; see [Far], [FK], [Ge], [Kh], [Ra], [St1], [Sh1]-[Sh3]. A systematic study of Radon transforms on matrix spaces was initiated by E.E. Petrov [Pe1] and continued in [Č], [Gr], [Pe2]-[Pe4], [Sh1], [Sh2]. These publications traditionally deal with Radon transforms $f \to \hat{f}$ of C^∞ rapidly decreasing functions, and employ decomposition in plane waves in order to recover f from $\hat{f}$.

We suggest another approach which is based on a matrix analogue of (1.1) (see Theorem 5.4) and allows to handle arbitrary continuous or locally integrable functions f subject to mild restrictions at infinity. These restrictions are minimal in a certain sense. We show that inequality $n - k \geq m$ is necessary and sufficient for injectivity of the matrix k-plane transform, and obtain two inversion formulas (in terms of the Fourier transform and in the form (1.2)).

Our key motivation is the following. Converting integral-geometrical entities into their matrix counterparts has specific "higher rank" features and sheds new light on classical problems in multidimensional spaces (just note the space $\mathbb{R}^N$ with $N = nm$ can be treated as a collection of $n \times m$ matrices).

Acknowledgements. We are grateful to Prof. E.E. Petrov and Dr. S.P. Khekalo for the papers they sent us, and useful discussions.

2. Preliminaries

We establish some notation and recall basic facts. Let $\mathfrak{M}_{n,m}$ be the space of real matrices $x = (x_{i,j})$ having n rows and m columns. We identify $\mathfrak{M}_{n,m}$ with the real Euclidean space $\mathbb{R}^{nm}$ and set $dx = \prod_{i=1}^n \prod_{j=1}^m dx_{i,j}$. In the following x' denotes the transpose of x, I_m is the identity $m \times m$ matrix, 0 stands for zero entries. Given square matrices a, b, q, s, etc., we denote by $|a|, |b|, |q|, |s|$ their determinants. $GL(m, \mathbb{R})$ is the group of real non-singular $m \times m$ matrices; $SO(n)$ is the group of orthogonal $n \times n$ real matrices of determinant one, endowed with the normalized invariant measure. We denote by $\mathcal{P}_m$ the cone of positive definite symmetric matrices $r = (r_{i,j})$; $\mathrm{tr}(r)$ is the trace of r, $dr = \prod_{i \leq j} dr_{i,j}$. The Lebesgue space $L^p(\mathfrak{M}_{n,m})$ and the Schwartz space $\mathcal{S}(\mathfrak{M}_{n,m})$ are identified with respective spaces on $\mathbb{R}^{nm}$.

Lemma 2.1. (see, e.g., [Mu, pp. 57–59]).
(i) If $x = ayb$, where $y \in \mathfrak{M}_{n,m}$, $a \in GL(n, \mathbb{R})$, $b \in GL(m, \mathbb{R})$, then $dx = |a|^m |b|^n dy$.

(ii) *If* $r = q'sq$, *where* $s \in \mathcal{P}_m$, $q \in GL(m, \mathbb{R})$, *then* $dr = |q|^{m+1} ds$.
(iii) *If* $r = s^{-1}$, *where* $s \in \mathcal{P}_m$, *then* $r \in \mathcal{P}_m$, *and* $dr = |s|^{-m-1} ds$.

The Siegel gamma function associated to the cone $\mathcal{P}_m$ is defined by

$$\Gamma_m(\alpha) = \int\limits_{\mathcal{P}_m} \exp(-\mathrm{tr}(r)) |r|^{\alpha - d} dr, \qquad d = (m+1)/2. \tag{2.1}$$

This integral converges absolutely if and only if $Re\, \alpha > d - 1$, and represents a product of ordinary Γ-functions:

$$\Gamma_m(\alpha) = \pi^{m(m-1)/4} \Gamma(\alpha) \Gamma(\alpha - \frac{1}{2}) \dots \Gamma(\alpha - \frac{m-1}{2}). \tag{2.2}$$

If $1 \le k < m$, $k \in \mathbb{N}$, then

$$\Gamma_m(\alpha) = \pi^{k(m-k)/2} \Gamma_k(\alpha) \Gamma_{m-k}(\alpha - k/2), \tag{2.3}$$

$$\frac{\Gamma_m(\alpha)}{\Gamma_m(\alpha + k/2)} = \frac{\Gamma_k(\alpha + (k-m)/2)}{\Gamma_k(\alpha + k/2)}. \tag{2.4}$$

For $n \ge m$, let $V_{n,m} = \{v \in \mathfrak{M}_{n,m} : v'v = I_m\}$ be the Stiefel manifold of orthonormal m-frames in $\mathbb{R}^n$; $V_{n,n} = O(n)$ is the orthogonal group in $\mathbb{R}^n$. We fix invariant measure dv on $V_{n,m}$ [Mu, p. 70] normalized by

$$\sigma_{n,m} \equiv \int\limits_{V_{n,m}} dv = \frac{2^m \pi^{nm/2}}{\Gamma_m(n/2)}\,. \tag{2.5}$$

Lemma 2.2. (polar decomposition; see, e.g., [Mu, pp. 66, 591], [Ma]). *Let* $x \in \mathfrak{M}_{n,m}$, $n \ge m$. *If* $\mathrm{rank}(x) = m$, *then*

$$x = v r^{1/2}, \qquad v \in V_{n,m}, \qquad r = x'x \in \mathcal{P}_m,$$

and $dx = 2^{-m} |r|^{(n-m-1)/2} dr dv$.

3. Riesz potentials

For $x \in \mathfrak{M}_{n,m}$, let $|x|_m = \det(x'x)^{1/2}$. The Riesz potential of order $\alpha \in \mathbb{C}$ is defined by

$$(I^\alpha f)(x) = \frac{1}{\gamma_{n,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,m}} f(x-y) |y|_m^{\alpha - n} dy, \tag{3.1}$$

$$\gamma_{n,m}(\alpha) = \frac{2^{\alpha m} \pi^{nm/2} \Gamma_m(\alpha/2)}{\Gamma_m((n-\alpha)/2)},$$

provided this expression is finite. Note that for $m \ge 2$, the sets of poles of $\Gamma_m(\alpha/2)$ and $\Gamma_m((n-\alpha)/2)$ are $\{m-1, m-2, \dots\}$, and $\{n-m+1, n-m+2, \dots\}$ respectively. These sets overlap if and only if $2m \geq n+2$ (keep this inequality in mind!). For $m=1$, the set of poles of $\Gamma_m((n-\alpha)/2)$ is $\{n,\, n+2,\, n+4, \ldots\}$. Thus if

$$\alpha = \begin{cases} n-m+1,\, n-m+2, \ldots & \text{for } m \geq 2, \\ n,\, n+2, \ldots & \text{for } m=1, \end{cases}$$

then the coefficient $1/\gamma_{n,m}(\alpha)$ is infinite. In the following we exclude these values of α and focus on the case $m \geq 2$.

The Riesz distribution corresponding to (3.1) is defined by

$$(h_\alpha, f) = a.c. \int\limits_{\mathfrak{M}_{n,m}} h_\alpha(x) f(x) dx, \qquad h_\alpha(x) = \frac{|x|_m^{\alpha-n}}{\gamma_{n,m}(\alpha)}, \tag{3.2}$$

where "*a.c.*" abbreviates analytic continuation in the α-variable, and $f \in \mathcal{S}(\mathfrak{M}_{n,m})$. The following lemma resumes basic properties of h_α.

Lemma 3.1. *Let* $m \geq 2$.
(i) *The integral in (3.2) absolutely converges if and only if* $Re\,\alpha > m-1$.
(ii) *The function* $\alpha \to h_\alpha$ *extends to all* $\alpha \in \mathbb{C}$ *as a meromorphic* $\mathcal{S}'$*-distribution with the only poles at the points* $\alpha = n-m+1, n-m+2, \ldots$. *The order of these poles is the same as in* $\Gamma_m((n-\alpha)/2)$.
(iii) h_α *is a positive measure if and only if* α *belongs to the set*

$$\mathcal{W} = \{0, 1, 2, \ldots, k_0\} \cup \{\alpha : Re\,\alpha > m-1;\ \alpha \neq n-m (\mathrm{mod}\, 1)\}, \tag{3.3}$$

$$k_0 = \min(m-1, n-m).$$

(iv) *If* $\alpha \neq n-m+1, n-m+2, \ldots$, *and*

$$(\mathcal{F}f)(y) = \int\limits_{\mathfrak{M}_{n,m}} \exp(\mathrm{tr}(iy'x)) f(x) dx \tag{3.4}$$

is the Fourier transform of $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, *then*

$$(h_\alpha, f) = (2\pi)^{-nm} (|y|_m^{-\alpha}, (\mathcal{F}f)(y)) \tag{3.5}$$

in the sense of analytic continuation. In particular,

$$(h_0, f) = f(0). \tag{3.6}$$

These statements can be found in [Kh]; see also [Sh1], [Sh3], [FK]. The set (3.3) is an analog of the Wallach set in [FK], p. 137. We adopt this name for our case too. The formula (3.5) is known as a functional equation for the zeta function $\alpha \to (h_\alpha, f)$. Different proofs of (3.5) (usually in a more general set-up) can be found in [Ge], Chapter IV; [Ra], Prop. II-9; [Sh1], [Far], [FK], Theorem XVI.4.3; see also [St1].

In the case $2m < n+2$, when poles of $\Gamma_m(\alpha/2)$ and $\Gamma_m((n-\alpha)/2)$ do not overlap, a simple proof of (3.5) can be given following slight modification of the argument from [St2], Chapter III, Sec. 3.4. Let $e_s(x) = \exp(\mathrm{tr}(-xsx')/4\pi)$, $s \in \mathcal{P}_m$. By the Plancherel formula,

$$(3.7) \quad |s|^{-n/2} \int\limits_{\mathfrak{M}_{n,m}} (\mathcal{F}f)(y) \exp(\mathrm{tr}(-\pi y s^{-1} y'))dy = \int\limits_{\mathfrak{M}_{n,m}} f(x) e_s(x) dx.$$

Multiplying (3.7) by $|s|^{(n-\alpha)/2-d}$, $d = (m+1)/2$, and integrating in $s \in \mathcal{P}_m$, after changing the order of integration we obtain

$$\int\limits_{\mathfrak{M}_{n,m}} (\mathcal{F}f)(y) a(y) dy = \int\limits_{\mathfrak{M}_{n,m}} f(x) b(x) dx,$$

where

$$\begin{aligned} a(y) &= \int\limits_{\mathcal{P}_m} |s|^{-\alpha/2-d} \exp(\mathrm{tr}(-\pi y s^{-1} y'))ds \qquad (s = t^{-1}) \\ &= \int\limits_{\mathcal{P}_m} |t|^{\alpha/2-d} \exp(\mathrm{tr}(-\pi t y' y))dt = \Gamma_m(\alpha/2)\pi^{-\alpha m/2}|y|_m^{-\alpha} \end{aligned}$$

if $Re\,\alpha > m-1$, and

$$b(x) = \int\limits_{\mathcal{P}_m} |s|^{(n-\alpha)/2-d} e_s(x) ds = \frac{(4\pi)^{m(n-\alpha)/2}\, \Gamma_m((n-\alpha)/2)}{|x|_m^{n-\alpha}}$$

if $Re\,\alpha < n-m+1$. A simple computation of these integrals is performed using (2.1) and Lemma 2.1. Thus (3.5) follows if the set $m-1 < Re\,\alpha < n-m+1$ is not vacuous, i.e., $2m < n+2$. We note that if $2m \geq n+2$ then distributions in both sides of (3.5) are not regular simultaneously.

Explicit representations for h_α in the discrete part of the Wallah set (3.3) play a vital role in our consideration.

Lemma 3.2. *Let* $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $k_0 = \min(m-1, n-m)$, $m \geq 2$. *Then for* $k = 1, 2, \ldots, k_0$,

$$(3.8) \qquad (h_k, f) = c \int\limits_{\mathfrak{M}_{k,m}} du \int\limits_{SO(n)} f\left(\gamma \begin{bmatrix} u \\ 0 \end{bmatrix}\right) d\gamma,$$

$$(3.9) \qquad c = 2^{-km}\pi^{-km/2}\,\Gamma_m\Big(\frac{n-k}{2}\Big)/\Gamma_m\Big(\frac{n}{2}\Big).$$

Proof. We split $x \in \mathfrak{M}_{n,m}$ in two blocks $x = [y; b]$ where $y \in \mathfrak{M}_{n,k}$ and $b \in \mathfrak{M}_{n,m-k}$. Then for $Re\,\alpha > m-1$,

$$(h_\alpha, f) = \frac{1}{\gamma_{n,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,k}} dy \int\limits_{\mathfrak{M}_{n,m-k}} f([y;b]) \begin{vmatrix} y'y & y'b \\ b'y & b'b \end{vmatrix}^{(\alpha-n)/2} db$$

where $\begin{vmatrix} * & * \\ * & * \end{vmatrix}$ denotes the determinant of the respective matrix $\begin{bmatrix} * & * \\ * & * \end{bmatrix}$. By passing to polar coordinates (see Lemma 2.2) $y = vr^{1/2}$, $v \in V_{n,k}$, $r \in \mathcal{P}_k$, we have

$$\begin{aligned} (h_\alpha, f) &= \frac{2^{-k}}{\gamma_{n,m}(\alpha)} \int\limits_{V_{n,k}} dv \int\limits_{\mathcal{P}_k} |r|^{(n-k-1)/2} dr \\ &\times \int\limits_{\mathfrak{M}_{n,m-k}} f([vr^{1/2}; b]) \begin{vmatrix} r & r^{1/2}v'b \\ b'vr^{1/2} & b'b \end{vmatrix}^{(\alpha-n)/2} db \\ &= \frac{2^{-k}\,\sigma_{n,k}}{\gamma_{n,m}(\alpha)} \int\limits_{SO(n)} d\gamma \int\limits_{\mathcal{P}_k} |r|^{(n-k-1)/2} dr \\ &\times \int\limits_{\mathfrak{M}_{n,m-k}} f_\gamma([\lambda_0 r^{1/2}; b]) \begin{vmatrix} r & r^{1/2}\lambda_0' b \\ b'\lambda_0 r^{1/2} & b'b \end{vmatrix}^{(\alpha-n)/2} db. \end{aligned}$$

Here

$$\lambda_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in V_{n,k}, \qquad f_\gamma(x) = f(\gamma x).$$

We write

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \qquad b_1 \in \mathfrak{M}_{k,m-k}, \qquad b_2 \in \mathfrak{M}_{n-k,m-k}.$$

Since $\lambda_0' b = b_1$, then

$$\begin{aligned} (h_\alpha, f) &= \frac{2^{-k}\,\sigma_{n,k}}{\gamma_{n,m}(\alpha)} \int\limits_{SO(n)} d\gamma \int\limits_{\mathcal{P}_k} |r|^{(n-k-1)/2} dr \int\limits_{\mathfrak{M}_{k,m-k}} db_1 \\ &\times \int\limits_{\mathfrak{M}_{n-k,m-k}} f_\gamma\left(\begin{bmatrix} r^{1/2} & b_1 \\ 0 & b_2 \end{bmatrix}\right) \begin{vmatrix} r & r^{1/2}b_1 \\ b_1' r^{1/2} & b_1'b_1 + b_2'b_2 \end{vmatrix}^{(\alpha-n)/2} db_2. \end{aligned}$$

Note that

$$\begin{bmatrix} r & r^{1/2}b_1 \\ b_1' r^{1/2} & b_1'b_1 + b_2'b_2 \end{bmatrix} = \begin{bmatrix} r & 0 \\ b_1' r^{1/2} & I_{m-k} \end{bmatrix} \begin{bmatrix} I_k & r^{-1/2}b_1 \\ 0 & b_2'b_2 \end{bmatrix},$$

and

$$\begin{vmatrix} r & r^{1/2}b_1 \\ b_1' r^{1/2} & b_1'b_1 + b_2'b_2 \end{vmatrix} = \det(r)\det(b_2'b_2),$$

see, e.g., [Mu], p. 577. Therefore,

$$(3.10)\quad (h_\alpha, f) = c_\alpha \int\limits_{SO(n)} d\gamma \int\limits_{\mathcal{P}_k} |r|^{(\alpha-k-1)/2} dr \int\limits_{\mathfrak{M}_{k,m-k}} \psi_{\alpha-k}(\gamma, r, b_1) db_1,$$

where

$$c_\alpha = \frac{2^{-k}\,\sigma_{n,k}\,\gamma_{n-k,m-k}(\alpha-k)}{\gamma_{n,m}(\alpha)},$$

$$\begin{aligned} \psi_{\alpha-k}(\gamma, r, b_1) &= \frac{1}{\gamma_{n-k,m-k}(\alpha-k)} \int\limits_{\mathfrak{M}_{n-k,m-k}} f_\gamma\left(\begin{bmatrix} r^{1/2} & b_1 \\ 0 & b_2 \end{bmatrix}\right) \\ &\times\ |b_2'b_2|^{\frac{(\alpha-k)-(n-k)}{2}} db_2. \end{aligned}$$

The last expression is the Riesz distribution of order $\alpha - k$ in the b_2-variable. Owing to (3.6), analytic continuation of (3.10) at $\alpha = k$ reads

$$(h_\alpha, f) = c_k \int\limits_{SO(n)} d\gamma \int\limits_{\mathcal{P}_k} |r|^{-1/2} dr \int\limits_{\mathfrak{M}_{k,m-k}} f_\gamma\left(\begin{bmatrix} r^{1/2} & b_1 \\ 0 & 0 \end{bmatrix}\right) db_1,$$

$c_k = \lim\limits_{\alpha\to k} c_\alpha$. To transform this expression, we replace γ by $\gamma \begin{bmatrix} \beta & 0 \\ 0 & I_{n-k} \end{bmatrix}$, $\beta \in SO(k)$ and integrate in β. This gives

$$\begin{aligned} (h_\alpha, f) &= c_k \int\limits_{\mathcal{P}_k} |r|^{-1/2} dr \int\limits_{SO(k)} d\beta \int\limits_{\mathfrak{M}_{k,m-k}} db_1 \int\limits_{SO(n)} f_\gamma\left(\begin{bmatrix} \beta r^{1/2} & \beta b_1 \\ 0 & 0 \end{bmatrix}\right) d\gamma \\ &\quad (\text{set } \zeta = \beta b_1, \quad \eta = b|r|^{1/2} \text{ and use Lemma 2.2}) \\ &= \frac{2^k\, c_k}{\sigma_{k,k}} \int\limits_{\mathfrak{M}_{k,k}} d\eta \int\limits_{\mathfrak{M}_{k,m-k}} d\zeta \int\limits_{SO(n)} f_\gamma\left(\begin{bmatrix} \eta & \zeta \\ 0 & 0 \end{bmatrix}\right) d\gamma \\ &= c \int\limits_{\mathfrak{M}_{k,m}} du \int\limits_{SO(n)} f\left(\gamma \begin{bmatrix} u \\ 0 \end{bmatrix}\right) d\gamma, \end{aligned}$$

$$c = \frac{\sigma_{n,k}}{\sigma_{k,k}} \lim_{\alpha\to k} \frac{\gamma_{n-k,m-k}(\alpha-k)}{\gamma_{n,m}(\alpha)} = 2^{-km}\,\pi^{-km/2}\,\Gamma_m\Big(\frac{n-k}{2}\Big)/\Gamma_m\Big(\frac{n}{2}\Big).$$

(here we used formulae (2.3) and (2.4)). $\square$

Definition 3.3. According to Lemma 3.2 and (3.6), we can redefine the Riesz potential $I^\alpha f$ for any locally integrable function f as

$$(I^\alpha f)(x) = \begin{cases} \dfrac{1}{\gamma_{n,m}(\alpha)} \displaystyle\int\limits_{\mathfrak{M}_{n,m}} f(x-y)|y|_m^{\alpha-n} dy \\ \text{if } Re\,\alpha > m-1; \quad \alpha \neq n-m+1, n-m+2, \dots, \\ c \displaystyle\int\limits_{\mathfrak{M}_{k,m}} du \int\limits_{SO(n)} f\left(x-\gamma\begin{bmatrix} u \\ 0 \end{bmatrix}\right) d\gamma \quad \text{if } \alpha = 1, \dots, k_0, \end{cases} \tag{3.11}$$

Here $m \geq 2$, c is the constant (3.9), $k_0 = \min(m-1, n-m)$. It is assumed that f is good enough, so that the corresponding integrals are absolutely convergent.

Conjecture 3.4. We state the following hypotheses.
(a) For $f \in L^p(\mathfrak{M}_{n,m})$, the integrals in (3.11) absolutely converge if and only if

$$1 \leq p < (n+m-1)/(Re\,\alpha + m - 1). \tag{3.12}$$

(b) If f is a continuous function satisfying $f(x) = O(|I_m + x'x|^{-\lambda/2})$, then the absolute convergence of (3.11) holds if and only if $\lambda > Re\,\alpha + m - 1$.

This conjecture is true if α is a positive integer; see the next section.
Remark 3.5. Another formula for h_k obtained in [Sh1] and [Kh] reads

$$(h_k, f) = c_1 \int\limits_{\mathfrak{M}_{n,k}} \frac{dy}{|y|_k^{n-m}} \int\limits_{\mathfrak{M}_{k,m-k}} f([y; yz]) dz, \quad k = 1, 2, \dots, k_0. \tag{3.13}$$

$$c_1 = 2^{-km} \pi^{k(k-n-m)/2}\, \Gamma_k\left(\frac{n-m}{2}\right) / \Gamma_k\left(\frac{k}{2}\right). \tag{3.14}$$

It can be derived from (3.8). Indeed,

$$\begin{aligned}
(h_k, f) &= c \int\limits_{\mathfrak{M}_{k,m}} du \int\limits_{SO(n)} f\left(\gamma \begin{bmatrix} u \\ 0 \end{bmatrix}\right) d\gamma \\
&= c \int\limits_{\mathfrak{M}_{k,m}} du \int\limits_{SO(n)} f(\gamma\lambda_0 u) d\gamma \qquad \left(\lambda_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in V_{n,k}\right) \\
&= \frac{c}{\sigma_{n,k}} \int\limits_{\mathfrak{M}_{k,m}} du \int\limits_{V_{n,k}} f(vu) dv.
\end{aligned}$$

Now we represent u in the block form $u = [\eta; \zeta]$, $\eta \in \mathfrak{M}_{k,k}$, $\zeta \in \mathfrak{M}_{k,m-k}$, and change the variable $\zeta = \eta z$. This gives

$$(h_k, f) = \frac{c}{\sigma_{n,k}} \int\limits_{\mathfrak{M}_{k,k}} |\eta|^{m-k} d\eta \int\limits_{\mathfrak{M}_{k,m-k}} dz \int\limits_{V_{n,k}} f(v[\eta; \eta z]) dv.$$

Using Lemma 2.2 repeatedly, and changing variables, we obtain

$$\begin{aligned}
(h_k, f) &= \frac{c\,\sigma_{k,k}\, 2^{-k}}{\sigma_{n,k}} \int\limits_{\mathcal{P}_k} |r|^{(m-k-1)/2} dr \int\limits_{\mathfrak{M}_{k,m-k}} dz \int\limits_{V_{n,k}} f(v[r^{1/2}, r^{1/2} z]) dv \\
&= c_1 \int\limits_{\mathfrak{M}_{n,k}} \frac{dy}{|y|_k^{n-m}} \int\limits_{\mathfrak{M}_{k,m-k}} f([y; yz]) dz
\end{aligned}$$

where by (3.9), (2.5) and (2.4),

$$c_1 = \frac{c\,\sigma_{k,k}}{\sigma_{n,k}} = 2^{-km} \pi^{k(k-n-m)/2}\, \Gamma_k\left(\frac{n-m}{2}\right) / \Gamma_k\left(\frac{k}{2}\right).$$

4. Radon transforms

4.1. **Matrix planes.** Let k, n, and m be positive integers, $0 < k < n$, $V_{n,n-k}$ be the Stiefel manifold of orthonormal $(n-k)$-frames in $\mathbb{R}^n$. For $\xi \in V_{n,n-k}$, $t \in \mathfrak{M}_{n-k,m}$, the linear manifold

$$\tau = \tau(\xi, t) = \{x \in \mathfrak{M}_{n,m} : \xi' x = t\} \tag{4.1}$$

will be called *a matrix k-plane* in $\mathfrak{M}_{n,m}$. We denote by $G(n,k,m)$ a variety of all such planes. The parameterization $\tau = \tau(\xi, t)$ by the points (ξ, t) of the "matrix cylinder" $V_{n,n-k} \times \mathfrak{M}_{n-k,m}$ is not one-to-one because for any orthogonal transformation $\theta \in O(n-k)$, the pairs (ξ, t) and $(\xi\theta', \theta t)$ define the same plane. We identify functions $\varphi(\tau)$ on $G(n,k,m)$ with functions $\varphi(\xi, t)$ on $V_{n,n-k} \times \mathfrak{M}_{n-k,m}$ satisfying $\varphi(\xi\theta', \theta t) = \varphi(\xi, t)$

for all $\theta \in O(n-k)$, and supply $G(n,k,m)$ with the measure $d\tau$ so that

$$\int\limits_{G(n,k,m)} \varphi(\tau)\, d\tau = \int\limits_{V_{n,n-k}\times\mathfrak{M}_{n-k,m}} \varphi(\xi,t)\, d\xi dt. \tag{4.2}$$

The matrix k-plane is, in fact, a usual km-dimensional plane in the Euclidean space $\mathbb{R}^{nm}$. To see this, we write $x = (x_{i,j}) \in \mathfrak{M}_{n,m}$ and $t = (t_{i,j}) \in \mathfrak{M}_{n-k,m}$ as column vectors

$$\bar{x} = \begin{pmatrix} x_{1,1} \\ x_{1,2} \\ \dots \\ x_{n,m} \end{pmatrix} \in \mathbb{R}^{nm}, \qquad \bar{t} = \begin{pmatrix} t_{1,1} \\ t_{1,2} \\ \dots \\ t_{n-k,m} \end{pmatrix} \in \mathbb{R}^{(n-k)m}, \tag{4.3}$$

and denote

$$\bar{\xi} = \mathrm{diag}(\xi, \dots, \xi) \in V_{nm,(n-k)m}. \tag{4.4}$$

Then (4.1) reads

$$\tau = \tau(\bar{\xi}, \bar{t}) = \{\bar{x} \in \mathbb{R}^{nm} : \bar{\xi}'\bar{x} = \bar{t}\}. \tag{4.5}$$

The km-dimensional planes (4.5) form a subset of measure zero in the affine Grassmann manifold of *all* km-dimensional planes in $\mathbb{R}^{nm}$.

The manifold $G(n,k,m)$ can be regarded as a fibre bundle the base of which is the ordinary Grassmann manifold $G_{n,k}$ of k-dimensional linear subspaces η of $\mathbb{R}^n$, and the fibres are homeomorphic to $\mathfrak{M}_{n-k,m}$. Indeed, let $\pi : G(n,k,m) \to G_{n,k}$ be the canonical projection which assigns to each matrix k-plane $\tau(\xi,t)$ the subspace

$$\eta = \eta(\xi) = \{y \in \mathbb{R}^n : \xi' y = 0\} \in G_{n,k}. \tag{4.6}$$

Let $\eta^\perp$ be the orthogonal complement of η in $\mathbb{R}^n$. The fiber $H_\eta = \pi^{-1}(\eta)$ is the set of all matrix planes (4.1), when t sweeps the space $\mathfrak{M}_{n-k,m}$.

Regarding $G(n,k,m)$ as a fibre bundle, one can utilize a parameterization which is alternative to (4.1) and one-to-one. Let

$$x = [x_1 \dots x_m], \quad x_i \in \mathbb{R}^n, \qquad t = [t_1 \dots t_m], \quad t_i \in \mathbb{R}^{n-k}. \tag{4.7}$$

For $\tau = \tau(\xi,t) \in G(n,k,m)$, we have

$$\tau = \{x \in \mathfrak{M}_{n,m} : \xi' x_i = t_i, \quad i = 1 \dots m\}.$$

Each k-dimensional plane $\tau_i = \{x_i \in \mathbb{R}^n : \xi' x_i = t_i\}$ can be parameterized by the pair (η, λ_i), where η is the subspace (4.6), and $\lambda_i \in \eta^\perp$, $i = 1, \dots, m$, are columns of the matrix $\lambda = \xi t \in \mathfrak{M}_{n,m}$. The corresponding parameterization

$$\tau = \tau(\eta, \lambda), \qquad \eta \in G_{n,k}, \quad \lambda = [\lambda_1 \dots \lambda_m], \quad \lambda_i \in \eta^\perp, \tag{4.8}$$

is one-to-one.

4.2. **Definition of the Radon transform.** The matrix k-plane Radon transform $\hat{f}$ of a function $f(x)$ on $\mathfrak{M}_{n,m}$ assigns to f a collection of integrals of f over all matrix planes $\tau \in G(n,k,m)$. Namely,

$$\hat{f}(\tau) = \int_{x \in \tau} f(x).$$

In order to give this integral precise meaning, we note that the matrix plane $\tau = \tau(\xi, t)$, $\xi \in V_{n,n-k}$, $t \in \mathfrak{M}_{n-k,m}$, consists of "points"

$$x = g_\xi \begin{bmatrix} u \\ t \end{bmatrix},$$

where $u \in \mathfrak{M}_{k,m}$, and $g_\xi \in SO(n)$ is a rotation satisfying

$$g_\xi \xi_0 = \xi, \qquad \xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k}. \tag{4.9}$$

This observation leads to the following

Definition 4.1. The Radon transform of a function $f(x)$ on $\mathfrak{M}_{n,m}$ is defined as a function on the "matrix cylinder" $V_{n,n-k} \times \mathfrak{M}_{n-k,m}$ by the formula

$$\hat{f}(\tau) \equiv \hat{f}(\xi, t) = \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} u \\ t \end{bmatrix}\right) du. \tag{4.10}$$

The reader is encouraged to check that (4.10) is independent of the choice of the rotation $g_\xi : \xi_0 \to \xi$. In terms of the one-to-one parameterization (4.8), where $\tau = \tau(\eta, \lambda)$, $\eta \in G_{n,k}$, $\quad \lambda = [\lambda_1 \dots \lambda_m] \in \mathfrak{M}_{n,m}$, and $\lambda_i \in \eta^\perp$, the Radon transform (4.10) reads

$$\hat{f}(\tau) = \int\limits_\eta dy_1 \dots \int\limits_\eta f([y_1 + \lambda_1 \dots y_m + \lambda_m])\, dy_m. \tag{4.11}$$

If $m = 1$, then $\hat{f}(\xi, t)$ is the ordinary k-plane Radon transform that assigns to a function $f(x)$ on $\mathbb{R}^n$ a collection of integrals of f over all k-dimensional planes [Hel]. A different definition of the matrix Radon transform was given by E.E. Petrov [Pe1]-[Pe3] (the case $n - k = m$), and L.P. Shibasov [Sh1], [Sh2] (the general case).

The following properties can be easily checked.

Lemma 4.2. *Suppose that the Radon transform*

$$f(x) \longrightarrow \hat{f}(\xi, t), \qquad x \in \mathfrak{M}_{n,m}, \quad (\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m},$$

exists (at least almost everywhere). Then

$$\hat{f}(\xi\theta', \theta t) = \hat{f}(\xi, t), \qquad \forall \theta \in O(n-k). \tag{4.12}$$

If $g(x) = \gamma x \beta + y$ *where* $\gamma \in O(n)$, $\beta \in O(m)$, $y \in \mathfrak{M}_{n,m}$, *then*

$$(f \circ g)^{\wedge}(\xi, t) = \hat{f}(\gamma\xi, t\beta + \xi'\gamma' y). \tag{4.13}$$

In particular, if $f_y(x) = f(x+y)$, *then*

$$\hat{f}_y(\xi, t) = \hat{f}(\xi, \xi' y + t). \tag{4.14}$$

The equality (4.12) is a matrix analog of the "evenness property" of the classical Radon transform, cf. [Hel], p. 3.

Lemma 4.3.
(i) *If* $f \in L^1(\mathfrak{M}_{n,m})$, *then the Radon transform* $\hat{f}(\xi, t)$ *exists for all* $\xi \in V_{n,n-k}$ *and almost all* $t \in \mathfrak{M}_{n-k,m}$. *Furthermore,*

$$\int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, t) dt = \int\limits_{\mathfrak{M}_{n,m}} f(x)\, dx, \qquad \forall\, \xi \in V_{n,n-k}. \tag{4.15}$$

(ii) *Let* $||x|| = (\mathrm{tr}(x'x))^{1/2} = (x_{1,1}^2 + \ldots + x_{n,m}^2)^{1/2}$. *If* f *is a continuous function satisfying*

$$f(x) = O(||x||^{-a}), \qquad a > km, \tag{4.16}$$

then $\hat{f}(\xi, t)$ *exists for all* $\xi \in V_{n,n-k}$ *and all* $t \in \mathfrak{M}_{n-k,m}$.

Proof. (i) is a consequence of the Fubini theorem:

$$\begin{aligned}
\int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, t) dt &= \int\limits_{\mathfrak{M}_{n-k,m}} dt \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} u \\ t \end{bmatrix}\right) du \\
&= \int\limits_{\mathfrak{M}_{n,m}} f(g_\xi x)\, dx = \int\limits_{\mathfrak{M}_{n,m}} f(x)\, dx.
\end{aligned}$$

(ii) becomes obvious if we regard $\tau = \tau(\xi, t)$ as a km-dimensional plane (4.5) in $\mathbb{R}^{nm}$. □

A much deeper result is contained in the following

Theorem 4.4. *If* $f \in L^p(\mathfrak{M}_{n,m})$ *then the Radon transform* $\hat{f}(\xi, t)$ *is finite for almost all* $(\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$ *provided*

$$1 \le p < p_0 = \frac{n+m-1}{k+m-1}. \tag{4.17}$$

If f *is a continuous function satisfying* $f(x) = O(|I_m + x'x|^{-\lambda/2})$, $\lambda > k+m-1$, *then* $\hat{f}(\xi, t)$ *is finite for all* $(\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$.

The proof of this theorem was given in [OR] using Abel type representation of the Radon transform of radial functions. The conditions for p and λ are sharp. For instance, one can show [OR] that

$$f_0(x) = |2I_m + x'x|^{-(n+m-1)/2p} (\log |2I_m + x'x|)^{-1} \tag{4.18}$$

belongs to $L^p(\mathfrak{M}_{n,m})$, and $\hat{f}_0(\xi, t) \equiv \infty$ if $p \geq p_0$. For $m = 1$, the result of Theorem 4.4 is due to Solmon [So]; see also [Ru2] for another proof.

4.3. **Connection with the Fourier transform.** The Fourier transform of a function $f \in L^1(\mathfrak{M}_{n,m})$ is defined by (3.4). The following statement is a matrix generalization of the so-called Central Slice Theorem. It links together the Fourier transform (3.4) and the Radon transform (4.10). In the case $m = 1$, this theorem can be found in [Na, p. 11] (for $k = n - 1$) and [Ke, p. 283] (for any $0 < k < n$).

For $y = [y_1 \dots y_m] \in \mathfrak{M}_{n,m}$, let $\mathcal{L}(y) = \mathrm{lin}(y_1, \dots, y_m)$ be the linear hull of the n-vectors $y_1 \dots y_m$, that is the smallest linear subspace containing $y_1 \dots y_m$. Suppose that $\mathrm{rank}(y) = \ell$. Then $\dim \mathcal{L}(y) = \ell \leq m$.

Theorem 4.5. *Let $f \in L^1(\mathfrak{M}_{n,m})$, $n - k \geq m$. If $y \in \mathfrak{M}_{n,m}$, and ζ is an $(n - k)$-dimensional plane in $\mathbb{R}^n$ containing $\mathcal{L}(y)$, then for any orthonormal frame $\xi \in V_{n,n-k}$ spanning ζ, there exists $b \in \mathfrak{M}_{n-k,m}$ so that $y = \xi b$. In this case*

$$(\mathcal{F}f)(y) = \int_{\mathfrak{M}_{n-k,m}} \exp(i\,\mathrm{tr}(b't))\, \hat{f}(\xi, t)\, dt, \tag{4.19}$$

or

$$(\mathcal{F}f)(\xi b) = \mathcal{F}[\hat{f}(\xi, \cdot)](b), \quad \xi \in V_{n,n-k}, \quad b \in \mathfrak{M}_{n-k,m}. \tag{4.20}$$

Proof. Since each vector y_j $(j = 1, \dots, m)$ lies in ζ, it decomposes as $y_j = \xi b_j$ for some $b_j \in \mathbb{R}^{n-k}$. Hence $y = \xi b$ where $b = [b_1 \dots b_m] \in \mathfrak{M}_{n-k,m}$. Thus it remains to prove (4.20). By (4.10),

$$\mathcal{F}[\hat{f}(\xi, \cdot)](b) = \int_{\mathfrak{M}_{n-k,m}} \exp(i\,\mathrm{tr}(b't))\, dt \int_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} u \\ t \end{bmatrix}\right) du.$$

If $x = g_\xi \begin{bmatrix} u \\ t \end{bmatrix}$, then, by (4.9),

$$\xi' x = \xi_0' g_\xi' g_\xi \begin{bmatrix} u \\ t \end{bmatrix} = \xi_0' \begin{bmatrix} u \\ t \end{bmatrix} = t, \qquad \xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k},$$

and the Fubini theorem yields

$$\mathcal{F}[\hat{f}(\xi,\cdot)](b) = \int\limits_{\mathfrak{M}_{n,m}} \exp(i\,\mathrm{tr}(b'\xi'x))\, f(x)\, dx = (\mathcal{F}f)(\xi b).$$

□

Remark 4.6. It is clear that matrices ξ and b in (4.19) are not uniquely defined. In the case $\mathrm{rank}(y) = m$, one can choose ξ and b as follows. By taking into account that $n-k \geq m$, we set

$$u_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in V_{n-k,m}, \qquad v_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in V_{n,m},$$

so that $\xi_0 u_0 = v_0$. Consider the polar decomposition

$$y = vr^{1/2}, \qquad v \in V_{n,n-k}, \quad r = y'y \in \mathcal{P}_m,$$

and let g_v be a rotation with the property $g_v v_0 = v$. Then

$$y = vr^{1/2} = g_v v_0 r^{1/2} = g_v \xi_0 u_0 r^{1/2} = \xi b,$$

where

$$\xi = g_v \xi_0 \in V_{n,n-k}, \qquad b = u_0 r^{1/2} \in \mathfrak{M}_{n-k,m}. \tag{4.21}$$

Theorem 4.7.
(*i*) *If $n-k \geq m$, then the Radon transform $f \to \hat{f}$ is injective on the Schwartz space $\mathcal{S}(\mathfrak{M}_{n,m})$, and f can be recovered by the formula*

$$\begin{aligned} f(x) \;&=\; \frac{2^{-m}}{(2\pi)^{nm}} \int\limits_{\mathcal{P}_m} |r|^{\frac{n-m-1}{2}} dr \\ &\times \int\limits_{V_{n,m}} \exp(-i\,\mathrm{tr}(x'vr^{1/2}))(\mathcal{F}\hat{f}(g_v\xi_0,\cdot))(\xi_0' v_0 r^{1/2}) dv. \end{aligned} \tag{4.22}$$

(*ii*) *For $n-k < m$, the Radon transform is non-injective.*

Proof. By Theorem 4.5, given the Radon transform $\hat{f}$ of $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, the Fourier transform $(\mathcal{F}f)(y)$ can be evaluated at each point $y \in \mathfrak{M}_{n,m}$ by the formula (4.19), so that if $\hat{f} \equiv 0$ then $\mathcal{F}f \equiv 0$. Since $\mathcal{F}$ is injective, then $f \equiv 0$, and we are done. Remark 4.6 allows to reconstruct f from $\hat{f}$, because (4.21) expresses ξ and b through $y \in \mathfrak{M}_{n,m}$ explicitly. This gives (4.22). To prove (*ii*), we denote

$$\mathfrak{L}_{n,m} = \{x \in \mathfrak{M}_{n,m} : \mathrm{rank}(x) = m\}.$$

This set is open in $\mathfrak{M}_{n,m}$. Let ψ be a Schwartz function with the Fourier transform supported in $\mathfrak{L}_{n,m}$. By (4.20),

$$\mathcal{F}[\hat{\psi}(\xi,\cdot)](b)=\hat{\psi}(\xi b)=0 \quad \forall\, \xi\in V_{n,n-k}, \forall\, b\in \mathfrak{M}_{n-k,m}, \tag{4.23}$$

because $\xi b\notin \mathfrak{L}_{n,m}$ (note that since $n-k<m$, then $\text{rank}(\xi b)<m$). By injectivity of the Fourier transform in (4.23), we obtain $\hat{\psi}(\xi,t)=0\ \forall\xi,t$. Thus, for $n-k<m$, the injectivity of the Radon transform fails. □

Remark 4.8. After the paper had been finished, we became aware of another account of the topic of this subsection in [Sh1], [Sh2], written in a different manner. Nevertheless, Remark 4.6, formula (4.22) and the statement (ii) of Theorem 4.7 seem to be new.

Remark 4.9. For $n-k>m$, the dimension of the manifold $G(n,k,m)$ of all matrix k-planes in $\mathfrak{M}_{n,m}$ is greater than that of the ambient space $\mathfrak{M}_{n,m}$, and the inversion problem is overdetermined. In the case $n-k=m$ both dimensions coincide. The problem of reducing overdeterminicity by fixing a certain "invertible" mn-dimensional complex of matrix planes, was studied in [Sh2].

5. The dual Radon transform and the Fuglede formula

Definition 5.1. Let $\tau=\tau(\xi,t)$ be a matrix plane (4.1), $(\xi,t)\in V_{n,n-k}\times\mathfrak{M}_{n-k,m}$. The dual Radon transform $\check{\varphi}(x)$ assigns to a function $\varphi(\tau)$ on $G(n,k,m)$ its mean value over all matrix planes τ through x. Namely,

$$\check{\varphi}(x)=\int\limits_{\tau\ni x}\varphi(\tau), \qquad x\in\mathfrak{M}_{n,m}.$$

This means that

$$\begin{aligned}\check{\varphi}(x) &= \frac{1}{\sigma_{n,n-k}}\int\limits_{V_{n,n-k}}\varphi(\xi,\xi'x)d\xi \\ &= \int\limits_{SO(n)}\varphi(\gamma\xi_0,\xi_0'\gamma'x)d\gamma, \qquad \xi_0=\begin{bmatrix}0\\ I_{n-k}\end{bmatrix}\in V_{n,n-k}.\end{aligned} \tag{5.1}$$

The mean value $\check{\varphi}(x)$ apparently exists for all $x\in\mathfrak{M}_{n,m}$ if φ is a continuous function. Moreover, $\check{\varphi}(x)$ is finite a.e. on $\mathfrak{M}_{n,m}$ for any locally integrable function φ [OR].

Remark 5.2. The dual Radon transform $\check{\varphi}(x)$ of a function $\varphi(\tau)$, $\tau\in G(n,k,m)$, is independent of the parameterization $\tau=\tau(\xi,t)$ in the sense that for any other parameterization $\tau = \tau(\xi\theta', \theta t)$, $\theta \in O(n-k)$, (see Sec. 4.1), (5.1) gives the same result:

$$\frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \varphi(\xi\theta', \theta\xi' x) d\xi = \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \varphi(\xi_1, \xi_1' x) d\xi_1 = \check{\varphi}(x).$$

Lemma 5.3. *The duality relation*

$$\int\limits_{\mathfrak{M}_{n,m}} f(x)\check{\varphi}(x)dx = \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi,t)\hat{f}(\xi,t)dt \tag{5.2}$$

holds provided that either side of (5.2) is finite for f and φ replaced by $|f|$ and $|\varphi|$, respectively.

Proof. By (4.10), the right hand side of (5.2) is

$$\frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi,t)\, dt \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} u \\ t \end{bmatrix}\right) du. \tag{5.3}$$

Changing variables $x = g_\xi \begin{bmatrix} u \\ t \end{bmatrix}$, we have

$$\xi' x = (g_\xi \xi_0)' g_\xi \begin{bmatrix} u \\ t \end{bmatrix} = \xi_0' \begin{bmatrix} u \\ t \end{bmatrix} = t.$$

Hence, by the Fubini theorem, (5.3) reads

$$\frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{n,m}} \varphi(\xi,\xi' x) f(x)\, dx = \int\limits_{\mathfrak{M}_{n,m}} \check{\varphi}(x) f(x)\, dx.$$

∎

Now we state the main theorem.

Theorem 5.4. *Let $f \in L^p(\mathfrak{M}_{n,m})$, $\quad 1 \le p < (n+m-1)/(k+m-1)$, or f is a continuous function satisfying $f(x) = O(|I_m + x'x|^{-\lambda/2})$, $\lambda > k+m-1$. Then*

$$c(\hat{f})^\vee(x) = (I^k f)(x), \qquad c = 2^{-km}\, \pi^{-km/2}\, \Gamma_m\Big(\frac{n-k}{2}\Big) / \Gamma_m\Big(\frac{n}{2}\Big), \tag{5.4}$$

(the generalized Fuglede formula).

Proof. The proof given below is applicable to any locally integrable function f for which either side of (5.4) is finite provided f is replaced by $|f|$. Let $f_x(y) = f(x+y)$. By (5.1) and (4.14),

$$\begin{aligned}(\hat{f})^{\vee}(x) &= \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \hat{f}(\xi, \xi' x) d\xi = \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \hat{f}_x(\xi, 0) d\xi \\ &= \int\limits_{\mathfrak{M}_{k,m}} du \int\limits_{SO(n)} f\left(x + \gamma \begin{bmatrix} u \\ o \end{bmatrix}\right) d\gamma.\end{aligned}$$

This coincides with $c^{-1}(I^k f)(x)$ for $k < m$, see Definition 3.3. If $k \geq m$, $d = (m+1)/2$, we pass to polar coordinates and get

$$\begin{aligned}(\hat{f})^{\vee}(x) &= 2^{-m} \int\limits_{V_{k,m}} dv \int\limits_{\mathcal{P}_m} |r|^{k/2-d} dr \int\limits_{SO(n)} f_x\left(\gamma \begin{bmatrix} vr^{1/2} \\ 0 \end{bmatrix}\right) d\gamma, \\ &= \frac{2^{-m}\,\sigma_{k,m}}{\sigma_{n,m}} \int\limits_{\mathcal{P}_m} |r|^{k/2-d} dr \int\limits_{V_{n,m}} f_x(wr^{1/2}) dw \\ &= \frac{\sigma_{k,m}}{\sigma_{n,m}} \int\limits_{\mathfrak{M}_{n,m}} f(x+y)|y'y|^{(k-n)/2} dy \\ &= c^{-1}(I^k f)(x).\end{aligned}$$

$\square$

Corollary 5.5. (cf. Conjecture 3.4) *For $f \in L^p(\mathfrak{M}_{n,m})$ and $k \in \mathbb{N}$, the Riesz potential $(I^k f)(x)$ is finite a.e. on $\mathfrak{M}_{n,m}$ if and only if*

$$1 \leq p < (n+m-1)/(k+m-1). \tag{5.5}$$

If f is a continuous function satisfying $f(x) = O(|I_m + x'x|^{-\lambda/2})$, then $(I^k f)(x)$ is finite for all $x \in \mathfrak{M}_{n,m}$ if and only if $\lambda > k+m-1$.

This statement holds by Theorem 4.4.

6. Inversion problem for the Radon transform

The Fuglede formula $c(\hat{f})^{\vee} = I^k f$ reduces the inversion problem for the Radon transform to Riesz potentials. This is exactly the same situation as in the rank-one case. Whereas for the ordinary k-plane transform and Riesz potentials a variety of pointwise inversion formulas is available in a large scale of function spaces [Ru1], [Ru2], in the higher rank case we cannot obtain pointwise inversion formulas rather than for Schwartz functions via the Fourier transform (see (4.22)) or in terms of divergent integrals understood somehow in the regularized sense. This is still an open problem.

Below we show how the unknown "rough" function f can be recovered in the framework of the theory of distributions. First we specify the space of test functions. From the Fourier transform formula $(h_\alpha, f) = (|y|_m^{-\alpha}, (\mathcal{F}f)(y))$, it is evident that the Schwartz class $\mathcal{S} \equiv \mathcal{S}(\mathfrak{M}_{n,m})$ does not suit well enough because it is not invariant under multiplication by $|y|_m^{-\alpha}$. To get around this difficulty, we follow an idea of V.I. Semyanistyi [Se] suggested for $m = 1$. Let $\Psi \equiv \Psi(\mathfrak{M}_{n,m})$ be the subspace of functions $\psi(y) \in \mathcal{S}$ vanishing on the set

$$\{y : y \in \mathfrak{M}_{n,m}, \ \mathrm{rank}(y) < m\} = \{y : y \in \mathfrak{M}_{n,m}, \ |y'y| = 0\} \tag{6.1}$$

with all derivatives (the coincidence of both sets in (6.1) is clear because $\mathrm{rank}(y) = \mathrm{rank}(y'y)$, see, e.g., [FZ], p. 5). The set Ψ is a closed linear subspace of $\mathcal{S}$. Therefore, it can be regarded as a linear topological space with the induced topology of $\mathcal{S}$. Let $\Phi \equiv \Phi(\mathfrak{M}_{n,m})$ be the Fourier image of Ψ. Since the Fourier transform $\mathcal{F}$ is an automorphism of $\mathcal{S}$ (i.e., a topological isomorphism of $\mathcal{S}$ onto itself), then Φ is a closed linear subspace of $\mathcal{S}$. Having been equipped with the induced topology of $\mathcal{S}$, the space Φ becomes a linear topological space isomorphic to Ψ under the Fourier transform. We denote by $\Phi' \equiv \Phi'(\mathfrak{M}_{n,m})$ the space of all linear continuous functionals (generalized functions) on Φ. Since for any complex α, multiplication by $|y|_m^{-\alpha}$ is an automorphism of Ψ, then, according to the general theory [GSh], I^α, as a convolution with h_α, is an automorphism of Φ, and we have

$$\mathcal{F}[I^\alpha f](y) = |y|_m^{-\alpha} \mathcal{F}[f](y)$$

for all Φ'-distributions f.

In the rank-one case, the spaces Φ, Ψ, their duals and generalizations were studied by P.I. Lizorkin, S.G. Samko and others in view of applications to the theory of function spaces and fractional calculus; see [Sa], [SKM], [Ru1] and references therein.

The Fuglede formula (5.4), Theorem 4.4, and Corollary 5.5 imply the following

Theorem 6.1. *Let $f \in L^p(\mathfrak{M}_{n,m})$, $1 \le p < (n+m-1)/(k+m-1)$ or f is a continuous function satisfying $f(x) = O(|I_m + x'x|^{-\lambda/2})$ for some $\lambda > k+m-1$. Then the Radon transform $g = \hat{f}$ is well defined, and f can be recovered from g in the sense of Φ'-distributions by the formula*

$$(f, \phi) = c(\check{g}, I^{-k}\phi), \qquad \phi \in \Phi, \tag{6.2}$$

where

$$(I^{-k}\phi)(x) = (\mathcal{F}^{-1}|y|_m^k \mathcal{F}\phi)(x), \quad c = 2^{-km}\,\pi^{-km/2}\,\Gamma_m\Big(\frac{n-k}{2}\Big)/\Gamma_m\Big(\frac{n}{2}\Big).$$

Remark 6.2. For k even, the Riesz potential $I^k f$ can be inverted (in the sense of Φ'-distributions) by repeated application of the Cayley-Laplace operator $\Delta_m = \det(\partial'\partial)$, $\partial = (\partial/\partial x_{i,j})$ [Kh]. In the Fourier terms, this operator agrees with multiplication by $(-1)^m|y|_m^2$, and therefore, $(-1)^m\Delta_m I^\alpha f = I^{\alpha-2} f$ in the Φ'-sense.

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