CHAPTER 14

Axiomatic approaches to coalitional bargaining

Sergiu Hart
TEL-AVIV UNIVERSITY

14.1 Introduction

The simplest bargaining situation is that of two persons who have to agree on the choice of an outcome from a given set of feasible outcomes; in case no agreement is reached, a specified disagreement outcome results. This *two-person pure bargaining problem* has been extensively analyzed, starting with Nash (1950).

When there are more than two participants, the n-person straightforward generalization considers either unanimous agreement or complete disagreement (see Roth (1979)). However, intermediate subsets of the players (i.e., more than one but not all) may also play an essential role in the bargaining. One is thus led to an *n-person coalitional bargaining problem,* where a set of feasible outcomes is specified for each coalition (i.e., subset of the players). This type of problem is known as a *game in coalitional form without side payments* (or, *with nontransferable utility*). It frequently arises in the analysis of various economic and other models; for references, see Aumann (1967, 1983*a*).

Solutions to such problems have been proposed by Harsanyi (1959, 1963, 1977), Shapley (1969), Owen (1972), and others. All of these were constructed to coincide with the Nash solution in the two-person case. Unlike the Nash solution, however, they were not defined (and determined) by a set of axioms.

Recently, Aumann (1983*b*) has provided an axiomatization for the Shapley solution. Following this work, further axiomatizations were obtained: for the Harsanyi solution by Hart (1983), and for a new class of monotonic solutions by Kalai and Samet (1983). The purpose of this chapter is to review and compare these three approaches.

The discussion is organized as follows. The mathematical model is described in Section 14.2, and is followed by the definitions of the solutions in Section 14.3. The axioms that determine these solutions are

Reprinted from
Game-theoretic models of bargaining
Edited by Alvin E. Roth

presented in Section 14.4, and Section 14.5 includes some general remarks together with a comparison of the solutions in terms of the axioms of Section 14.4.

It should be emphasized that this chapter includes only a minimal discussion of the various concepts; it is intended as a summary and a directory to the existing literature on the subject. In particular, the reader should consult the papers of Aumann (1983*b*), Hart (1983), and Kalai and Samet (1983) for extensive presentations and comments.

14.2 The mathematical model

We start by introducing the notations. The real line is denoted $\mathbb{R}$. For a finite set I, let $|I|$ be the number of elements of I, and let $\mathbb{R}^I$ be the $|I|$-dimensional euclidean space with coordinates indexed by the elements of I (when I is the empty set ϕ, $\mathbb{R}^\phi$ contains just one element, namely, ϕ). We will thus write $x = (x^i)_{i \in I} \in \mathbb{R}^I$ and, for $J \subset I$, $x^J = (x^i)_{i \in J} \in \mathbb{R}^J$ (hence, $x = x^I$; note that we use the symbol $\subset$ for weak inclusion). Some distinguished vectors in $\mathbb{R}^I$ are: the origin $0 = (0, \ldots, 0)$; and for every $J \subset I$, its indicator 1_J, with $1_J^i = 1$ if $i \in J$ and $1_J^i = 0$ if $i \notin J$.

For x and y in $\mathbb{R}^I$, the inequalities $x \geq y$ and $x > y$ are to be understood coordinatewise: $x^i \geq y^i$ and $x^i > y^i$, respectively, for all $i \in I$. The nonnegative, the positive, and the nonpositive orthants of $\mathbb{R}^I$ (defined by the inequalities $x \geq 0$, $x > 0$, and $x \leq 0$, respectively) are denoted $\mathbb{R}^I_+$, $\mathbb{R}^I_{++}$, and $\mathbb{R}^I_-$. For λ and x in $\mathbb{R}^I$, we write $\lambda \cdot x$ for the real number $\Sigma_{i \in I} \lambda^i x^i$ (their scalar product), and λx for that element of $\mathbb{R}^I$ given by $(\lambda x)^i = \lambda^i x^i$ for all $i \in I$.

Let A be a closed subset of $\mathbb{R}^I$; its boundary is denoted ∂A. For λ in $\mathbb{R}^I$, the set λA is $\{\lambda a \mid a \in A\}$; for another closed subset B of $\mathbb{R}^I$, $A + B$ is the *closure* of $\{a + b \mid a \in A, b \in B\}$. Note that $A - B$ will denote the set difference $\{x \in A \mid x \notin B\}$.

A *coalitional bargaining problem* – *c.b.p.*, for short – is an ordered pair (N, V), where N is a finite set and V is a set-valued function that assigns to every $S \subset N$ a subset $V(S)$ of $\mathbb{R}^S$. The set N is the set of *players*; a subset S of N is a *coalition*; and V is the *characteristic function.*

The interpretation is as follows. Let A be the set of all possible outcomes. For each player $i \in N$, let $u^i: A \to \mathbb{R}$ be his utility function. Finally, for every coalition $S \subset N$, let $A(S) \subset A$ be the set of outcomes that can be reached by S. Then, $V(S)$ is the set of utility payoff vectors that are feasible for S, namely,

$$V(S) = \{(u^i(a))_{i \in S} \in \mathbb{R}^S \mid a \in A(S)\}.$$

In game theory, such a pair (N,V) is called a *game in coalitional* (or, *characteristic function*) *form with nontransferable utility* (or, *without side payments*).

The set N of players will be fixed throughout this chapter; a c.b.p. will thus be given by its characteristic function V. The space $\Gamma \equiv \Gamma(N)$ of all c.b.p.'s that we will consider consists of all members of V that satisfy the following conditions:

(1) For every $S \subset N$, the set $V(S)$ is
  a. A non-empty subset of $\mathbb{R}^I$,
  b. Closed,
  c. Convex,
  d. Comprehensive (i.e., $x \in V(S)$ and $x \geq y$ imply $y \in V(S)$).

We will also consider the following additional regularity conditions:

(2) The set $V(N)$ is
  a. Smooth (i.e., $V(N)$ has a unique supporting hyperplane at each point of its boundary $\partial V(N)$),
  b. Nonlevel (i.e., $x, y \in \partial V(N)$ and $x \geq y$ imply $x = y$).

(3) For every $S \subset N$, there exists $x \in \mathbb{R}^N$ such that

$$V(S) \times \mathbb{R}_{-}^{N-S} \subset V(N) + \{x\}.$$

(4) For every $S \subset N$ and every sequence $\{x_m\}_{m=1}^{\infty} \subset V(S)$ that is non-decreasing (i.e., $x_{m+1} \geq x_m$ for all $m \geq 1$), there exists $y \in \mathbb{R}^S$ such that $x_m \leq y$ for all $m \geq 1$.

Denote the set of all V in Γ that satisfy (2) by Γ_1, those that satisfy (4) by Γ_2, and those that satisfy (2) and (3) by Γ_3.

Conditions (1) are standard. Condition (2b) is a commonly used regularity condition, meaning that weak and strong Pareto-optimality coincide for $V(N)$. The smoothness of $V(N)$ is an essential condition; (2a) implies that, for every $x \in \partial V(N)$, there exists a unique normalized vector λ in $\mathbb{R}^N$ such that $\lambda \cdot x \geq \lambda \cdot y$ for all y in $V(N)$. Note that λ must be positive (i.e., $\lambda \in \mathbb{R}_{++}^N$) by (1d) and (2b). Condition (3) may be viewed as an extremely weak kind of monotonicity: There exists some translate of $V(N)$ that includes all of the payoff vectors that are feasible for S and assign zero to the players outside S. Finally, (4) is a boundedness-from-above condition. A thorough discussion on assumptions and their impact can be found in Sections 9 and 10 in Aumann (1983*b*).

For a coalition $S \subset N$, an *S-payoff vector* is simply an element of $\mathbb{R}^S$; when $S = N$, it is a *payoff vector*. A collection $\mathbf{x} = (x_S)_{S \subset N}$ of S-payoff vectors for all coalitions S is called a *payoff configuration* (thus, $x_S = (x_S^i)_{i \in S} \in \mathbb{R}^S$ for all $S \subset N$). The space of payoff configurations

$\Pi_{S \subset N} \mathbb{R}^S$ will be denoted X. In particular, the payoff configuration $\mathbf{x}$ with $x_S = 0$ for all S will be denoted $\mathbf{0}$.

Note that every c.b.p. V may be regarded as a (rectangular) subset of X, namely, $\Pi_{S \subset N} V(S)$. Operations are thus always understood coalitionwise. Hence, $V + W$ is given by $(V + W)(S) = V(S) + W(S)$ for all $S \subset N$; $V \subset W$ means $V(S) \subset W(S)$ for all S; and ∂V is $\Pi_{S \subset N} \partial V(S) \subset X$. If λ is a vector in $\mathbb{R}^N_{++}$, then λV is defined by

$$(\lambda V)(S) = \lambda^S V(S) = \{(\lambda^i x^i)_{i \in S} \mid x = (x^i)_{i \in S} \in V(S)\}$$

(recall that $\lambda^S = (\lambda^i)_{i \in S}$ is the restriction of λ to $\mathbb{R}^S_{++}$). Moreover, for a subset Y of X, we write λY for $\{(\lambda^S y_S)_{S \subset N} \mid \mathbf{y} = (y_S)_{S \subset N} \in Y\}$.

14.3 Solutions

In this section, we will define the three solutions of Harsanyi, Shapley, and Kalai and Samet. The following conditions will be considered, where V is a coalitional bargaining problem; λ is a vector in $\mathbb{R}^N_{++}$; and for each $S \subset N$, $x_S \in \mathbb{R}^S$ is an S-payoff vector and $\xi_S \in \mathbb{R}$ a real number:

(5) $x_S \in \partial V(S)$,
(6) $\lambda^S \cdot x_S \geq \lambda^S \cdot y$ for all $y \in V(S)$,
(7) $\lambda^i x^i_S = \Sigma_{T \subset S, i \in T} \xi_T$ for all $i \in S$.

The solutions are then defined as follows:

Definition 1. A payoff vector $x \in \mathbb{R}^N$ is a *Harsanyi (NTU) solution* of a c.b.p. V if there exist $\lambda \in \mathbb{R}^N_{++}$, $\mathbf{x} = (x_S)_{S \subset N} \in X$ with $x_N = x$, and $\xi_S \in \mathbb{R}$ for all $S \subset N$ such that the following are satisfied:

Condition (5) for all $S \subset N$,
Condition (6) for $S = N$,
Condition (7) for all $S \subset N$.

Definition 2. A payoff vector $x \in \mathbb{R}^N$ is a *Shapley (NTU) solution* of a c.b.p. V if there exist $\lambda \in \mathbb{R}^N_{++}$, $\mathbf{x} = (x_S)_{S \subset N} \in X$ with $x_N = x$, and $\xi_S \in \mathbb{R}$ for all $S \subset N$ such that the following are satisfied:

Condition (5) for all $S \subset N$,
Condition (6) for all $S \subset N$,
Condition (7) for $S = N$.

Definition 3. Let λ be a vector in $\mathbb{R}^N_{++}$. A payoff vector $x \in \mathbb{R}^N$ is the *Kalai and Samet λ-egalitarian solution* of a c.b.p. V if there exist

$\mathbf{x} = (x_S)_{S \subset N} \in X$ with $x_N = x$ and $\xi_S \in \mathbb{R}$ for all $S \subset N$ such that the following are satisfied:

Condition (5) for all $S \subset N$,
Condition (7) for all $S \subset N$.

Note that Kalai and Samet define a *class* of solutions, parameterized by λ (which is thus *exogenously* given, besides V); moreover, each c.b.p. can have at most one λ-egalitarian solution (for every λ; see the end of this section). In both the Harsanyi and Shapley solutions, λ is *endogenously* determined, and there may be no solution or more than one. When the λ^i are all equal (i.e., $\lambda = c1_N$ for some $c \in \mathbb{R}_+$), the corresponding Kalai and Samet solution is called the *symmetric egalitarian solution.* The vector λ yields an interpersonal comparison of the utility scales of the players; whether it is obtained from the bargaining problem itself, or it is an additional datum of the problem, is thus an essential distinction (see also Section 14.5).

The associated payoff configuration $\mathbf{x} = (x_S)_{S \subset N}$ specifies for *every* coalition S a feasible (and even efficient, by condition (1)) outcome x_S. One may view x_S as the payoff vector that the members of S agree upon from their feasible set $V(S)$; if coalition S "forms," then x_S^i is the amount that player i (in S) will receive (note that these are contingent payoffs – *if* S forms). Following Harsanyi, one may furthermore regard x_S as an optimal *threat* of coalition S (against its complement $N - S$), in the bargaining problem. More discussion on these interpretations can be found at the end of section 5 in Hart (1983).

The three conditions (5), (6), and (7) may be interpreted as *efficiency,* λ*-utilitarity,* and λ*-equity* (or *fairness*), respectively. Indeed, condition (5) means that the S-payoff vector x_S is Pareto-efficient for the coalition S: There is no vector $y \in \mathbb{R}^S$ that is feasible for S (i.e., $y \in V(S)$) such that all members of S prefer y to x_S (i.e., $y > x_S$). Condition (6) means that x_S is λ-utilitarian for S, since it maximizes the sum of the payoffs for members of S, weighted according to λ, over their feasible set $V(S)$. And, finally, the weighted payoff $\lambda^i x_S^i$ of each member of the coalition S is the sum of the *dividends* ξ_T that player i has accumulated from all subcoalitions T of S to which he belongs; because the dividend ξ_T is exactly the same for all members of T (for each T), x_S is thus λ-equitable or λ-fair.

In the two-person simple bargaining problem, the Nash solution is efficient, and for an appropriate vector $\lambda > 0$, it is both λ-utilitarian and λ-equitable. Both the Harsanyi and the Shapley solutions to the general coalitional bargaining problem require efficiency (5) for all coalitions together with utilitarity (6) and equity (7) for the grand coalition N. They differ in the condition imposed on subcoalitions (other than N and the singletons): The Harsanyi solution requires equity, and the Shapley solution utilitarity. The Kalai and Samet solutions do not consider utilitarity at all, but only equity (for all coalitions). Thus, the weights λ have to be given exogenously, whereas in the other two solutions they are determined by the conjunction of (6) and (7). Note further that, in the two-person case, all of the egalitarian solutions (including the symmetric one) differ from the Nash solution (they are the *proportional solutions* of Kalai (1977)).

In general, some of the coordinates of λ may be zero (and thus $\lambda \in \mathbb{R}^N_+ - \{0\}$ instead of $\lambda \in \mathbb{R}^N_{++}$); the simplifying assumption (2b) rules out this for the Harsanyi and the Shapley solutions (by (6) for $S = N$); for the egalitarian solutions, the positivity of λ is part of the definition.

The three definitions have been stated in order to facilitate comparison among the solutions. For alternative (and more constructive) definitions, we need the following.

A *transferable utility game* (*TU game*, for short) consists of a finite set of players N and a real function v that assigns to each coalition $S \subset N$ its *worth* $v(S)$, with $v(\phi) = 0$. The *Shapley* (1953) *value* of such a TU game (N,v) is a payoff vector $x \in \mathbb{R}^N$, which will be denoted $Sh(N,v) = (Sh^i(N,v))_{i \in N}$. It is defined by a set of axioms, and it equals the vector of average marginal contributions of each player to those preceding him in a random order of all the players.

Using this concept, one can rewrite the condition "there exist real numbers $\xi_T \in \mathbb{R}$ for all $T \subset S$ such that (7) is satisfied for all $T \subset S$" as:

(8) $\lambda^i x^i_S = Sh^i(S,v)$, where $v(T) = \lambda^T \cdot x_T = \Sigma_{i \in T} \lambda^i x^i_T$ for all $T \subset S$.

We then obtain the following.

Definition 1′. A payoff vector $x \in \mathbb{R}^N$ is a *Harsanyi solution* of a c.b.p. V if there exist $\lambda \in \mathbb{R}^N_{++}$ and $\mathbf{x} = (x_S)_{S \subset N} \in X$ with $x_N = x$ such that conditions (5) and (8) are satisfied for all $S \subset N$ and condition (6) is satisfied for $S = N$.

Definition 2′. A payoff vector $x \in \mathbb{R}^N$ is a *Shapley solution* of a c.b.p. V if $x \in V(N)$ and there exists $\lambda \in \mathbb{R}^N_{++}$ such that

$$\lambda^i x^i = Sh^i(N,v),$$

where $v(S) = \max\{\lambda^S \cdot y \mid y \in V(S)\}$ for all $S \subset N$.

(Note that (6) implies (5) and that the TU game v in (8) coincides with v defined previously.)

Definition 3′. Let $\lambda \in \mathbb{R}^N_{++}$. A payoff vector $x \in \mathbb{R}^N$ is the *Kalai and Samet λ-egalitarian solution* of a c.b.p. V if there exists $\mathbf{x} = (x_S)_{S \subset N} \in X$ with $x_N = x$ such that conditions (5) and (8) are satisfied for all $S \subset N$.

How are the solutions of a given c.b.p. V constructed? In the case of the Shapley solution, for each $\lambda \in \mathbb{R}^N_{++}$, one computes the TU game v_λ by

$$v_\lambda(S) = \max\{\lambda^S \cdot y \mid y \in V(S)\}$$

for all $S \subset N$, and then obtains its Shapley TU value $Sh(N,v_\lambda)$. If the payoff vector $x = (x^i)_{i \in N}$ given by $x^i = Sh^i(N,v_\lambda)/\lambda^i$ for all $i \in N$ belongs to $V(N)$, then x is a Shapley solution of V.

The Kalai and Samet λ-egalitarian solution (for a given $\lambda \in \mathbb{R}^N_{++}$) is obtained as follows. Inductively on the lattice of all subsets S of N, we define

$$\xi_S = \max\{t \in \mathbb{R} \mid z_S(t) \in V(S)\},$$

where $z_S(t) = (z^i_S(t))_{i \in S} \in \mathbb{R}^S$ is given by

$$z^i_S(t) = \frac{1}{\lambda^i}\Big(t + \sum_{\substack{T \subset S \\ T \neq S \\ i \in T}} \xi_T\Big)$$

If we put

$$x_S = z_S(\xi_S),$$

then x_N is the λ-egalitarian solution of V.

Finally, to obtain the Harsanyi solution, we compute the λ-egalitarian solution $x(\lambda)$ for each $\lambda \in \mathbb{R}^N_{++}$; then, $x = x(\lambda)$ is a Harsanyi solution of V whenever $\lambda \cdot x(\lambda) = \max\{\lambda \cdot y \mid y \in V(N)\}$.

Thus, both the Harsanyi and the Shapley approaches require essentially a fixed-point construction, whereas the Kalai and Samet approach does not (again, λ is exogenously given there). It is now clear that each V in Γ has at most one λ-egalitarian solution, and precisely one (for each λ) if V belongs to Γ_2.

14.4 Axiomatizations

In this section, we will present an axiomatization for each one of the three solutions defined previously. By *axiomatization* is meant a set of axioms that uniquely characterize the corresponding solution.

As is usually the case, there are various ways of choosing an appropriate set of axioms. We will follow here the choices made by each author in the papers we review. This will enable us to exhibit a large number of axioms emphasizing a variety of reasonable principles for solving coalitional bargaining problems.

The first solution studied will be the Shapley solution, according to Aumann's (1983*b*) pioneering paper. For every c.b.p. V in Γ, let $\Lambda(V)$ denote the set of Shapley solutions of V; $\Lambda(V)$ is thus a (possibly empty) subset of $\mathbb{R}^N$.

Let Γ_4 denote the set of all c.b.p.'s in Γ_3 that possess at least one Shapley solution (i.e., $\Gamma_4 = \{V \in \Gamma \mid V$ satisfies (2), (3), and $\Lambda(V) \neq \phi\}$). The set-valued function Λ from Γ_4 to $\mathbb{R}^N$ will be called the *Shapley function.* The following axioms will characterize it.

A0. Φ is a non-empty-set-valued function from Γ_4 to $\mathbb{R}^N$.

(For each $V \in \Gamma_4$, $\Phi(V)$ is a nonempty subset of $\mathbb{R}^N$.)

A1. $\Phi(V) \subset \partial V(N)$ for all $V \in \Gamma_4$.

(*Efficiency:* Every solution must be Pareto-efficient [for the grand coalition].)

A2. $\Phi(\lambda V) = \lambda \Phi(V)$ for all $\lambda \in \mathbb{R}^N_{++}$ and $V \in \Gamma_4$.

(*Scale covariance:* If the payoffs of the players are rescaled independently, all solutions will be rescaled accordingly.)

A3. If $U = V + W$, then $\Phi(U) \supset (\Phi(V) + \Phi(W)) \cap \partial U(N)$ for all $U, V, W \in \Gamma_4$.

(*Conditional additivity:* If $x \in \Phi(V)$ is a solution of $V, y \in \Phi(W)$ is a solution of W, and $z = x + y$ is efficient for $U = V + W$, then z is a solution of U[i.e., $z \in \Phi(U)$].)

A4. If $V(N) \subset W(N)$ and $V(S) = W(S)$ for all $S \neq N$, then $\Phi(V) \supset \Phi(W) \cap V(N)$ for all $V, W \in \Gamma_4$.

(*Independence of irrelevant alternatives:* If V is obtained from W by restricting the feasible set of the grand coalition, then any solution of W that remains feasible in V will be a solution of V as well.)

For the next axiom, we define a class of c.b.p.'s usually referred to as *unanimity games.* For every nonempty $T \subset N$ and every real number c, let $u_{T,c}$ be the TU game given by $u_{T,c}(S) = c$ if $S \supset T$ and $u_{T,c}(S) = 0$ otherwise; then, $U_{T,c}(S) = \{x \in \mathbb{R}^S \mid \Sigma_{i \in S} x^i \leq u_{T,c}(S)\}$ for all $S \subset N$.

A5. $\Phi(U_{T,1}) = \{1_T/|T|\}$ for all $T \subset N$, $T \neq \phi$.

(*Unanimity games:* $U_{T,1}$ models the situation where each coalition S can arbitrarily divide among its members the amount 1, if it contains *all* the players of T – or nothing, otherwise. This c.b.p. has a unique solution, where the members of T receive equal shares ($1/|T|$), and the other players zero.)

Theorem A (Aumann (1983*b*)). There exists a unique function Φ satisfying A0 through A5; it is the Shapley function Λ.

We continue next with the Harsanyi solution, according to Hart (1983). We will consider here not only the payoff vector $x = x_N$ of the grand coalition, but rather the full payoff configuration $\mathbf{x} = (x_S)_{S \subset N}$. Let $H(V)$ stand for the set of all $\mathbf{x} = (x_S)_{S \subset N} \in X$ associated with a Harsanyi solution of a c.b.p. V; that is, $\mathbf{x} = (x_S)_{S \subset N} \in H(V)$ if there exists $\lambda \in \mathbb{R}^N_{++}$ such that conditions (5) and (8) are satisfied for all $S \subset$ N and condition (6) is satisfied for $S = N$; see definition 2′). The set-valued function H from Γ_1 to X will be called the *Harsanyi function;* we will refer to $\mathbf{x} = (x_S)_{S \subset N} \in H(V)$ as a Harsanyi solution of V (rather than just to x_N). Note that $H(V)$ may well be an empty set for some $V \in \Gamma_1$. Consider now the following axioms.

B0. Ψ is a set-valued function from Γ_1 to X.

(For each $V \in \Gamma_1$, $\Psi(V)$ is a subset of X.)

B1. $\Psi(V) \subset \partial V$ for all $V \in \Gamma_1$.

(*Efficiency:* Every solution $\mathbf{x} = (x_S)_{S \subset N} \in H(V)$ must be Pareto-efficient for all coalitions: $x_S \in \partial V(S)$ for all $S \subset N$.)

B2. $\Psi(\lambda V) = \lambda \Psi(V)$ for all $\lambda \in \mathbb{R}^N_{++}$ and $V \in \Gamma_1$.

(*Scale covariance:* If the payoffs of the players are rescaled independently, all solutions will be rescaled accordingly.)

B3. If $U = V + W$, then $\Psi(U) \supset (\Psi(V) + \Psi(W)) \cap \partial U$ for all $U, V, W \in \Gamma_1$.

(*Conditional additivity:* If $\mathbf{x} = (x_S)_{S \subset N} \in \Psi(V)$ is a solution of V, $\mathbf{y} = (y_S)_{S \subset N} \in \Phi(W)$ is a solution of W, and $\mathbf{z} = \mathbf{x} + \mathbf{y}$ is efficient for $U = V + W$ [i.e., $z_S \in \partial V(S)$ for all $S \subset N$], then z is a solution of U [i.e., $\mathbf{z} \in \Psi(U)$].)

B4. If $V \subset W$, then $\Psi(V) \supset \Psi(W) \cap V$ for all $V, W \in \Gamma_1$.

(*Independence of irrelevant alternatives:* If V is obtained from W by restricting the feasible set of some coalition(s), then any solution of W that remains feasible in V will be a solution of V as well: $\mathbf{x} = (x_S)_{S \subset N} \in \Psi(W)$ and $x_S \in V(S) \subset W(S)$ for all $S \subset N$ imply $\mathbf{x} \in \Psi(V)$.)

B5. $\Psi(U_{T,c}) = \{\mathbf{z}(T,c)\} = \{z_S(T,c))_{S \subset N}\}$ for all $T \subset N$, $T \neq \phi$, and all $c \in \mathbb{R}$, where $z_S(T,c) = c1_T/|T|$ if $S \supset T$ and $z_S(T,c) = 0$ otherwise.

(*Unanimity games:* $U_{T,c}$ models the situation where each coalition S can arbitrarily divide among its members the amount c if it contains *all* the players of T – or nothing, otherwise. This c.b.p. has a unique solution $\mathbf{z} \equiv \mathbf{z}(T,c)$; the S-payoff vector z_S of a coalition S that contains T assigns equal shares $(c/|T|)$ to all members of T and zero to the rest; if S does not contain T, everyone gets zero.)

B6. If $\mathbf{0} \in \partial V$, then $\mathbf{0} \in \Psi(V)$.

(*Inessential games:* A c.b.p. is called *zero inessential* if $\mathbf{0} \in \partial V(S)$ for all $S \subset N$, that is, if the zero payoff vector is efficient for all coalitions. This means that for all coalitions, $\mathbf{0}$ is feasible, whereas no positive vector is feasible. In particular, $V(\{i\}) = \{x^i \in \mathbb{R}^{\{i\}} \mid x^i \leq 0\}$. For such a game, where there is nothing to bargain on, the payoff configuration zero is a solution.)

Theorem B (Hart (1983)). There exists a unique function Ψ satisfying B0 through B6; it is the Harsanyi function H.

It is remarkable that the two solutions of Shapley and Harsanyi are determined by very similar sets of axioms. The main difference lies in the range: payoff vectors for the former versus payoff configurations for the latter (see Section 5 in Hart (1983) for further discussion of this subject).

We come now to the class of Kalai and Samet solutions. For every $\lambda \in \mathbb{R}^N_{++}$, let E^λ be the function from Γ_2 into $\mathbb{R}^N$ that assigns to each $V \in \Gamma_2$ its λ-egalitarian solution $E^\lambda(V)$. We will refer to E^λ as the *λ-egalitarian function.* Consider now the following axioms, according to Kalai and Samet (1983).

C0. F is a function from Γ_2 into $\mathbb{R}^N$.

As usual, we will write $F(V) = (F^i(V))_{i \in N} \in \mathbb{R}^N$ and $F^T(V) = (F^i(V))_{i \in T} \in \mathbb{R}^T$ for all $T \subset N$. Given a c.b.p. V, a coalition $T \subset N$ is a *carrier* of V if $V(S) = V(S \cap T) \times \mathbb{R}^{S-T}_{-}$ for all $S \subset N$.

C1. If T is a carrier of V, then $F^T(V) \in \partial V(T)$ for all $V \in \Gamma_2$ and $T \subset N$.

(*Carrier:* The solution is Pareto-efficient for any carrier of V.) This axiom implies both an *efficiency* axiom and a *dummy* axiom (where any player outside a carrier is a dummy). Note that if T is a carrier of V, then any $T' \supset T$ (in particular $T' = N$, for all V) is also a carrier, and thus $F^{T'}(V) \in \partial V(T')$.

Given a c.b.p. V, let V_+ denote its *individually rational restriction,* defined as follows:

$$\vartheta^i = \max\{x \mid x \in V(\{i\})\} \text{ for all } i \in N,$$

$$V_+(S) = \{x \in V(S) \mid x^i \geq \vartheta^i \text{ for all } i \in S\},$$

for $S \subset N$. Note that V_+ does not belong to Γ (it does not satisfy comprehensiveness nor possibly nonemptiness [see (1) in Section 14.2]). A c.b.p. V is *individually rational monotonic* (*monotonic* in Kalai and Samet (1983)) if for all $S \subset N$ and $i \notin S$.

$$V_+(S \cup \{i\}) \supset V_+(S) \times V_+(\{i\}).$$

In such a c.b.p., the contribution of a player is never detrimental, so long as only individually rational outcomes are considered. Note that $V_+(\{i\}) = \{\vartheta^i\}$, and repeated applications of the preceding inclusion imply $\vartheta^S \in V_+(S) \subset V(S)$ for all $S \subset N$.

C2. If V is individually rational monotonic, then $F(V) \geq \vartheta$ for all $V \in \Gamma_2$.

(*Individual rationality:* The solution is individually rational (i.e., $F^i(V) \geq \vartheta^i$ for all $i \in N$), provided that every player always contributes nonnegatively to all coalitions he may join.)

Given a nonempty coalition $T \subset N$ and a payoff vector $a \in \mathbb{R}^N$ with $a^i = 0$ for all $i \notin T$, let A_T be the c.b.p. defined by

$$A_T = \begin{cases} \{a^S\} + \mathbb{R}^S_{-} & \text{if } S \supset T, \\ \mathbb{R}^S_{-} & \text{otherwise.} \end{cases}$$

C3. If $F(A_T) = a$, then $F(V + A_T) = F(V) + a$ for all $V \in \Gamma_2$.

(*Translation invariance:* If the payoffs of each member i of T in each coalition that contains T are translated by a^i, and moreover the vector a is acceptable (meaning that the solution of the c.b.p. A_T is precisely a), then the solution of the new c.b.p. will be translated by a; note that the c.b.p. $W = V + A_T$ is given by $W(S) = V(S) + \{a^S\}$ for $S \supset T$ and $W(S) = V(S)$ otherwise.)

C4. If $T \subset N$ is such that $V(S) = W(S)$ for all $S \neq T$ and $V(T) \subset W(T)$, then $F^T(V) \leq F^T(W)$ for all $V, W \in \Gamma_2$.

(*Monotonicity:* If the feasible set of some coalition T is enlarged (and all the other feasible sets remain unchanged), then the payoff of each member of T does not decrease.)

On the space of c.b.p.'s, consider the product of the Hausdorff topologies for all $S \subset N$: A sequence $\{V_m\}_{m=1}^{\infty}$ converges to V if, for each $S \subset N$, the sequence $\{V_m(S)\}_{m=1}^{\infty}$ of subsets of $\mathbb{R}^S$ converges in the Hausdorff topology to $V(S)$.

C5. The function F is continuous on Γ_2.

(*Continuity:* If a sequence $\{V_m\}_{m=1}^{\infty}$ in Γ_2 converges to $V \in \Gamma_2$, then $F^i(V_m)$ converges to $F^i(V)$ for all $i \in N$.)

Theorem C (Kalai and Samet (1983)). A function F satisfies axioms C0 through C5 if and only if there exists $\lambda \in \mathbb{R}^N_{++}$ such that $F = E^\lambda$ is the λ-egalitarian function.

Thus, each E^λ (for $\lambda \in \mathbb{R}^N_{++}$) satisfies C0 through C5; moreover, those are the only functions to do so.

This completes the presentation of the three axiomatic systems. We note again that we have considered here only one of the various ways of characterizing the three solutions; there are other combinations of the postulates given (and others) that could do as well.

14.5 Discussion

This last section will include some remarks on the solutions presented earlier, together with a comparison of their properties in terms of the axioms presented in Section 14.4.

The basic assumption underlying all the approaches is that the only information available is the characteristic function of the coalitional bargaining problem. Thus, nothing is given on the extensive form of the bargaining process: how the players are discussing, who talks to whom, whether there are procedures for making proposals – or threats – and for rejecting them, for making coalitions, and so on. The solutions thus cannot depend on such data; moreover, any particular assumptions of this kind must necessarily be ad hoc. This implies that the solutions proposed must be based on rather general principles, which should hold in all situations described by the same characteristic function. The Nash solution in the two-person case is an example of such an axiomatic approach applied to the characteristic-function form. (See Sections 5 through 7 in Aumann (1983*a*) for further discussion on this point; and see Binmore (1983) for an "extensive-form" approach to three-person coalitional bargaining.)

The solutions discussed in this chapter are of two types. The Harsanyi and Shapley solutions depend only on the cardinal representation of the utility of each player separately. Thus, if a positive linear transformation is applied to some player's payoffs, the same transformation applies to the solutions as well (for rescaling, see axioms A2 and B2). The Kalai and Samet solutions are different. If only one player's utility is rescaled, the solution may change completely. Only *joint rescaling* of the utilities of *all* players leads to the same change in the solution as well. In the former two solutions, *independent* rescaling is allowed; here, only *common* rescaling. Formally, for every $i \in N$, let $l_i: \mathbb{R} \to \mathbb{R}$ be a positive linear transformation given by $l_i(x) = a^i x + b^i$, where $a^i > 0$ and b^i are real constants. For each c.b.p. V, let $W = l(V)$ be the transformed c.b.p., namely,

$$W(S) = \{l(x) = (l^i(x^i))_{i\in S} \in \mathbb{R}^S \mid x = (x^i)_{i\in S} \in V(S)\}$$

for all $S \subset N$. If a payoff vector $x \in \mathbb{R}^N$ is a Harsanyi or a Shapley solution of V, then the payoff vector $l(x) \in \mathbb{R}^N$ is a Harsanyi or a Shapley solution, respectively, of $W = l(V)$. Let $\lambda \in \mathbb{R}^N_{++}$; if a payoff vector $x \in \mathbb{R}^N$ is the Kalai and Samet λ-egalitarian solution of V, and *moreover* all the a^i are identical (i.e., $a^i = a$ for all $i \in N$), then the payoff vector $l(x)$ is the λ-egalitarian solution of $W = l(V)$.

According to Shapley's (1983) classification, the Harsanyi and Shapley solutions are both of category $CARD^N$, whereas the egalitarian solutions are of category $CARD_N$. The interpersonal comparison of utility is obtained endogenously in the former, and it is exogenously given in the latter.

We conclude this paper with a comparison of the three solutions via the axioms presented in Section 14.4. Table 14.1 points out for each solution function which axioms it satisfies and which it does not. The domain is Γ_1 for the Shapley and the Harsanyi functions, and Γ_2 for the Kalai and Samet functions. Note that axioms A1 through A5 regard the solution as a set-valued function that assigns a subset of $\mathbb{R}^N$ to each c.b.p. (possibly empty, since we consider Γ_1 and not Γ_4); similarly for axioms B1 through

Table 14.1.

Axiom	Solution function: Shapley	Harsanyi	Kalai and Samet
A1 (efficiency)	Yes	Yes	Yes
A2 (scale)	Yes	Yes	No[a]
A3 (additivity)	Yes	No[b]	No[b]
A4 (irrelevant alternatives)	Yes	Yes	Yes
A5 (unanimity games)	Yes	Yes	No[c]
B1 (efficiency)	Yes	Yes	Yes
B2 (scale)	Yes	Yes	No[a]
B3 (additivity)	Yes	Yes	Yes
B4 (irrelevant alternatives)	Yes	Yes	Yes
B5 (unanimity games)	No[d]	Yes	No[c]
B6 (inessential games)	No[e]	Yes	Yes
C1 (carrier)	Yes	Yes	Yes
C2 (individual rationality)	Yes	Yes	Yes
C3 (translation)	No[f]	No[f]	Yes
C4 (monotonicity)	No[g]	No[g]	Yes
C5 (continuity)	No[h]	No[h]	Yes

The instances where axioms are satisfied ("yes" in the table) follow easily from theorems A, B, and C, and straightforward arguments (see also Section 5 in Hart (1983) and Section 8 in Kalai and Samet (1983)).

[a] If $\lambda = c1_N$ for some $c \in \mathbb{R}_+$, then "yes."

[b] Let $N = \{1,2,3\}$, $U = U_{N,0}$, and $W = U_{\{1,2\},1}$; let $V(S) = U_{N,1}(S)$ for $S \neq \{1,2\}$ and $V(\{1,2\}) = \{x \in \mathbb{R}^{\{1,2\}} \mid x^1 + 2x^2 \leq 0 \text{ and } 2x^1 + x^2 \leq 3\}$. Then, $W = V + U$; each of the three games W, V, and U has a unique Harsanyi solution, namely, $z = (\frac{1}{2},\frac{1}{2},0)$, $x = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$, and $y = (0,0,0)$, respectively; and $z \neq x + y$. Furthermore, the symmetric egalitarian solution coincides with the Harsanyi solution for each of the three games.

[c] "Yes" for the symmetric egalitarian solution.

[d] See proposition 5.4 and axiom B5 in Hart (1983).

[e] See example 5.6 in Hart (1983).

[f] "No" already in the two-person case (the Nash solution): Let $N = \{1,2\}$, $a = (2,1)$, $T = N$, $V(\{i\}) = \mathbb{R}^{\{i\}}_{-}$, and $V(N) = \{x \in \mathbb{R}^N \mid x^1 + x^2 \leq 2\}$; the Nash (=Shapley = Harsanyi) solutions of A_N, V, and $V + A_N$ are, respectively, (2,1), (1,1), and (2.5,2.5).

[g] See Sections 1 and 2 in Kalai and Samet (1983).

[h] Upper-semi-continuity only: If x_m is a solution of V_m for every $m \geq 1$, $x_m \to x$, and $V_m \to V$, then x is a solution of V (recall condition (2b) in Section 14.2).

B6, with the range being the subsets of X (the set of payoff configurations). Axioms C1 through C5 refer to a (point-valued) function into $\mathbb{R}^N$, although they can be extended in a straightforward manner to apply to a set-valued function.

REFERENCES

Aumann, R. J. (1967): A Survey of Cooperative Games without Side Payments. In *Essays in Mathematical Economics in Honor of Oskar Morgenstern*, ed. by M. Shubik. Princeton University Press, Princeton, New Jersey, 3-27.

(1983*a*): *On the Non-Transferable Utility Value.* RM-55, Center for Research in Mathematical Economics and Game Theory, The Hebrew University, Jerusalem, forthcoming in *Econometrica.*

(1983*b*): *An Axiomatization of the Non-Transferable Utility Value.* RM-57, Center for Research in Mathematical Economics and Game Theory, The Hebrew University, Jerusalem, forthcoming in *Econometrica.*

Binmore, K. (1983): *Bargaining and Coalitions I.* Theoretical Economics DP-83/71, ICERD, London School of Economics.

Harsanyi, J. C. (1959): A Bargaining Model for the Cooperative *n*-Person Game. In *Contributions to the Theory of Games IV* (Annals of Mathematics Studies 40), ed. by A. W. Tucker and R. D. Luce. Princeton University Press, Princeton, New Jersey, 325-55.

(1963): A Simplified Bargaining Model for the *n*-Person Cooperative Game. *International Economic Review,* 4, 194-220.

(1977): *Rational Behavior and Bargaining Equilibrium in Games and Social Situations.* Cambridge University Press, Cambridge.

Hart, S. (1983): *An Axiomatization of Harsanyi's Non-Transferable Utility Solution.* DP-573, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, Illinois, forthcoming in *Econometrica.*

Kalai, E. (1977): Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons. *Econometrica,* 45, 1623-30.

Kalai, E., and D. Samet (1983): *A Monotonic Solution to General Cooperative Games.* DP-567, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, Illinois, forthcoming in *Econometrica.*

Nash, J. F. (1950): The Bargaining Problem. *Econometrica,* 18, 155-62.

Owen, G. (1972): Values of Games without Side Payments. *International Journal of Game Theory,* 1, 94-109.

Roth, A. E. (1979): *Axiomatic Models of Bargaining.* Springer-Verlag, Berlin, Heidelberg, New York.

Shapley, L. S. (1953): A Value for *n*-Person Games. In *Contributions to the Theory of Games II* (Annals of Mathematics Studies 28), ed. by H. W. Kuhn and A. W. Tucker. Princeton University Press, Princeton, New Jersey, 307-17.

(1969): Utility Comparison and the Theory of Games. In *La Décision: Aggrégation et Dynamique des Ordres de Préférence,* Editions du Centre National de la Recherche Scientifique, Paris, 251-63.

(1983): Bargaining without Cardinal Utility. Paper presented at the Conference on Game-Theoretic Models of Bargaining, University of Pittsburgh.