# Two(!) Good To Be ... Monotonic

Sergiu Hart

September 2024

CONFERENCE IN HONOR OF
PHIL RENY

# Two(!) Good To Be ... Monotonic

*Sergiu Hart*

Center for the Study of Rationality
Dept of Economics    Dept of Mathematics
The Hebrew University of Jerusalem

hart@huji.ac.il
http://www.ma.huji.ac.il/hart

# Joint work with

**_Phil Reny_**
Department of Economics
University of Chicago

**_Noam Nisan_**
Department of Computer Science
Hebrew University of Jerusalem

# Joint work with


**Ran Ben Moshe**

M.Sc., Department of Mathematics

Hebrew University of Jerusalem

**Yannai Gonczarowski**

Department of Economics

Department of Computer Science

Harvard University

# Papers

# Papers

- **Sergiu Hart and Phil Reny**
  "Revenue Maximization in Two Dimensions"
  (2010)

# Papers

- **Sergiu Hart and Phil Reny**
  "Revenue Maximization in Two Dimensions" (2010)

- **Sergiu Hart and Phil Reny**
  "Maximal Revenue with Multiple Goods: Nonmonotonicity and Other Observations" (2011; *Theoretical Economics* 2015)
  www.ma.huji.ac.il/hart/abs/monot-m.html

# Papers

# Papers

- **Sergiu Hart and Phil Reny**
  "Implementation of Reduced Form Mechanisms: A Simple Approach and a New Characterization"
  (2011; *Economic Theory Bulletin* 2015)
  www.ma.huji.ac.il/hart/abs/q-mech.html

# Papers

- **Sergiu Hart and Phil Reny**
  "Implementation of Reduced Form Mechanisms: A Simple Approach and a New Characterization"
  (2011; *Economic Theory Bulletin* 2015)
  www.ma.huji.ac.il/hart/abs/q-mech.html

- **Sergiu Hart and Phil Reny**
  "The Better Half of Selling Separately"
  (2016; *ACM Trans on Economics and Computation* 2019)
  www.ma.huji.ac.il/hart/abs/srev.html

# Papers

# Papers

- **Sergiu Hart and Noam Nisan**
  "Approximate Revenue Maximization with Multiple Items"
  (2012; *J Econ Theory* 2017)
  www.ma.huji.ac.il/hart/abs/m-approx.html

# Papers

- **Sergiu Hart and Noam Nisan**
  "Approximate Revenue Maximization with Multiple Items"
  (2012; *J Econ Theory* 2017)

  www.ma.huji.ac.il/hart/abs/m-approx.html

- **Sergiu Hart and Noam Nisan**
  "Selling Multiple Correlated Goods: Revenue Maximization and Menu-Size Complexity"
  (2013; *J Econ Theory* 2019)

  www.ma.huji.ac.il/hart/abs/m-corr.html

# Papers

# Papers

- **Ran Ben Moshe, Sergiu Hart and Noam Nisan**
  "Monotonic Mechanisms for Selling Multiple Goods"
  (2021)
  www.ma.huji.ac.il/hart/abs/mech-monot.html

# Papers

- **Ran Ben Moshe, Sergiu Hart and Noam Nisan**
  "Monotonic Mechanisms for Selling Multiple Goods"
  (2021)
  www.ma.huji.ac.il/hart/abs/mech-monot.html

- **Sergiu Hart and Noam Nisan**
  "Two Good To Be ... Three" (TT)
  (in preparation)

# Papers

# Papers

- **Yannai Gonczarowski and Sergiu Hart**
  “Buying Multiple Goods: Committing to Overbid" (TT)
  (in preparation)

# A Simple Problem

# A Simple Problem

- 1 **SELLER**

# A Simple Problem

- 1 SELLER
- 1 BUYER

# A Simple Problem

- $1$ SELLER
- $1$ BUYER
- $k$ GOODS (ITEMS)

# A Simple Problem

- $1$ **SELLER**
- $1$ **BUYER**
- $k$ **GOODS (ITEMS)**

**OBJECTIVE**:

**MAXIMIZE** the **REVENUE** of the **SELLER**

# A Simple Problem

- $1$ SELLER
- $1$ BUYER
- $k$ GOODS (ITEMS)

# A Simple Problem

- $1$ SELLER
- $1$ BUYER
- $k$ GOODS (ITEMS)
  - **values** of GOODS to BUYER :
    $X = (X_1, X_2, ..., X_k)$

# A Simple Problem

- 1 **SELLER**
- 1 **BUYER**
- $k$ **GOODS (ITEMS)**
  - **values** of **GOODS** to **BUYER** :
    $X = (X_1, X_2, ..., X_k)$
  - **additive** valuation
    (good 1 and good 2 = $X_1 + X_2$)

# A Simple Problem

- 1 **SELLER**
- 1 **BUYER**
- $k$ **GOODS (ITEMS)**
  - **values** of **GOODS** to **BUYER** :
    $X = (X_1, X_2, ..., X_k)$
  - **additive** valuation
    (good 1 and good 2 = $X_1 + X_2$)
  - **BUYER** **knows** the value $X$

# A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)
  - **values** of GOODS to BUYER :
    $X = (X_1, X_2, ..., X_k)$
  - **additive** valuation
    (good 1 and good 2 = $X_1 + X_2$)
  - BUYER **knows** the value $X$
  - SELLER does **not know** the value $X$

# A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)
  - **values** of GOODS to BUYER :
    $X = (X_1, X_2, ..., X_k)$
  - **additive** valuation
    (good 1 and good 2 = $X_1 + X_2$)
  - BUYER **knows** the value $X$
  - SELLER does **not know** the value $X$
  - $X$ **distributed** according to c.d.f. $\mathcal{F}$ on $\mathbb{R}^k_+$

# A Simple Problem

- $1$ SELLER
- $1$ BUYER
- $k$ GOODS (ITEMS)
  - **values** of GOODS to BUYER :
    $X = (X_1, X_2, ..., X_k)$
  - **additive** valuation
    (good $1$ and good $2$ = $X_1 + X_2$)
  - BUYER **knows** the value $X$
  - SELLER does **not know** the value $X$
  - $X$ **distributed** according to c.d.f. $\mathcal{F}$ on $\mathbb{R}^k_+$
  - SELLER **knows** the distribution $\mathcal{F}$ of $X$

# A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)
  - **values** of GOODS to BUYER :
    $X = (X_1, X_2, ..., X_k)$ (random variable)
  - **additive** valuation
    (good 1 and good 2 = $X_1 + X_2$)
  - BUYER **knows** the value $X$
  - SELLER does **not know** the value $X$
  - $X$ **distributed** according to c.d.f. $\mathcal{F}$ on $\mathbb{R}^k_+$
  - SELLER **knows** the distribution $\mathcal{F}$ of $X$

# A Simple Problem

- $1$ SELLER
- $1$ BUYER
- $k$ GOODS (ITEMS)

# A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)

SELLER and BUYER :

- **quasi-linear** utilities (i.e., additive in monetary payments)

# A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)

SELLER and BUYER :

- **quasi-linear** utilities (i.e., additive in monetary payments)
- **risk-neutral** (i.e., linear in probabilities)

# A Simple Problem

- 1 **SELLER**
- 1 **BUYER**
- $k$ **GOODS (ITEMS)**

**SELLER** and **BUYER** :

- **quasi-linear** utilities (i.e., additive in monetary payments)
- **risk-neutral** (i.e., linear in probabilities) (or: linear in quantities)

# A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)

SELLER and BUYER :

- **quasi-linear** utilities (i.e., additive in monetary payments)
- **risk-neutral** (i.e., linear in probabilities) (or: linear in quantities)

SELLER :

- **no value** and **no cost** for the GOODS

# A Simple Problem

- $1$ SELLER
- $1$ BUYER
- $k$ GOODS (ITEMS)

# A Simple Problem

- $1$ **SELLER**
- $1$ **BUYER**
- $k$ **GOODS (ITEMS)**

**OBJECTIVE**:

**MAXIMIZE** the **REVENUE** of the **SELLER**

# One Good

**ONE GOOD ($k = 1$):**

# One Good: Solution

**ONE GOOD ($k = 1$):**

---

Myerson 1981, Riley and Zeckhauser 1983, ...

# One Good: Solution

**ONE GOOD** ($k = 1$):

- **SELLER posts** a **PRICE** $p$

Myerson 1981, Riley and Zeckhauser 1983, ...

# One Good: Solution

**ONE GOOD ($k = 1$):**

- **SELLER posts** a **PRICE** $p$
- **BUYER chooses** between:
  - get the good and pay $p$, or
  - get nothing and pay nothing

---

Myerson 1981, Riley and Zeckhauser 1983, ...

# One Good: Solution

**ONE GOOD ($k = 1$):**

- **SELLER** **posts** a **PRICE** $p$
- **BUYER** **chooses** between:
  - get the good and pay $p$, or
  - get nothing and pay nothing
- $p$ such that **REVENUE** $R = p \cdot \Pr[X > p]$ $= p \cdot (1 - \mathcal{F}(p))$ is **MAXIMAL**

---

Myerson 1981, Riley and Zeckhauser 1983, ...

# One Good: Solution

**ONE GOOD ($k = 1$):**

- **SELLER posts** a **PRICE** $p$
- **BUYER chooses** between:
  - get the good and pay $p$, or
  - get nothing and pay nothing
- $p$ such that **REVENUE** $R = p \cdot \Pr[X > p]$ $= p \cdot (1 - \mathcal{F}(p))$ is **MAXIMAL**

$$\textrm{Rev}(X) = \max_p p \cdot (1 - \mathcal{F}(p))$$

---

Myerson 1981, Riley and Zeckhauser 1983, ...

# One Good: Solution

**ONE GOOD ($k = 1$):**

# One Good: Solution

**ONE GOOD ($k = 1$):**

**SELLER** and/or **BUYER** are **RISK-AVERSE** (rather than risk-neutral):

# One Good: Solution

**ONE GOOD ($k = 1$):**

**SELLER** and/or **BUYER** are **RISK-AVERSE** (rather than risk-neutral):

**Tomer Siedner** (2020)
"Optimal Selling With Risk-Averse Agents"
www.ma.huji.ac.il/hart/students.html#tomers

# One Good: Example

# One Good: Example

$$X \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases}$$

# One Good: Example

$$X \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases}$$

- $p = 10 \longrightarrow R = 10 \cdot 1 = 10$

# One Good: Example

$$X \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases}$$

- $p = 10 \longrightarrow R = 10 \cdot 1 = 10$
- $p = 22 \longrightarrow R = 22 \cdot 1/2 = 11$

# One Good: Example

$$X \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases}$$

- $p = 10 \longrightarrow R = 10 \cdot 1 = 10$
- $p = 22 \longrightarrow R = 22 \cdot 1/2 = 11$ $\longleftarrow$

# One Good: Example

$$X \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases}$$

- $p = 10 \longrightarrow R = 10 \cdot 1 = 10$
- $p = 22 \longrightarrow R = 22 \cdot 1/2 = 11$ $\longleftarrow$

$$\text{REV}(X) = 11 \qquad p = 22$$

# Two Goods

# Two Goods

**Two Goods ($k = 2$)**

# Two Goods

## Two Goods ($k = 2$), Independent

# Two Goods

**Two Goods ($k = 2$), Independent**

- sell separately:

# Two Goods

## Two Goods ($k = 2$), Independent

- sell separately:

  **PRICE** $= p_1$ for good $1$
  **PRICE** $= p_2$ for good $2$

# Two Goods

**Two Goods ($k = 2$), Independent**

# Two Goods: Example

**Two Goods ($k = 2$), Independent**

# Two Goods: Example

**Two Goods ($k = 2$), Independent**

$$\boldsymbol{X_1, X_2} \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases}$$

# Two Goods: Example

**Two Goods ($k = 2$), Independent**

$$X_1, X_2 \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases}$$

- $\textsc{Rev}(X_1) = \textsc{Rev}(X_2) = 11$

# Two Goods: Example

**Two Goods ($k = 2$), Independent**

$$X_1, X_2 \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases}$$

- $\text{REV}(X_1) = \text{REV}(X_2) = 11$
- $\text{REV}(X_1) + \text{REV}(X_2) = 11 + 11 = 22$

# Two Goods: Example

**Two Goods ($k = 2$), Independent**

$$X_1, X_2 \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases}$$

- $\text{REV}(X_1) = \text{REV}(X_2) = 11$
- $\text{REV}(X_1) + \text{REV}(X_2) = 11 + 11 = 22$
- sell the two goods together (**"bundle"**) for the price $p_{12} = 32$ :

# Two Goods: Example

**Two Goods ($k = 2$), Independent**

$$X_1, X_2 \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases}$$

- $\text{Rev}(X_1) = \text{Rev}(X_2) = 11$
- $\text{Rev}(X_1) + \text{Rev}(X_2) = 11 + 11 = 22$
- sell the two goods together (**"bundle"**) for the price $p_{12} = 32$ :

$$R = 32 \cdot 3/4 = 24 > 22$$

# Two Goods: Example

**Two Goods ($k = 2$), Independent**

$$X_1, X_2 \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases}$$

- $\text{REV}(X_1) = \text{REV}(X_2) = 11$
- $\text{REV}(X_1) + \text{REV}(X_2) = 11 + 11 = 22$
- sell the two goods together (**"bundle"**) for the price $p_{12} = 32$ :

$$R = 32 \cdot 3/4 = \boxed{24} > 22$$

# Two Goods: Example

**Two Goods ($k = 2$), Independent**

$$X_1, X_2 \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases}$$

- $\text{REV}(X_1) = \text{REV}(X_2) = 11$
- $\text{REV}(X_1) + \text{REV}(X_2) = 11 + 11 = 22$
- sell the two goods together (**"bundle"**) for the price $p_{12} = 32$ :

$$R = 32 \cdot 3/4 = \boxed{24} > 22$$

# General Mechanism

# General Mechanism

OUTCOME:

# General Mechanism

**OUTCOME**:

- $q_i$ = probability that **BUYER** gets good $i$

# General Mechanism

**OUTCOME**:

- $q_i$ = probability that **BUYER** gets good $i$
  $q = (q_1, ..., q_k) \in [0, 1]^k$

# General Mechanism

**OUTCOME**:

- $q_i$ = probability that **BUYER** gets good $i$
  $q = (q_1, ..., q_k) \in [0, 1]^k$
- $s$ = payment from **BUYER** to **SELLER** ("**REVENUE**")

# General Mechanism

**OUTCOME**:

- $q_i$ = probability that **BUYER** gets good $i$
  $q = (q_1, ..., q_k) \in [0, 1]^k$
- $s$ = payment from **BUYER** to **SELLER**
  (**"REVENUE"**)

**PAYOFF** (utility) of **BUYER** when his valuation is $x = (x_1, ..., x_k)$:

# General Mechanism

**OUTCOME**:

- $q_i$ = probability that **BUYER** gets good $i$
  $q = (q_1, ..., q_k) \in [0, 1]^k$
- $s$ = payment from **BUYER** to **SELLER** ("**REVENUE**")

**PAYOFF** (utility) of **BUYER** when his valuation is $x = (x_1, ..., x_k)$:

- $b = q_1 \cdot x_1 + ... + q_k \cdot x_k - s$

# General Mechanism

**OUTCOME**:

- $q_i$ = probability that **BUYER** gets good $i$
  $q = (q_1, ..., q_k) \in [0, 1]^k$

- $s$ = payment from **BUYER** to **SELLER**
  ("**REVENUE**")

**PAYOFF** (utility) of **BUYER** when his valuation is $x = (x_1, ..., x_k)$:

- $b = q_1 \cdot x_1 + ... + q_k \cdot x_k - s = q \cdot x - s$

# Simple Mechanism

# Simple Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

# Simple Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0,1]^k \times \mathbb{R}$$

# Simple Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0,1]^k \times \mathbb{R}$$

- **SELLER posts** a **MENU** $M$

# Simple Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}$$

- **SELLER posts** a **MENU** $M$
- **BUYER chooses** one **OUTCOME** in **MENU** $M$:

# Simple Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0,1]^k \times \mathbb{R}$$

- **SELLER posts** a **MENU** $M$
- **BUYER chooses** one **OUTCOME** in **MENU** $M$:
  - **OUTCOME** chosen by **BUYER** when his valuation is $x$: $(q(x), s(x)) \in M$

# Simple Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0,1]^k \times \mathbb{R}$$

- **SELLER posts** a **MENU** $M$
- **BUYER chooses** one **OUTCOME** in **MENU** $M$:
  - **OUTCOME** chosen by **BUYER** when his valuation is $x$: $(q(x), s(x)) \in M$
  - payoff of **SELLER**: $s(x)$

# Simple Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0,1]^k \times \mathbb{R}$$

- **SELLER posts** a **MENU** $M$
- **BUYER chooses** one **OUTCOME** in **MENU** $M$:
  - **OUTCOME** chosen by **BUYER** when his valuation is $x$: $(q(x), s(x)) \in M$
  - payoff of **SELLER**: $s(x)$
  - payoff of **BUYER**: $b(x) = q(x) \cdot x - s(x)$

# "Menu" Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0,1]^k \times \mathbb{R}$$

- **SELLER posts** a **MENU** $M$
- **BUYER chooses** one **OUTCOME** in **MENU** $M$:
  - **OUTCOME** chosen by **BUYER** when his valuation is $x$: $(q(x), s(x)) \in M$
  - payoff of **SELLER**: $s(x)$
  - payoff of **BUYER**: $b(x) = q(x) \cdot x - s(x)$

# "Menu" Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0,1]^k \times \mathbb{R}$$

- **SELLER posts** a **MENU** $M$
- **BUYER chooses** one **OUTCOME** in **MENU** $M$:
  - **OUTCOME** chosen by **BUYER** when his valuation is $x$: $(q(x), s(x)) \in M$
  - payoff of **SELLER**: $s(x)$
  - payoff of **BUYER**: $b(x) = q(x) \cdot x - s(x)$

---

***The Revelation Principle***:

# "Menu" Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}$$

- **SELLER posts** a **MENU** $M$
- **BUYER chooses** one **OUTCOME** in **MENU** $M$:
  - **OUTCOME** chosen by **BUYER** when his valuation is $x$: $(q(x), s(x)) \in M$
  - payoff of **SELLER**: $s(x)$
  - payoff of **BUYER**: $b(x) = q(x) \cdot x - s(x)$

---

***The Revelation Principle***: Every mechanism is equivalent to a **MENU MECHANISM**

# "Menu" Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}$$

- **SELLER posts** a **MENU** $M$
- **BUYER chooses** one **OUTCOME** in **MENU** $M$:
  - **OUTCOME** chosen by **BUYER** when his valuation is $x$: $(q(x), s(x)) \in M$
  - payoff of **SELLER**: $s(x)$
  - payoff of **BUYER**: $b(x) = q(x) \cdot x - s(x)$

---

***The Revelation Principle***: Every mechanism is equivalent to a **MENU MECHANISM** (*"direct mechanism"*)

# Buyer

# Buyer

- **Incentive Compatibility (IC)**

# Buyer

- **Incentive Compatibility (IC)**

$$q(x) \cdot x - s(x) \geq q(y) \cdot x - s(y)$$

(for all $x$ and $y$)

# Buyer

- **Incentive Compatibility (IC)**

  $$q(x) \cdot x - s(x) \geq q(y) \cdot x - s(y)$$

  (for all $x$ and $y$)

- **Individual Rationality (IR) / Participation**

# Buyer

- **Incentive Compatibility (IC)**

  $$q(x) \cdot x - s(x) \geq q(y) \cdot x - s(y)$$

  (for all $x$ and $y$)

- **Individual Rationality (IR) / Participation**

  $$q(x) \cdot x - s(x) \geq 0$$

  (for all $x$)

# Seller

# Seller

- **Maximize Revenue:**

# Seller

- **Maximize Revenue:**

  maximize

$$R = \mathrm{E}[s(X)] = \int s(x)\,\mathrm{d}\mathcal{F}(x)$$

# Seller

- **Maximize Revenue**:

  maximize

  $$R = \mathrm{E}[s(X)] = \int s(x) \mathrm{d}\mathcal{F}(x)$$

  subject to

  $(q, s)$ satisfies IC & IR

# Multiple Goods

# Multiple Goods

**Revenue maximizing mechanisms:**

# Multiple Goods

**Revenue maximizing mechanisms**:

1. post a price for each good separately

# Multiple Goods

**Revenue maximizing mechanisms**:

1. post a price for each good separately
2. post a price for the bundle

# Multiple Goods

**Revenue maximizing mechanisms**:

1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle

# Multiple Goods

**Revenue maximizing mechanisms**:

1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle
4. post prices for various lotteries

# Multiple Goods

**Revenue maximizing mechanisms**:

1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle
4. post prices for various lotteries

1 – 3: deterministic mechanisms
4: probabilistic mechanisms

# Multiple Goods

**Revenue maximizing mechanisms**:

1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle
4. post prices for various lotteries

# Multiple Goods

**Revenue maximizing mechanisms**:

1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle
4. post prices for various lotteries

---

Thanassoulis 2004, Pycia 2006,
Manelli & Vincent 2006, 2007, 2012
Pavlov 2011, Hart & Reny 2010, 2012, ...

# Multiple Goods, I.I.D. Uniform

# Multiple Goods, I.I.D. Uniform

$X_1, X_2, ..., X_k \sim \text{Uniform } [0, 1]$, i.i.d.

# Multiple Goods, I.I.D. Uniform

$X_1, X_2, \ldots, X_k \sim \text{Uniform}\ [0,1]$, i.i.d.

- $k = 1$: **MENU** $= \{0,\ x_1 - \frac{1}{2}\}$

# Multiple Goods, I.I.D. Uniform

$X_1, X_2, ..., X_k \sim \text{Uniform}\,[0,1]$, i.i.d.

- $k = 1$: **MENU** $= \{0,\ x_1 - \frac{1}{2}\}$
- $k = 2$: **MENU** $= \{0,\ x_i - \frac{2}{3},\ x_1 + x_2 - \frac{4-\sqrt{2}}{3}\}$

# Multiple Goods, I.I.D. Uniform

$X_1, X_2, \ldots, X_k \sim$ Uniform $[0,1]$, i.i.d.

- $k = 1$: **MENU** $= \{0,\ x_1 - \frac{1}{2}\}$
- $k = 2$: **MENU** $= \{0,\ x_i - \frac{2}{3},\ x_1 + x_2 - \frac{4-\sqrt{2}}{3}\}$
- $k = 3$: **MENU** $=$
  $\{0,\ x_i - \frac{3}{4},\ x_i + x_j - \frac{6-\sqrt{2}}{4},\ x_1 + x_2 + x_3 - s\}$

# Multiple Goods, I.I.D. Uniform

$X_1, X_2, \ldots, X_k \sim \text{Uniform}\,[0,1]$, i.i.d.

- $k = 1$: **MENU** $= \{0,\ x_1 - \frac{1}{2}\}$
- $k = 2$: **MENU** $= \{0,\ x_i - \frac{2}{3},\ x_1 + x_2 - \frac{4-\sqrt{2}}{3}\}$
- $k = 3$: **MENU** $=$
  $\{0,\ x_i - \frac{3}{4},\ x_i + x_j - \frac{6-\sqrt{2}}{4},\ x_1 + x_2 + x_3 - s\}$

  where $s = \frac{9}{4} - \frac{\sqrt{6}}{4}\cos\left(\frac{1}{3}\arctan\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right) - \frac{3\sqrt{2}}{4}\sin\left(\frac{1}{3}\arctan\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right)$

# Multiple Goods, I.I.D. Uniform

$X_1, X_2, ..., X_k \sim \text{Uniform } [0, 1]$, i.i.d.

- $k = 1$: **MENU** $= \{0,\ x_1 - \frac{1}{2}\}$
- $k = 2$: **MENU** $= \{0,\ x_i - \frac{2}{3},\ x_1 + x_2 - \frac{4-\sqrt{2}}{3}\}$
- $k = 3$: **MENU** $=$
  $\{0,\ x_i - \frac{3}{4},\ x_i + x_j - \frac{6-\sqrt{2}}{4},\ x_1 + x_2 + x_3 - s\}$
  where $s \approx 1.2257... =$ solution of 3rd degree equation with coefficients in $\mathbb{Q}[\sqrt{2}]$

# Multiple Goods, I.I.D. Uniform

$X_1, X_2, \ldots, X_k \sim$ Uniform $[0,1]$, i.i.d.

- $k = 1$: **MENU** $= \{0,\ x_1 - \frac{1}{2}\}$
- $k = 2$: **MENU** $= \{0,\ x_i - \frac{2}{3},\ x_1 + x_2 - \frac{4-\sqrt{2}}{3}\}$
- $k = 3$: **MENU** $=$ $\{0,\ x_i - \frac{3}{4},\ x_i + x_j - \frac{6-\sqrt{2}}{4},\ x_1 + x_2 + x_3 - s\}$
- . . .

# Multiple Goods, I.I.D. Uniform

$X_1, X_2, \ldots, X_k \sim$ Uniform $[0, 1]$, i.i.d.

- $k = 1$: **MENU** $= \{0,\ x_1 - \frac{1}{2}\}$
- $k = 2$: **MENU** $= \{0,\ x_i - \frac{2}{3},\ x_1 + x_2 - \frac{4-\sqrt{2}}{3}\}$
- $k = 3$: **MENU** $=$
  $\{0,\ x_i - \frac{3}{4},\ x_i + x_j - \frac{6-\sqrt{2}}{4},\ x_1 + x_2 + x_3 - s\}$
- . . .

---

Manelli & Vincent 2006, Hart & Reny 2010,
Giannakopoulos & Koutsoupias 2014,
Daskalakis, Deckelbaum & Tzamos 2017

# Monotonicity

# Monotonicity

**Valuations ("willingness to pay")** of **BUYER increase**

# Monotonicity

Valuations ("willingness to pay") of BUYER increase

$\Rightarrow$ Maximal revenue of SELLER increases

# Monotonicity

**Valuations ("willingness to pay")** of
**BUYER increase**
$\Longrightarrow$ **Maximal revenue** of **SELLER increases**

**Proof for $k = 1$.**

# Monotonicity

**Valuations ("willingness to pay")** of **BUYER increase**
$\Longrightarrow$ **Maximal revenue** of **SELLER increases**

**Proof for $k = 1$.** For every $y > x$:

# Monotonicity

Valuations ("willingness to pay") of BUYER **increase**

$\Longrightarrow$ Maximal revenue of SELLER **increases**

**Proof for** $k = 1$. For every $y > x$:

$$q(x)\,x - s(x) \geq q(y)\,x - s(y) \qquad \text{(IC: } x \nrightarrow y\text{)}$$

# Monotonicity

Valuations ("willingness to pay") of BUYER increase

$\Rightarrow$ Maximal revenue of SELLER increases

**Proof for $k = 1$.** For every $y > x$:

$$q(x)\,x - s(x) \ge q(y)\,x - s(y) \qquad (\text{IC: } x \not\to y)$$

$$q(y)\,y - s(y) \ge q(x)\,y - s(x) \qquad (\text{IC: } y \not\to x)$$

# Monotonicity

**Valuations ("willingness to pay")** of **BUYER increase**

$\Rightarrow$ **Maximal revenue** of **SELLER increases**

**Proof for $k = 1$.** For every $y > x$:

$$q(x)\,x - s(x) \geq q(y)\,x - s(y) \qquad \text{(IC: } x \nrightarrow y\text{)}$$

$$q(y)\,y - s(y) \geq q(x)\,y - s(x) \qquad \text{(IC: } y \nrightarrow x\text{)}$$

$$\Rightarrow\ q(y)(y - x) \geq q(x)(y - x) \qquad \text{(add)}$$

# Monotonicity

**Valuations ("willingness to pay")** of **BUYER increase**
$\Rightarrow$ **Maximal revenue** of **SELLER increases**

**Proof for $k = 1$.** For every $y > x$:

$$q(x)\,x - s(x) \ge q(y)\,x - s(y) \qquad (\text{IC: } x \not\to y)$$

$$q(y)\,y - s(y) \ge q(x)\,y - s(x) \qquad (\text{IC: } y \not\to x)$$

$$\Rightarrow\ q(y)(y - x) \ge q(x)(y - x) \qquad (\text{add})$$

$$\Rightarrow\ q(y) \ge q(x) \qquad (y > x)$$

# Monotonicity

**Valuations ("willingness to pay")** of **BUYER increase**
$\Longrightarrow$ **Maximal revenue** of **SELLER increases**

**Proof for $k = 1$.** For every $y > x$:

$$q(x)\,x - s(x) \geq q(y)\,x - s(y) \qquad \text{(IC: } x \not\to y\text{)}$$

$$q(y)\,y - s(y) \geq q(x)\,y - s(x) \qquad \text{(IC: } y \not\to x\text{)}$$

$$\Rightarrow\ q(y)(y - x) \geq q(x)(y - x) \qquad \text{(add)}$$

$$\Rightarrow\ q(y) \geq q(x) \qquad (y > x)$$

$$s(y) - s(x) \geq (q(y) - q(x))\,x \qquad \text{(IC: } x \not\to y\text{)}$$

# Monotonicity

Valuations ("willingness to pay") of BUYER increase
$\Rightarrow$ Maximal revenue of SELLER increases

**Proof for $k = 1$.** For every $y > x$:

$$q(x)\,x - s(x) \geq q(y)\,x - s(y) \qquad \text{(IC: } x \not\to y\text{)}$$

$$q(y)\,y - s(y) \geq q(x)\,y - s(x) \qquad \text{(IC: } y \not\to x\text{)}$$

$$\Rightarrow\ q(y)(y - x) \geq q(x)(y - x) \qquad \text{(add)}$$

$$\Rightarrow\ q(y) \geq q(x) \qquad (y > x)$$

$$s(y) - s(x) \geq (q(y) - q(x))\,x \qquad \text{(IC: } x \not\to y\text{)}$$

$$\Rightarrow\ s(y) - s(x) \geq 0$$

# Monotonicity

**Valuations ("willingness to pay")** of **BUYER** **increase**
$\Longrightarrow$ **Maximal revenue** of **SELLER** **increases**

**Proof for $k = 1$.** For every $y > x$:

$\Rightarrow \; s(y) - s(x) \geq 0$

# Monotonicity

Valuations ("willingness to pay") of BUYER increase
$\Rightarrow$ Maximal revenue of SELLER increases

Proof for $k = 1$.

- $y > x \Rightarrow s(y) \geq s(x)$

# Monotonicity

**Valuations ("willingness to pay")** of **BUYER increase**
$\Rightarrow$ **Maximal revenue** of **SELLER increases**

**Proof for $k = 1$.**

- $y > x \Rightarrow s(y) \geq s(x)$
- Every **IC** mechanism has **monotonic** $s$

# Monotonicity

**Valuations ("willingness to pay")** of **BUYER increase**
$\Rightarrow$ **Maximal revenue** of **SELLER increases**

**Proof for $k = 1$.**

- $y > x \;\Rightarrow\; s(y) \geq s(x)$
- Every **IC** mechanism has **monotonic** $s$
- $\Rightarrow$ Revenue of every **IC** mechanism is **monotonic** w.r.t. to **BUYER** valuations

# Monotonicity

**Valuations ("willingness to pay")** of **BUYER** **increase**
$\Rightarrow$ **Maximal revenue** of **SELLER** **increases**

**Proof for** $k = 1$.

- $y > x \;\Rightarrow\; s(y) \geq s(x)$
- Every **IC** mechanism has **monotonic** $s$
- $\Rightarrow$ Revenue of every **IC** mechanism is **monotonic** w.r.t. to **BUYER** valuations
- $\Rightarrow$ Maximal revenue is **monotonic** w.r.t. **BUYER** valuations

# Monotonicity

**Valuations ("willingness to pay")** of **BUYER increase**
$\Rightarrow$ **Maximal revenue** of **SELLER increases**

**Proof for $k = 1$.**

- $y > x \;\Rightarrow\; s(y) \geq s(x)$
- Every **IC** mechanism has **monotonic** $s$
- $\Rightarrow$ Revenue of every **IC** mechanism is **monotonic** w.r.t. to **BUYER** valuations
- $\Rightarrow$ Maximal revenue is **monotonic** w.r.t. **BUYER** valuations

**Proof for $k > 1$ ?**

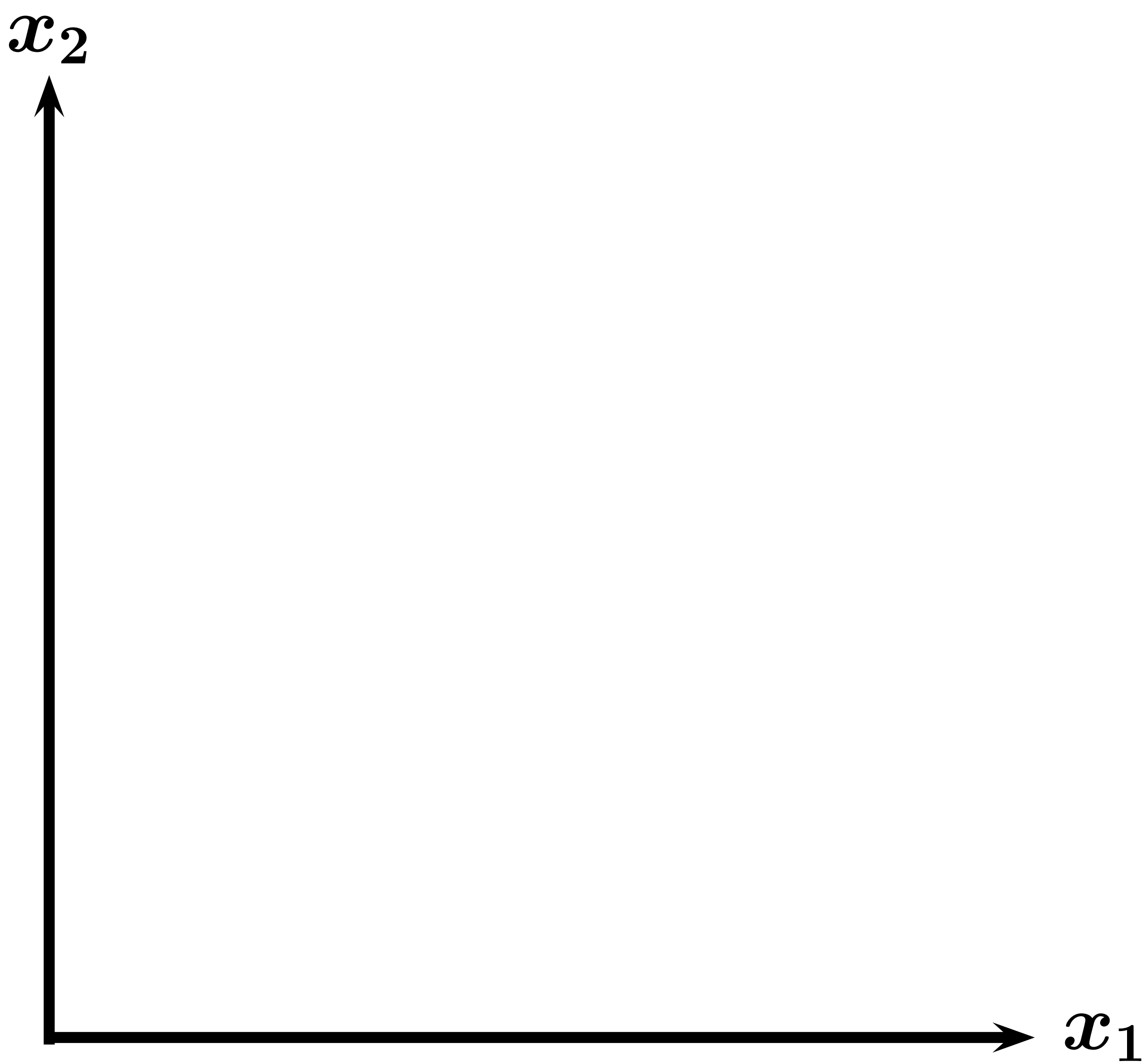
# Non-Monotonic Mechanism ($k = 2$)

# Non-Monotonic Mechanism ($k = 2$)

MENU $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$

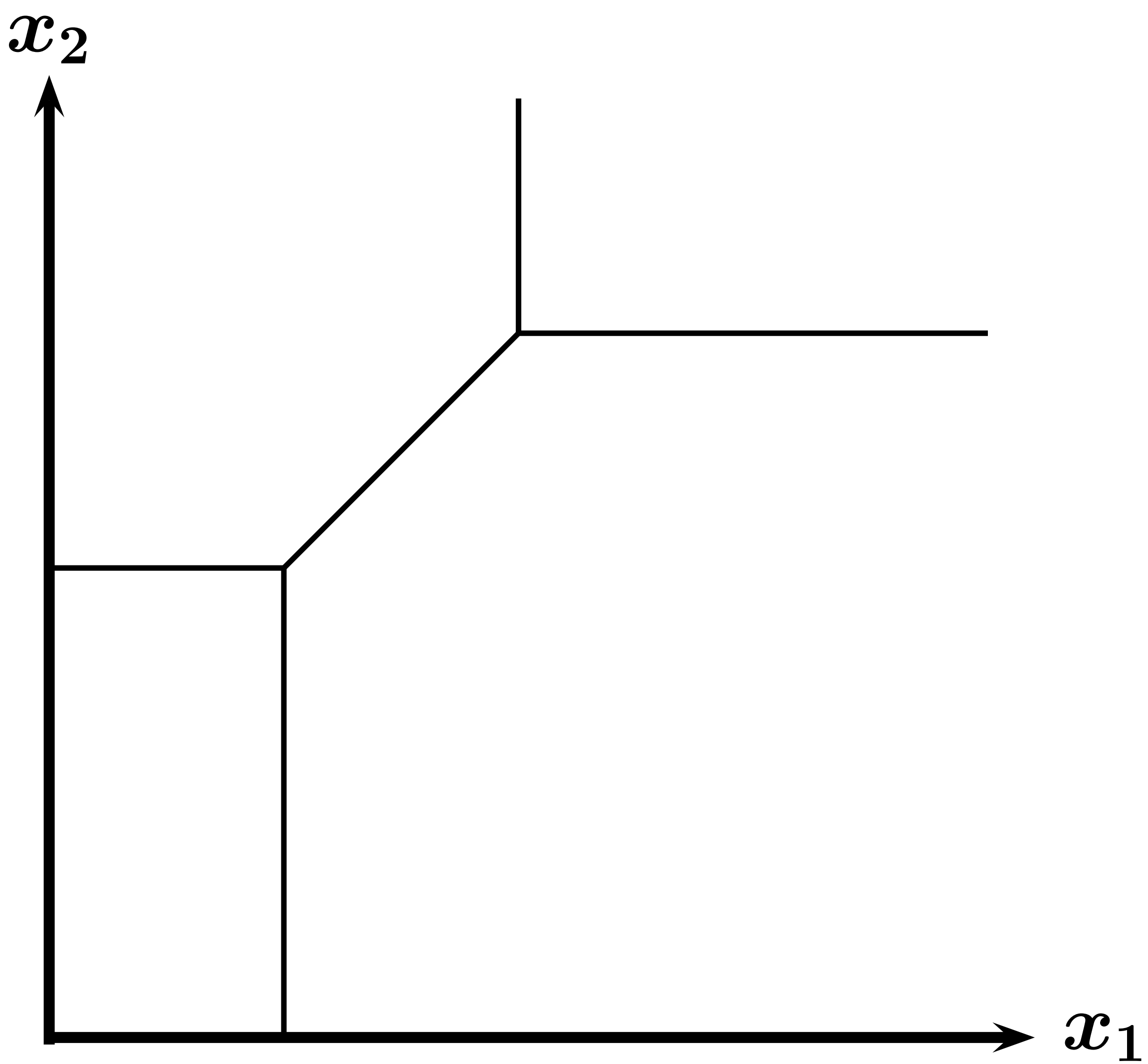
# Non-Monotonic Mechanism ($k = 2$)

**MENU** $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$



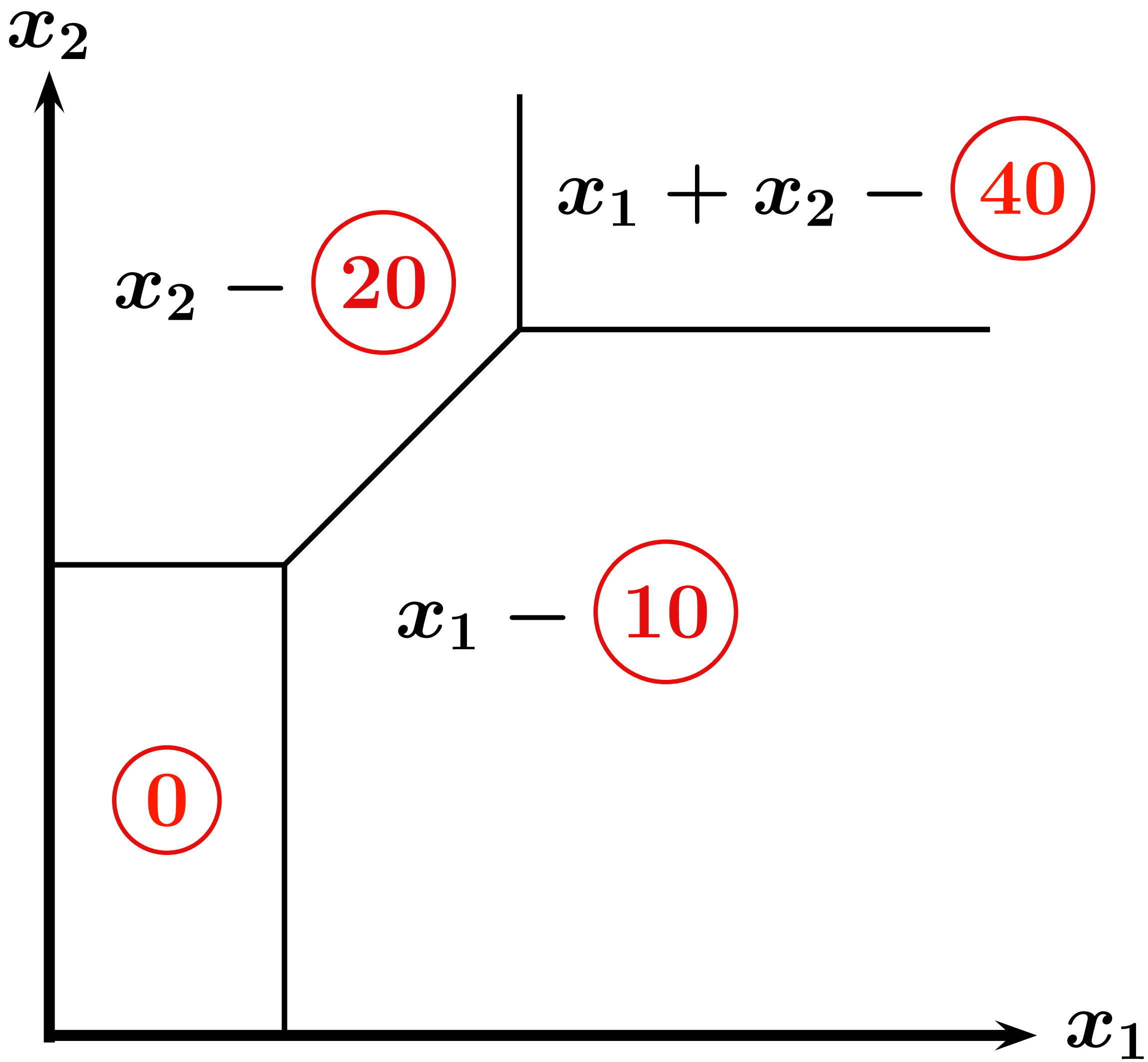
# Non-Monotonic Mechanism ($k = 2$)

$\textsf{MENU} = \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$



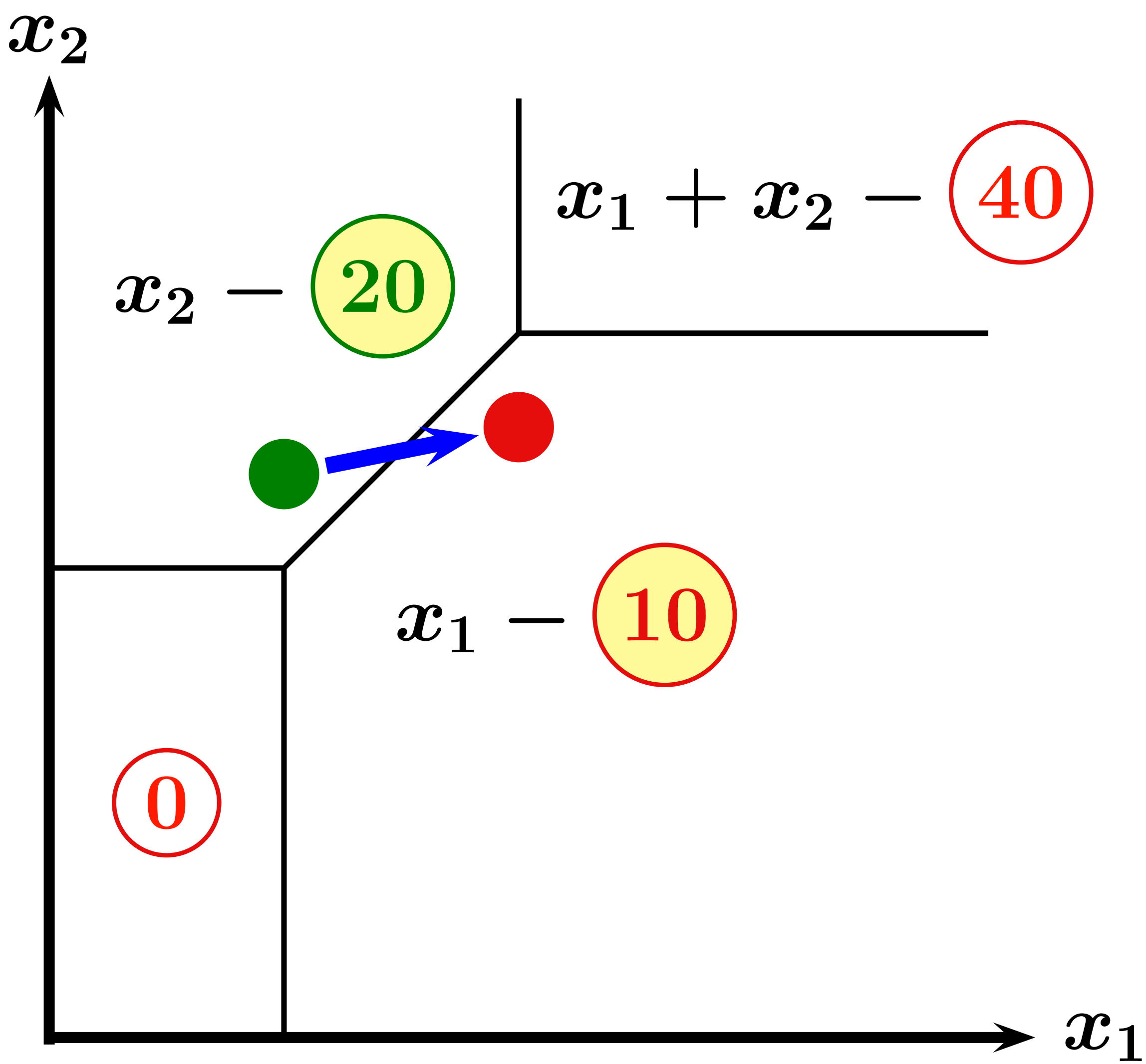
# Non-Monotonic Mechanism ($k = 2$)

MENU $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$



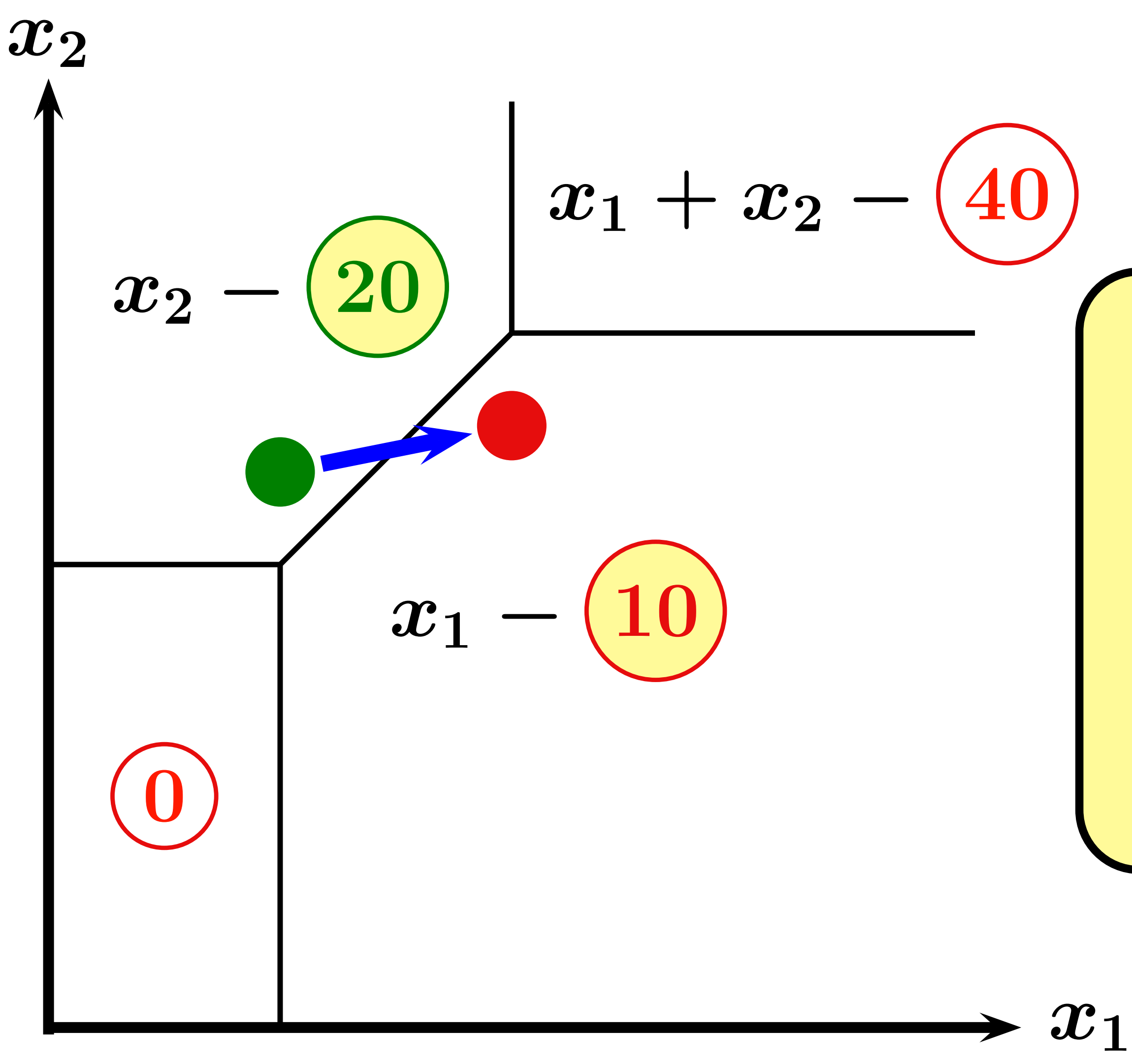
# Non-Monotonic Mechanism ($k = 2$)

MENU $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$



# Non-Monotonic Mechanism ($k = 2$)

**MENU** $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$

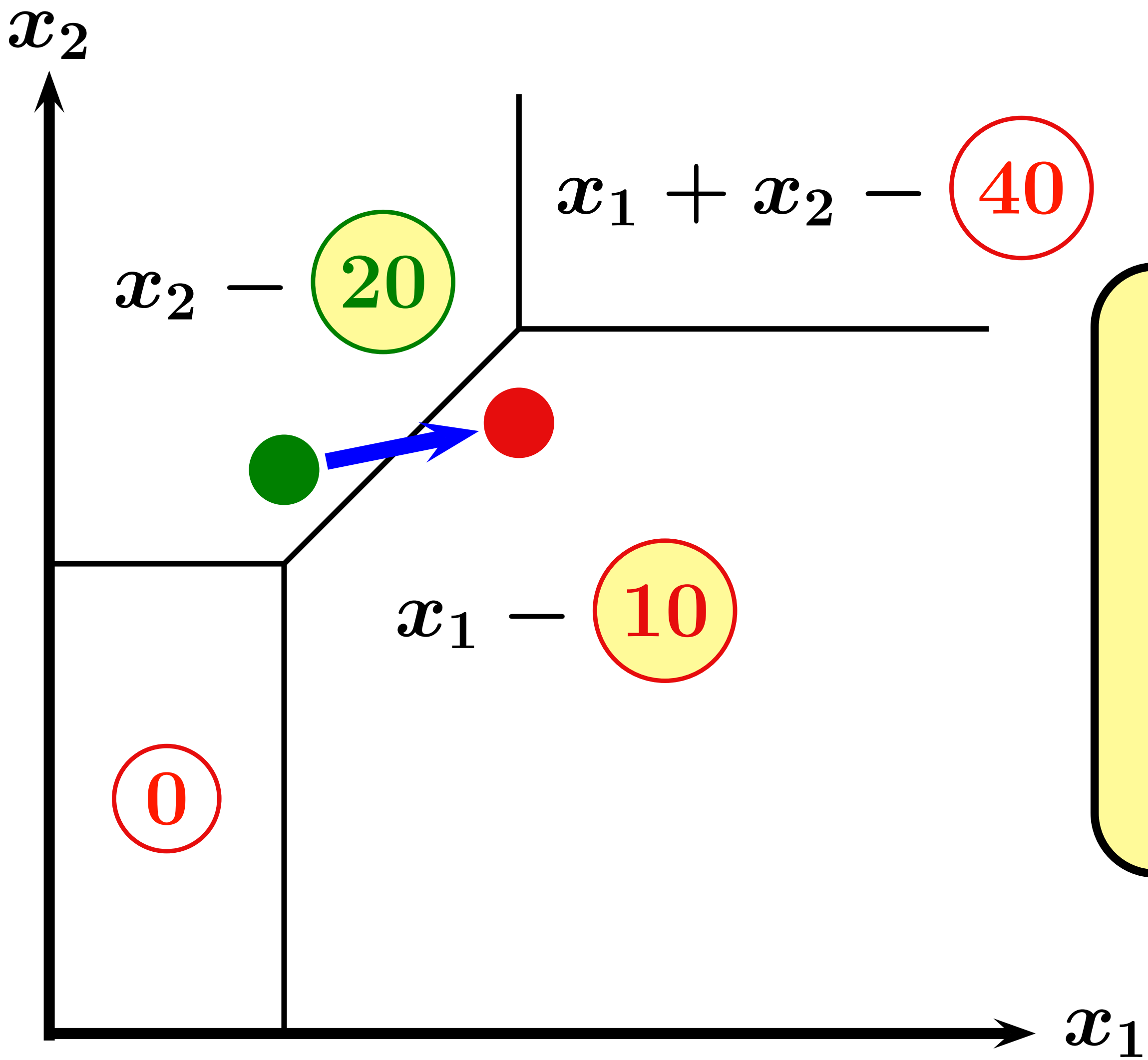



$(10, 24)$ : $x_2 - 20$

$(20, 26)$ : $x_1 - 10$

# Non-Monotonic Mechanism ($k = 2$)

**MENU** $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$




$(10, 24)$ : $x_2 - 20$

$(20, 26)$ : $x_1 - 10$

$x_1$ **increases**

$x_2$ **increases**

$s$ **DECREASES !**

# Non-Monotonic Mechanism

**MENU** $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$

# Non-Monotonic Mechanism

**MENU** $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$

- There exist 2-good valuations $X$ for which this mechanism **MAXIMIZES REVENUE**

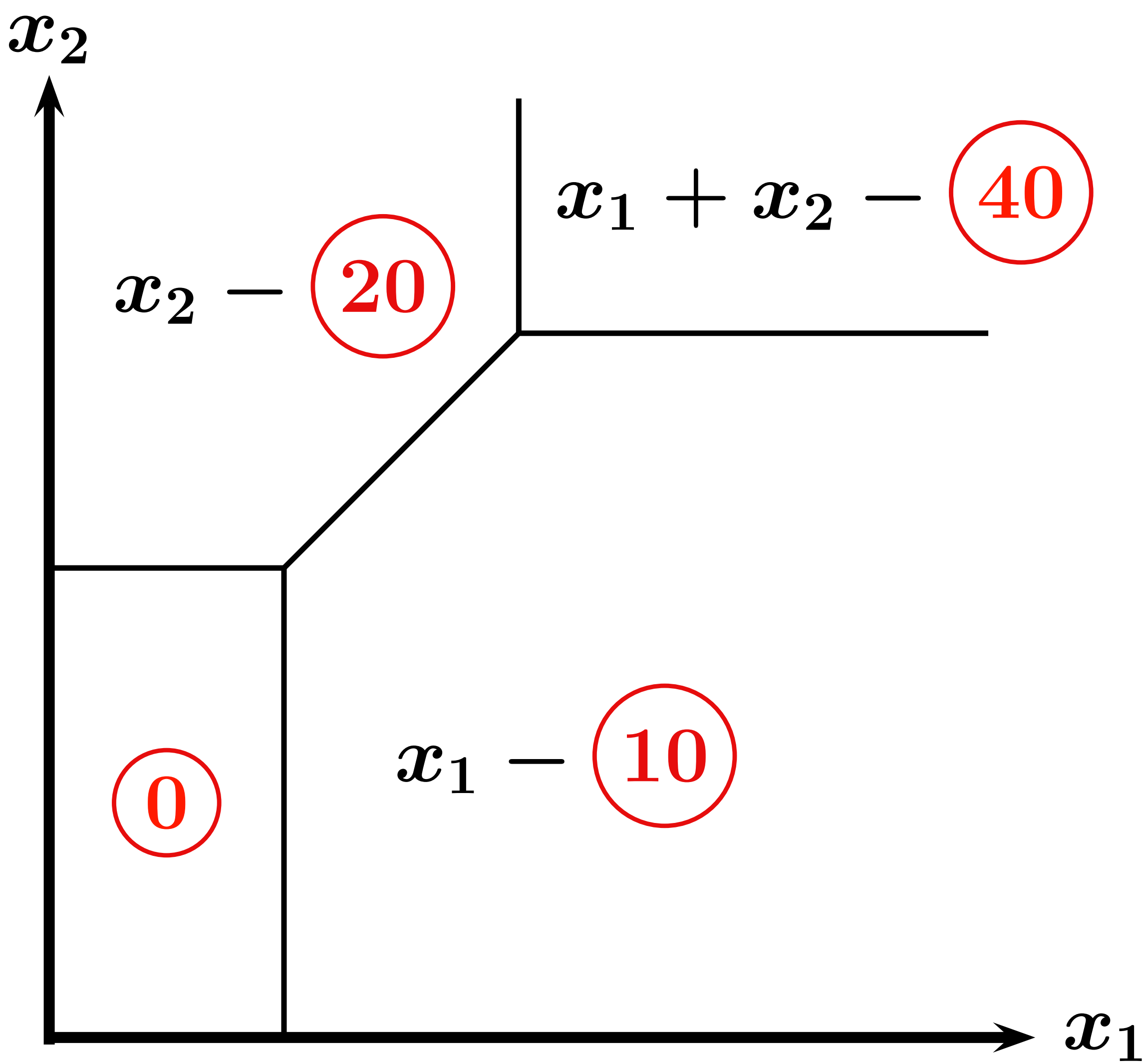
# Non-Monotonic Mechanism

**MENU** $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$

- There exist 2-good valuations $X$ for which this mechanism **MAXIMIZES REVENUE** (moreover: unique maximizer; robust)

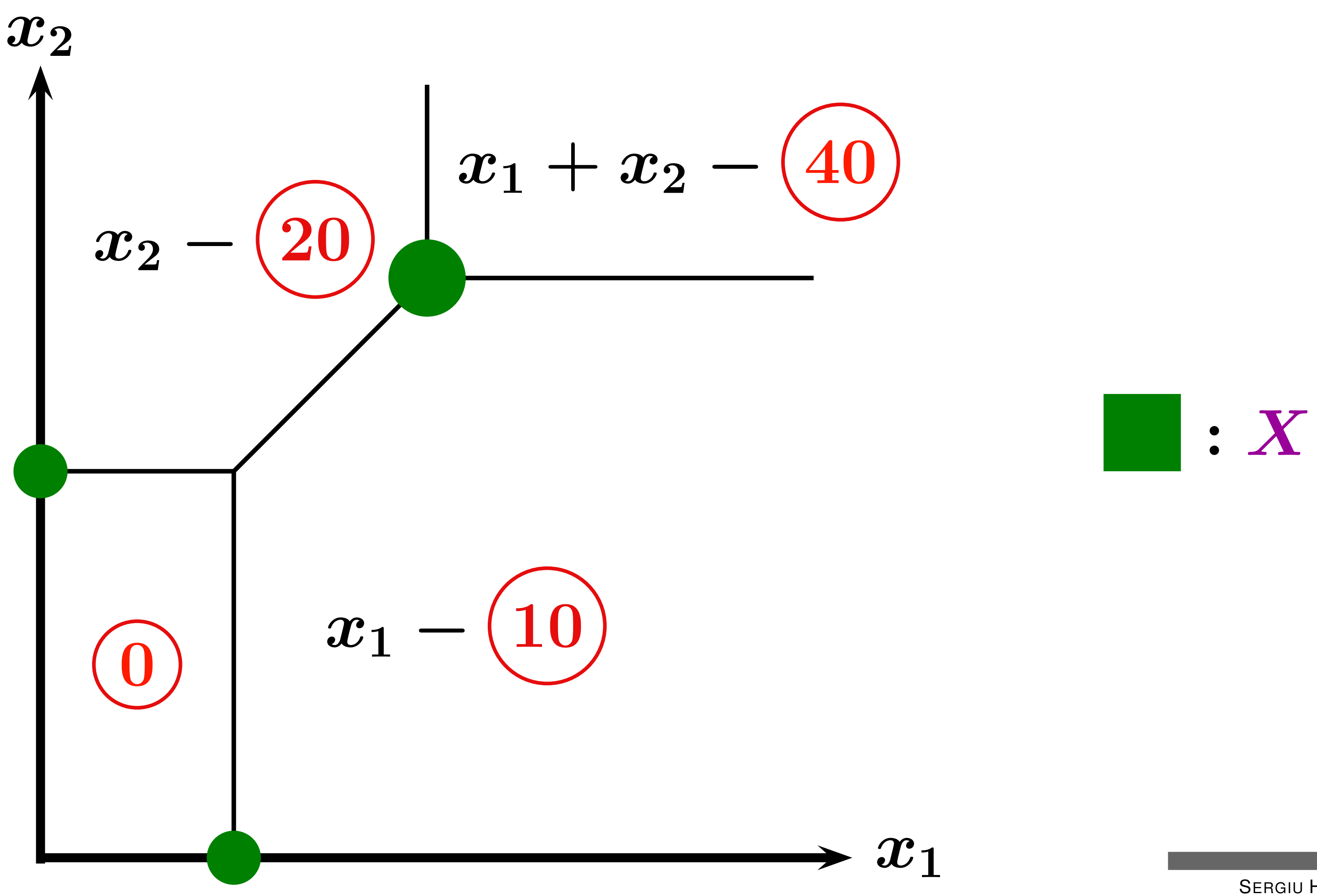
# Non-Monotonic Mechanism

**MENU** $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$



# Non-Monotonic Mechanism

**MENU** $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$



# Non-Monotonic Mechanism

**MENU** $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$

- There exist 2-good valuations $X$ for which this mechanism **MAXIMIZES REVENUE** (moreover: unique maximizer; robust)

# Non-Monotonic Mechanism

**MENU** $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$

- There exist 2-good valuations $X$ for which this mechanism **MAXIMIZES REVENUE** (moreover: unique maximizer; robust)

- There exist 2-good valuations $X, Y$ s.t.

$$Y \geq X \quad \text{and} \quad \text{REV}(Y) < \text{REV}(X)$$

# Non-Monotonic Revenue

**MENU** $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$

- There exist 2-good valuations $X$ for which this mechanism **MAXIMIZES REVENUE** (moreover: unique maximizer; robust)

- There exist 2-good valuations $X, Y$ s.t.

$$Y \geq X \quad \text{and} \quad \text{REV}(Y) < \text{REV}(X)$$

($Y \geq X$: first-order stochastic dominance)

# Non-Monotonic Revenue

MENU $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$

- There exist 2-good valuations $X$ for which this mechanism **MAXIMIZES REVENUE** (moreover: unique maximizer; robust)
- There exist 2-good valuations $X, Y$ s.t.

$$Y \geq X \quad \text{and} \quad \text{REV}(Y) < \text{REV}(X)$$

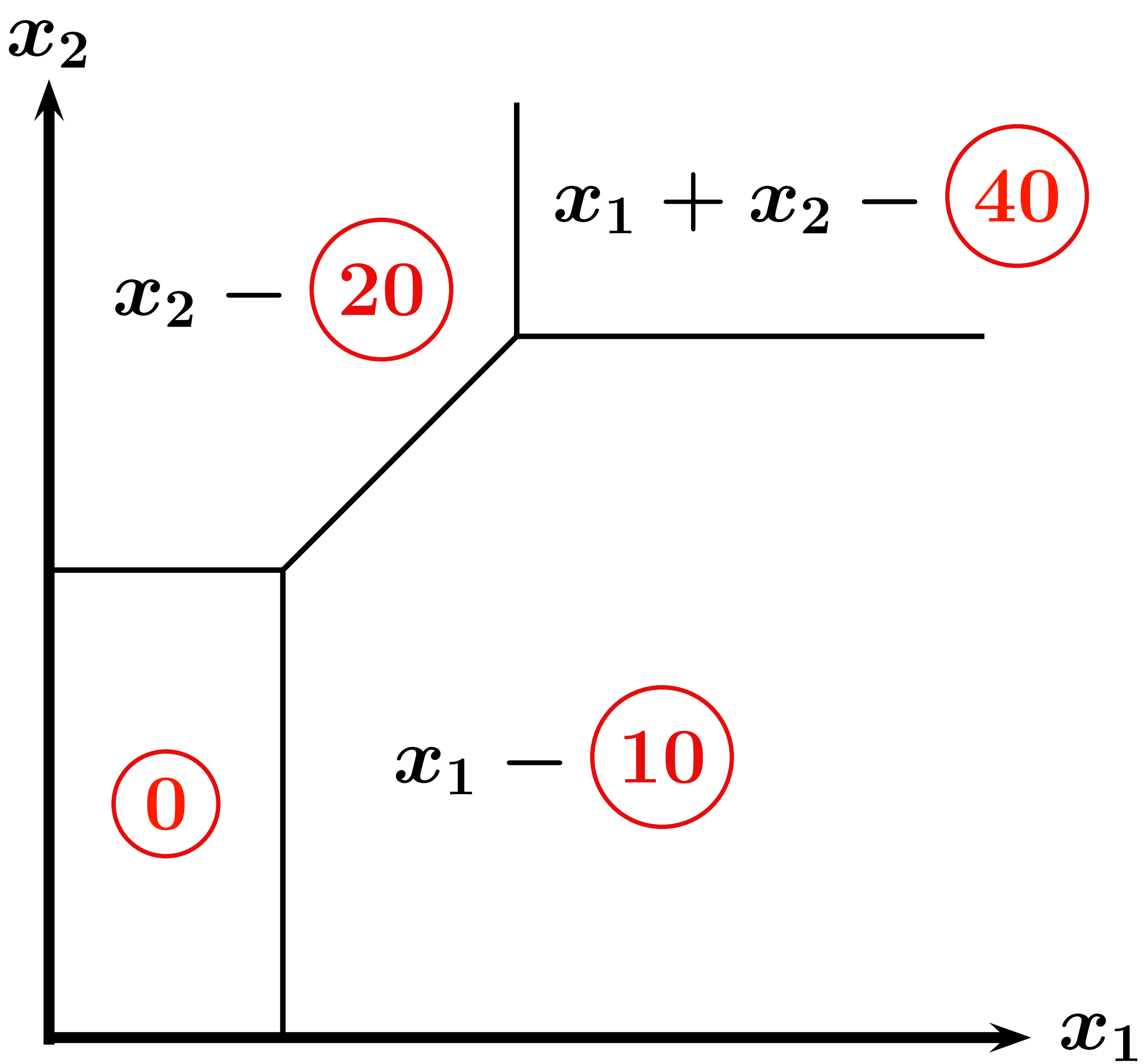
($Y \geq X$: first-order stochastic dominance)

---

Hart & Reny 2015

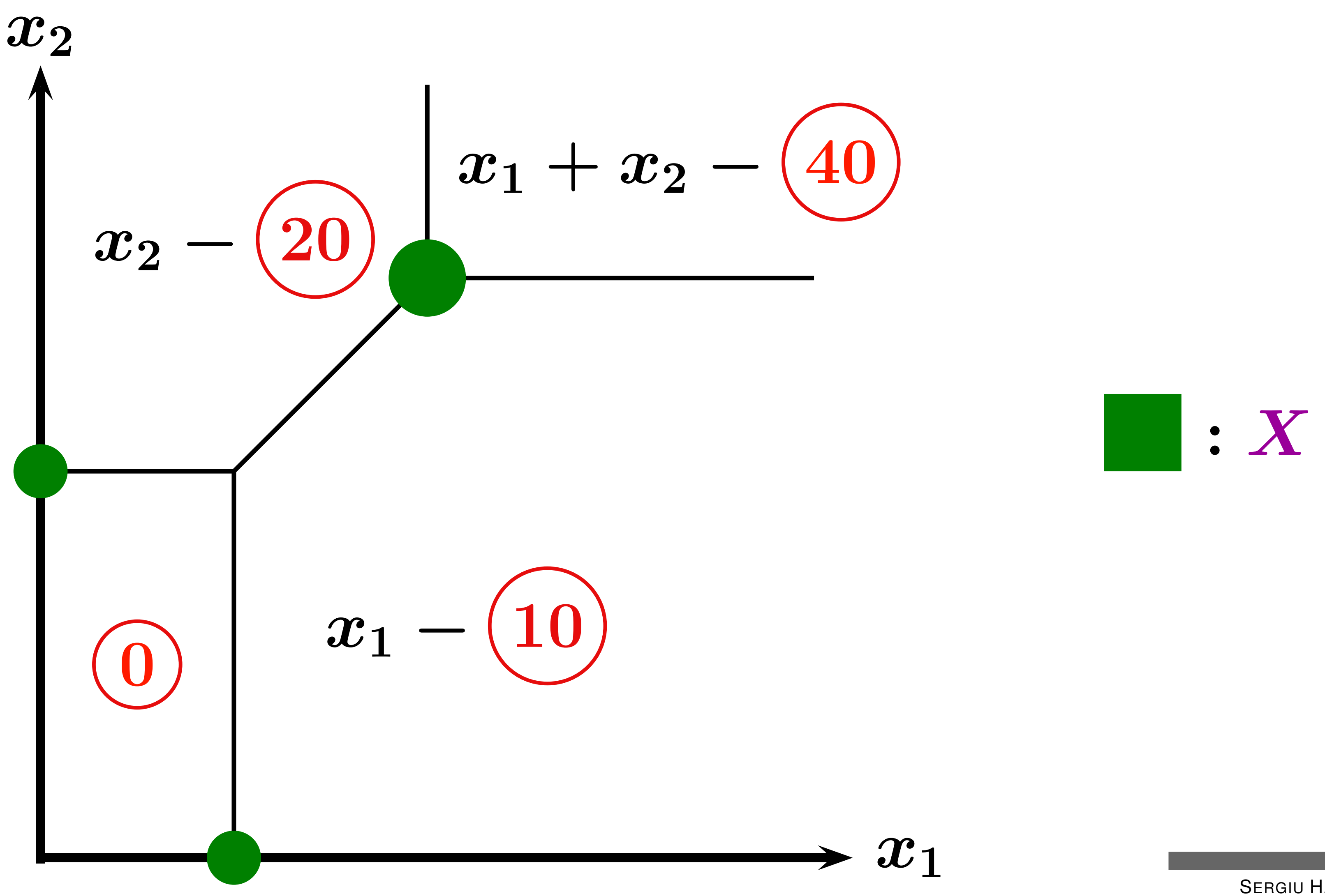
# Non-Monotonic Revenue

MENU $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$



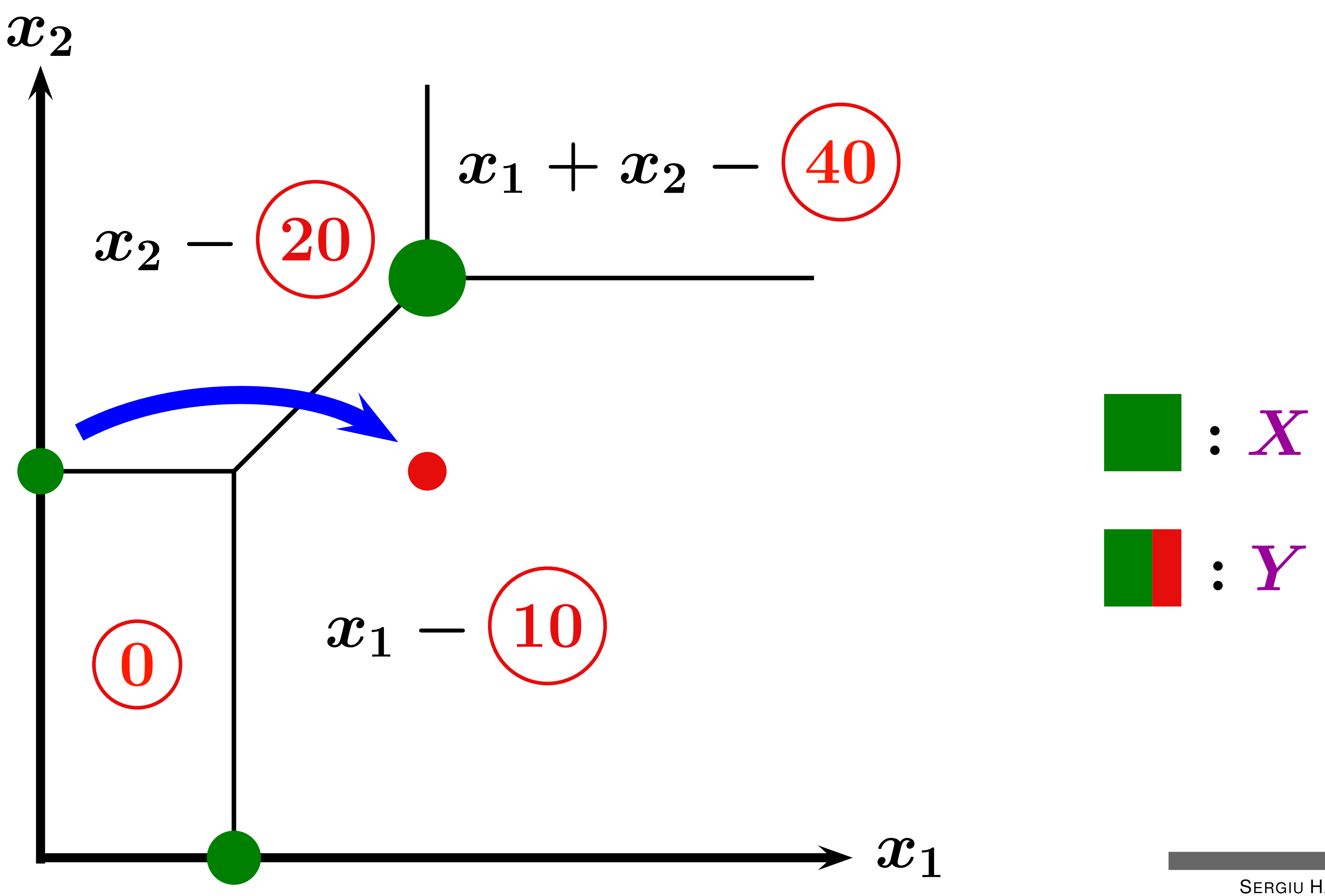
# Non-Monotonic Revenue

$\textsf{MENU} = \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$



# Non-Monotonic Revenue

MENU $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$



# Non-Monotonic Revenue

**MENU** $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$

# Non-Monotonic Revenue

MENU $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$

$$X_\alpha \sim \begin{cases} (10, 0) & \text{w/probability } 1/4 \\ (0, 20) & \text{w/probability } 1/4 - \alpha \\ (20, 20) & \text{w/probability } \alpha \\ (20, 30) & \text{w/probability } 1/2 \end{cases}$$

# Non-Monotonic Revenue

$\text{MENU} = \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$

$$X_\alpha \sim \begin{cases} (10, 0) & \text{w/probability } 1/4 \\ (0, 20) & \text{w/probability } 1/4 - \alpha \\ (20, 20) & \text{w/probability } \alpha \\ (20, 30) & \text{w/probability } 1/2 \end{cases}$$

$$\text{REV}(X_\alpha) = 27.5 - \alpha \qquad (0 \le \alpha \le 1/12)$$

# Non-Monotonicity: Questions

# Non-Monotonicity: Questions

When the willingness to pay of **BUYER** increases:

# Non-Monotonicity: Questions

When the willingness to pay of **BUYER** increases:

- How large can the loss in revenue be?

# Non-Monotonicity: Questions

When the willingness to pay of **BUYER** increases:

- How large can the loss in revenue be?
- Is there a way for **SELLER** to avoid this loss in revenue?

# Non-Monotonicity Loss

# Non-Monotonicity Loss

**QUESTION.** How large can the loss in revenue be when the willingness to pay of **BUYER** increases?

# Non-Monotonicity Loss

**QUESTION.** How large can the loss in revenue be when the willingness to pay of **BUYER** increases?

$$\lambda := \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)}$$

# Non-Monotonicity Loss

**Question.** How large can the loss in revenue be when the willingness to pay of **buyer** increases?

$$\lambda := \inf_{Y \geq X} \frac{\text{Rev}(Y)}{\text{Rev}(X)}$$

$(0 \leq \lambda \leq 1)$

# Non-Monotonicity Loss

**QUESTION.** How large can the loss in revenue be when the willingness to pay of **BUYER** increases?

$$\lambda := \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)}$$

$(0 \leq \lambda \leq 1)$

- 1 good: $\lambda = 1$

# Non-Monotonicity Loss

**QUESTION.** How large can the loss in revenue be when the willingness to pay of **BUYER** increases?

$$\lambda := \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)}$$

$(0 \leq \lambda \leq 1)$

- 1 good: $\lambda = 1$
- 2 or more goods: $\lambda = 0$

# Non-Monotonicity Loss ($k \geq 2$)

# Non-Monotonicity Loss ($k \geq 2$)

$$\lambda = \inf_{Y \geq X} \frac{\textsc{Rev}(Y)}{\textsc{Rev}(X)} = 0$$

# Non-Monotonicity Loss ($k \geq 2$)

$$\lambda = \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)} = 0$$

For every $\varepsilon > 0$ there are random valuations $X$ and $Y$ in $[0, 1]^k$ such that

# Non-Monotonicity Loss ($k \geq 2$)

$$\lambda = \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)} = 0$$

For every $\varepsilon > 0$ there are random valuations $X$ and $Y$ in $[0,1]^k$ such that

$$Y \geq X$$

$$\text{REV}(Y) < \varepsilon \cdot \text{REV}(X)$$

# Non-Monotonicity Loss ($k \geq 2$)

$$\lambda = \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)} = 0$$

For every $\varepsilon > 0$ there are random valuations $X$ and $Y$ in $[0, 1]^k$ such that

$$Y \geq X$$

$$\text{REV}(Y) < \varepsilon \cdot \text{REV}(X)$$

---

Hart & Nisan (2023)

# Non-Monotonicity Loss: Proof

$$\lambda = \inf_{Y \geq X} \frac{\textsc{Rev}(Y)}{\textsc{Rev}(X)} = 0$$

# Non-Monotonicity Loss: Proof

$$\lambda = \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)} = 0$$

- $Z := \sum_i X_i$

# Non-Monotonicity Loss: Proof

$$\lambda = \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)} = 0$$

- $Z := \sum_i X_i$
- $Y := (Z, ..., Z)$

# Non-Monotonicity Loss: Proof

$$\lambda = \inf_{Y \geq X} \frac{\textsc{Rev}(Y)}{\textsc{Rev}(X)} = 0$$

- $Z := \sum_i X_i$
- $Y := (Z, ..., Z) \geq X$

# Non-Monotonicity Loss: Proof

$$\lambda = \inf_{Y \geq X} \frac{\text{Rev}(Y)}{\text{Rev}(X)} = 0$$

- $Z := \sum_i X_i$
- $Y := (Z, ..., Z) \geq X$
- $\text{Rev}(Y) = \text{Rev}(Z, ..., Z) = k \cdot \text{Rev}(Z)$

# Non-Monotonicity Loss: Proof

$$\lambda = \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)} = 0$$

- $Z := \sum_i X_i$
- $Y := (Z, ..., Z) \geq X$
- $\text{REV}(Y) = \text{REV}(Z, ..., Z) = k \cdot \text{REV}(Z)$
  $= k \cdot \text{REV}(\sum_i X_i) = k \cdot \text{BREV}(X)$

# Non-Monotonicity Loss: Proof

$$\lambda = \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)} = 0$$

- $Z := \sum_i X_i$
- $Y := (Z, ..., Z) \geq X$
- $\text{REV}(Y) = \text{REV}(Z, ..., Z) = k \cdot \text{REV}(Z)$
  $= k \cdot \text{REV}(\sum_i X_i) = k \cdot \text{BREV}(X)$
- Take $X$ s.t. $\text{BREV}(X) < (\varepsilon / k) \cdot \text{REV}(X)$

# Non-Monotonicity Loss: Proof

$$\lambda = \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)} = 0$$

- $Z := \sum_i X_i$
- $Y := (Z, ..., Z) \geq X$
- $\text{REV}(Y) = \text{REV}(Z, ..., Z) = k \cdot \text{REV}(Z)$
  $= k \cdot \text{REV}(\sum_i X_i) = k \cdot \text{BREV}(X)$
- Take $X$ s.t. $\text{BREV}(X) < (\varepsilon / k) \cdot \text{REV}(X)$
  (Hart & Nisan 2013 and Briest & al 2010)

# Non-Monotonicity Loss: Proof

$$\lambda = \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)} = 0$$

- $Z := \sum_i X_i$
- $Y := (Z, ..., Z) \geq X$
- $\text{REV}(Y) = \text{REV}(Z, ..., Z) = k \cdot \text{REV}(Z)$ $= k \cdot \text{REV}(\sum_i X_i) = k \cdot \text{BREV}(X)$
- Take $X$ s.t. $\text{BREV}(X) < (\varepsilon/k) \cdot \text{REV}(X)$ (Hart & Nisan 2013 and Briest & al 2010)

$\Rightarrow \text{REV}(Y) < \varepsilon \cdot \text{REV}(X)$

# Non-Monotonicity Loss

# Non-Monotonicity Loss

$$\lambda := \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)}$$

# Independent Goods

$$\lambda := \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)}$$

**INDEPENDENT** goods:

# Independent Goods

$$\lambda := \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)}$$

**INDEPENDENT** goods:

- $k = 2$ $\qquad \lambda \geq 0.62 \qquad \left(\frac{\sqrt{e}}{\sqrt{e}+1}\right)$

# Independent Goods

$$\lambda := \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)}$$

**INDEPENDENT** goods:

- $k = 2$ $\lambda \geq 0.62$ $\left(\frac{\sqrt{e}}{\sqrt{e}+1}\right)$
- $k = 2$ regular $\lambda \geq 0.73$ $\left(\frac{e}{e+1}\right)$

# Independent Goods

$$\lambda := \inf_{Y \geq X} \frac{\text{REV}(Y)}{\text{REV}(X)}$$

INDEPENDENT goods:

- $k = 2$ $\quad \lambda \geq 0.62 \quad \left(\frac{\sqrt{e}}{\sqrt{e}+1}\right)$
- $k = 2$ regular $\quad \lambda \geq 0.73 \quad \left(\frac{e}{e+1}\right)$
- $k \geq 2$ $\quad \lambda \geq 1/6$

# Independent Goods: Proof

# Independent Goods: Proof

- For every $Y \geq X$:

# Independent Goods: Proof

- For every $Y \geq X$:
  - $\text{REV}(Y) \geq \text{SREV}(Y) \geq \text{SREV}(X)$

# Independent Goods: Proof

- For every $Y \geq X$:
  - $\textsf{Rev}(Y) \geq \textsf{SRev}(Y) \geq \textsf{SRev}(X)$
  - $\textsf{Rev}(Y) \geq \textsf{BRev}(Y) \geq \textsf{BRev}(X)$

# Independent Goods: Proof

- For every $Y \geq X$:
  - $\textsf{REV}(Y) \geq \textsf{SREV}(Y) \geq \textsf{SREV}(X)$
  - $\textsf{REV}(Y) \geq \textsf{BREV}(Y) \geq \textsf{BREV}(X)$

$$\Rightarrow \lambda \geq \frac{\max\{\textsf{SREV}(X), \textsf{BREV}(X)\}}{\textsf{REV}(X)}$$

# Independent Goods: Proof

- For every $Y \geq X$:
  - $\text{REV}(Y) \geq \text{SREV}(Y) \geq \text{SREV}(X)$
  - $\text{REV}(Y) \geq \text{BREV}(Y) \geq \text{BREV}(X)$

$$\Rightarrow \lambda \geq \frac{\max\{\text{SREV}(X), \text{BREV}(X)\}}{\text{REV}(X)}$$

- $k = 2$: Hart & Reny 2016/2019

  $\frac{\text{SREV}(X)}{\text{REV}(X)} \geq 0.62$, $\frac{\text{SREV}(X)}{\text{REV}(X)} \geq 0.73$ (regular)

# Independent Goods: Proof

- For every $Y \geq X$:
  - $\text{REV}(Y) \geq \text{SREV}(Y) \geq \text{SREV}(X)$
  - $\text{REV}(Y) \geq \text{BREV}(Y) \geq \text{BREV}(X)$

$$\Rightarrow \lambda \geq \frac{\max\{\text{SREV}(X), \text{BREV}(X)\}}{\text{REV}(X)}$$

- $k = 2$: Hart & Reny 2016/2019

  $\frac{\text{SREV}(X)}{\text{REV}(X)} \geq 0.62$, $\frac{\text{SREV}(X)}{\text{REV}(X)} \geq 0.73$ (regular)

- $k \geq 2$: Babaioff & al 2014

  $\frac{\max\{\text{SREV}(X),\text{BREV}(X)\}}{\text{REV}(X)} \geq 1/6$

# Non-Monotonicity

# Non-Monotonicity

Define $$\text{gREV}(X) := \inf_{Y \geq X} \text{REV}(Y)$$

# Non-Monotonicity

Define $$\text{gRev}(X) := \inf_{Y \geq X} \text{Rev}(Y)$$

**Theorem.** There exists a mechanism $\mu$ such that

$$R(\mu; Y) \geq \text{gRev}(X)$$

for every $Y \geq X$.

# Non-Monotonicity

Define $$\text{gREV}(X) := \inf_{Y \geq X} \text{REV}(Y)$$

**Theorem.** There exists a mechanism $\mu$ such that

$$R(\mu; Y) \geq \text{gREV}(X)$$

for every $Y \geq X$.

**Proof.** **MINIMAX** theorem (assume: $X$ integrable)

# Guaranteed Revenue

Define $\quad \textsf{gRev}(X) := \inf_{Y \geq X} \textsf{Rev}(Y)$

**gRev = *guaranteed Revenue***

**Theorem.** There exists a mechanism $\mu$ such that

$$R(\mu; Y) \geq \textsf{gRev}(X)$$

for every $Y \geq X$.

**Proof.** **MINIMAX** theorem (assume: $X$ integrable)

# Guaranteed Revenue

$$\textsc{gRev}(X) := \inf_{Y \geq X} \textsc{Rev}(Y)$$

# Guaranteed Revenue for 2 Goods

$$\text{gREV}(X) := \inf_{Y \geq X} \text{REV}(Y)$$

For $k = 2$:

# Guaranteed Revenue for 2 Goods

$$\text{gREV}(X) := \inf_{Y \geq X} \text{REV}(Y)$$

For $k = 2$:

- $\text{gREV}(X) = \text{MONREV}(X) =$ the maximal revenue obtainable from $X$ using ***monotonic*** mechanisms

# Guaranteed Revenue for 2 Goods

$$\textsf{gRev}(X) := \inf_{Y \geq X} \textsf{Rev}(Y)$$

For $k = 2$:

- $\textsf{gRev}(X) = \textsf{MonRev}(X) =$ the maximal revenue obtainable from $X$ using ***monotonic*** mechanisms
- For $X$ with finite support: $\textsf{gRev}(X)$ is computed by Linear Programming, using the ***"conical grid"*** generated by $X$

# Non-Monotonicity: Questions

# Non-Monotonicity: Questions

When the willingness to pay of **BUYER** increases:

# Non-Monotonicity: Questions

When the willingness to pay of **BUYER** increases:

- How large can the loss in revenue be?

# Non-Monotonicity: Answers

When the willingness to pay of **BUYER** increases:

- How large can the loss in revenue be?
  - ***Extremely large !*** (almost all revenue)

# Non-Monotonicity: Answers

When the willingness to pay of **BUYER** increases:

- How large can the loss in revenue be?
  - ***Extremely large !*** (almost all revenue)
- Is there a way for **SELLER** to avoid this loss in revenue?

# Non-Monotonicity: Answers

When the willingness to pay of **BUYER** increases:

- How large can the loss in revenue be?
  - ***Extremely large !*** (almost all revenue)
- Is there a way for **SELLER** to avoid this loss in revenue?
  - ***No way !*** (?)

# Selling Goods

# Selling Goods

***ex ante***:

# Selling Goods

***ex ante***:

- **SELLER** chooses mechanism $\mu = (q, s)$

# Selling Goods

***ex ante***:

- **SELLER** chooses mechanism $\mu = (q, s)$

***ex post*** ($X$ realized):

# Selling Goods

***ex ante***:

- **SELLER** chooses mechanism $\mu = (q, s)$

***ex post*** ($X$ realized):

- **BUYER** payoff $= b(X) = q(X) \cdot X - s(X)$

# Selling Goods

***ex ante***:

- **SELLER** chooses mechanism $\mu = (q, s)$

***ex post*** ($X$ realized):

- **BUYER** payoff $= b(X) = q(X) \cdot X - s(X)$
- **SELLER** payoff $= \mathbf{E}[s(Y)]$

# Selling Goods

***ex ante***:

$\Rrightarrow$ **BUYER** announces $Y \geq X$
and ***commits*** to act according to $Y$

- **SELLER** chooses mechanism $\mu = (q, s)$

***ex post*** ($X$ realized):

- **BUYER** payoff $= b(X) = q(X) \cdot X - s(X)$
- **SELLER** payoff $= \mathbf{E}[s(Y)]$

# Committing to Overbid

***ex ante***:

$\Rrightarrow$ **BUYER** announces $Y \geq X$
and ***commits*** to act according to $Y$

- **SELLER** chooses mechanism $\mu = (q, s)$

***ex post*** ($X$ realized):

- **BUYER** payoff $= b(X) = q(X) \cdot X - s(X)$
- **SELLER** payoff $= \mathbf{E}[s(Y)]$

# Committing to Overbid

***ex ante***:

$\Rrightarrow$ **BUYER** announces $Y \geq X$
and ***commits*** to act according to $Y$

- **SELLER** chooses mechanism $\tilde{\mu} = (\tilde{q}, \tilde{s})$

***ex post*** ($X$ realized):

- **BUYER** payoff $= b(X) = q(X) \cdot X - s(X)$
- **SELLER** payoff $= \mathbf{E}[s(Y)]$

# Committing to Overbid

***ex ante***:

$\Rrightarrow$ **BUYER** announces $Y \geq X$
and ***commits*** to act according to $Y$

- **SELLER** chooses mechanism $\tilde{\mu} = (\tilde{q}, \tilde{s})$

***ex post*** ($X$ realized):

- **BUYER** payoff $= \tilde{b}(Y|X) = \tilde{q}(Y) \cdot X - \tilde{s}(Y)$
- **SELLER** payoff $= \mathbf{E}[s(Y)]$

# Committing to Overbid

***ex ante***:

$\Rrightarrow$ **BUYER** announces $Y \geq X$
and ***commits*** to act according to $Y$

- **SELLER** chooses mechanism $\tilde{\mu} = (\tilde{q}, \tilde{s})$

***ex post*** ($X$ realized):

- **BUYER** payoff $= \tilde{b}(Y|X) = \tilde{q}(Y) \cdot X - \tilde{s}(Y)$
- **SELLER** payoff $= \mathbf{E}[\tilde{s}(Y)]$

# Committing to Overbid

***ex ante***:

$\Rightarrow$ **BUYER** announces $Y \geq X$
and ***commits*** to act according to $Y$

- **SELLER** chooses mechanism $\tilde{\mu} = (\tilde{q}, \tilde{s})$

***ex post*** ($X$ realized):

- **BUYER** payoff $= \tilde{b}(Y|X) = \tilde{q}(Y) \cdot X - \tilde{s}(Y)$
- **SELLER** payoff $= \mathrm{E}[\tilde{s}(Y)]$

**PROFITABLE** to **BUYER** :

- $\tilde{b}(Y|X) \geq b(X)$ with some strict inequalities

# Committing to Overbid: Example

# Committing to Overbid: Example

| Prob |
|---|
| $\mathbf{0.3}$ |
| $\mathbf{0.3}$ |
| $\mathbf{0.4}$ |

# Committing to Overbid: Example

| Prob | $X$ |
| --- | --- |
| $\mathbf{0.3}$ | $\mathbf{(10, 0)}$ |
| $\mathbf{0.3}$ | $\mathbf{(10, 20)}$ |
| $\mathbf{0.4}$ | $\mathbf{(20, 30)}$ |

# Committing to Overbid: Example

| Prob | $X$ | $b(X)$ |
|---|---|---|
| $0.3$ | $(10, 0)$ | $0$ |
| $0.3$ | $(10, 20)$ | $0$ |
| $0.4$ | $(20, 30)$ | $10$ |

$\mu$ = unique optimal mechanism for $X$:

MENU $= \{0,\ x_1 - 10,\ x_2 - 20,\ x_1 + x_2 - 40\}$

REV$(X) = 25$

# Committing to Overbid: Example

| Prob | $X$ | $b(X)$ |
|---|---|---|
| $0.3$ | $(10, 0)$ | $0$ |
| $0.3$ | $(10, 20)$ | $0$ |
| $0.4$ | $(20, 30)$ | $10$ |

# Committing to Overbid: Example

| Prob | $X$ | $b(X)$ | $Y$ |
|---|---|---|---|
| $0.3$ | $(10, 0)$ | $0$ | $(10, 0)$ |
| $0.3$ | $(10, 20)$ | $0$ | $(12, 20)$ |
| $0.4$ | $(20, 30)$ | $10$ | $(20, 30)$ |

# Committing to Overbid: Example

| Prob | $X$ | $b(X)$ | $Y$ | $\tilde{b}(Y\|X)$ |
|---|---|---|---|---|
| $0.3$ | $(10, 0)$ | $0$ | $(10, 0)$ | $0$ |
| $0.3$ | $(10, 20)$ | $0$ | $(12, 20)$ | $0$ |
| $0.4$ | $(20, 30)$ | $10$ | $(20, 30)$ | $20$ |

$\tilde{\mu} =$ unique optimal mechanism for $Y$:

MENU$^{\sim} = \{0,\ x_1 - 10,\ x_1 + x_2 - 30\}$

REV$(Y) = 24$

# Committing to Overbid: Example

| Prob | $X$ | $b(X)$ | $Y$ | $\tilde{b}(Y\|X)$ |
|---|---|---|---|---|
| 0.3 | $(10, 0)$ | 0 | $(10, 0)$ | 0 |
| 0.3 | $(10, 20)$ | 0 | $(12, 20)$ | 0 |
| 0.4 | $(20, 30)$ | 10 | $(20, 30)$ | 20 |

# Committing to Overbid: Example

| Prob | $X$ | $b(X)$ | $Y$ | $\tilde{b}(Y\|X)$ |
|---|---|---|---|---|
| $0.3$ | $(10, 0)$ | $0$ | $(10, 0)$ | $0$ |
| $0.3$ | $(10, 20)$ | $0$ | $(12, 20)$ | $0$ |
| $0.4$ | $(20, 30)$ | $10$ | $(20, 30)$ | $20$ |

**COMMITMENT TO $Y$ IS PROFITABLE FOR $X$:**
$\tilde{b}(Y|X)$ **DOMINATES** $b(X)$

# Committing to Overbid: Example

# Committing to Overbid: Example

**BUYER** ***commits to higher valuations***

# Committing to Overbid: Example

**BUYER** ***commits to higher valuations***

- **BUYER** : no types lose and some types gain (according to the original valuations)

# Committing to Overbid: Example

**BUYER** ***commits to higher valuations***

- **BUYER** : no types lose and some types gain (according to the original valuations)
- **SELLER** : loses (revenue non-monotonicity)

# Committing to Overbid: Example

**BUYER** ***commits to higher valuations***

- **BUYER** : no types lose and some types gain (according to the original valuations)
- **SELLER** : loses (revenue non-monotonicity)
- **MARKET** : is more efficient (allocations increase)

# Committing to Overbid: Example

**BUYER *commits to higher valuations***

- **BUYER** : no types lose and some types gain (according to the original valuations)
- **SELLER** : loses (revenue non-monotonicity)
- **MARKET** : is more efficient (allocations increase)

---

Gonczarowski & Hart 2024

# Non-Monotonicity: Answers

# Non-Monotonicity: Answers

When the willingness to pay of **BUYER** increases:

# Non-Monotonicity: Answers

When the willingness to pay of **BUYER** increases:

- How large can the loss in revenue be?

# Non-Monotonicity: Answers

When the willingness to pay of **BUYER** increases:

- How large can the loss in revenue be?
  - ***Extremely large !*** (up to entire revenue)

# Non-Monotonicity: Answers

When the willingness to pay of **BUYER** increases:

- How large can the loss in revenue be?
  - ***Extremely large !*** (up to entire revenue)

- Is there a way for **SELLER** to avoid this loss in revenue?

# Non-Monotonicity: Answers

When the willingness to pay of **BUYER** increases:

- How large can the loss in revenue be?
  - ***Extremely large !*** (up to entire revenue)
- Is there a way for **SELLER** to avoid this loss in revenue?
  - ***No way !***

# Two(!) Good To Be ... Monotonic

# Two(!) Good To Be ... Monotonic

# Non-Monotonicity

# Non-Monotonicity

Working with Phil is
as exciting and **NON-MONOTONIC**
as it can get !

# Non-Monotonicity

Working with Phil is
as exciting and **NON-MONOTONIC**
as it can get !

***Thank you, Phil***