ON TENSOR PRODUCTS OF SEMISTABLE LATTICES

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Abstract. We give an elementary proof of low rank cases of the conjecture that the tensor product of two semistable Euclidean lattices is again semistable.

The degree of a lattice L in Euclidean space $\mathbb{R}^n$ is defined by the expression

$$\deg(L) = -\log vol(\mathbb{R}^n/L), \tag{0.1}$$

where *vol* stands for volume. Its rank is $rk(L) = n$, and the *slope* $\mu(L)$ is the ratio

$$\mu(L) = \frac{\deg(L)}{rk(L)}. \tag{0.2}$$

The lattice L is called *semistable* if for any (lower rank) sublattice S, $\mu(S) \leq \mu(L)$. This notion is invariant under scaling, so if we normalize L to be unimodular ($\mu(L) = 0$), it means that the volume of a fundamental parallelopiped of any sublattice S is at least 1. Thus the shortest nonzero vector in L has norm at least 1, every two independent vectors span a parallelogram of area at least 1, and so on.

The following problem seems to be open. The first author learned about it more than ten years ago from J.-B. Bost.

Question. Is $L \otimes M$ semistable whenever L and M are?

The notion of semistability and the question can be easily generalized to "metrized vector bundles over $Spec(\mathcal{O}_K)$" for any number field K. They come up naturally in Arakelov geometry. However, it is already interesting, and probably not less difficult, over $Spec(\mathbb{Z})$, where it amounts to the statement made above. It is modeled on similar results for vector bundles over curves, and for filtered vector spaces (see [Fa],[To]). The corresponding result for filtered vector spaces is surprisingly difficult, although it is only a statement in linear algebra, and this leads one to suspect that the question for lattices is not easy either.

The purpose of this note is to give an affirmative answer in some special cases.

Theorem 0.1. *Let L and M be two semistable lattices. If S is a sublattice of $L \otimes M$ and $rk(S) \leq 3$ then*

$$\mu(S) \leq \mu(L \otimes M). \tag{0.3}$$

Bost had obtained other special cases using Geometric Invariant Theory (unpublished). We should also stress that our elementary methods do not generalize to "metrized vector bundles over $Spec(\mathcal{O}_K)$" when K is not $\mathbb{Q}$.

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1. The proof

1.1. **Preliminaries on Euclidean spaces.** If V and W are Euclidean spaces (finite dimensional inner product spaces over $\mathbb{R}$) then so is $V\otimes W$ with the inner product

$$(v\otimes w, v'\otimes w')_{V\otimes W}=(v,v')_V(w,w')_W. \tag{1.1}$$

If $\{e_i\}$ is an orthonormal basis of V and $\left\{e'_j\right\}$ an orthonormal basis of W then $\left\{e_i\otimes e'_j\right\}$ is an orthonormal basis of $V\otimes W$. In this way $V^{\otimes k}$ is endowed with a Euclidean structure.

The k-th exterior power $\bigwedge^k V$ is the quotient of $V^{\otimes k}$ by the subspace $\mathcal{N}$ spanned by tensors $v_1\otimes\cdots\otimes v_k$ in which, for some $i\neq j$, $v_i=v_j$. The orthogonal complement $\mathcal{N}^\perp$ is the space of *alternating k-tensors*, spanned by

$$\frac{1}{k!}\sum_{\sigma\in S_k} sgn(\sigma)v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k)}, \tag{1.2}$$

and this tensor projects modulo $\mathcal{N}$ to $v_1\wedge\cdots\wedge v_k$, the image modulo $\mathcal{N}$ of $v_1\otimes\cdots\otimes v_k$. We may therefore identify $\bigwedge^k V$ with $\mathcal{N}^\perp$. As such it inherits from $V^{\otimes k}$ a Euclidean structure. If $\{e_i\}$ is an orthonormal basis of V then $e_I=e_{i_1}\wedge\cdots\wedge e_{i_k}$, for $I=(i_1<\cdots<i_k)$, is a basis of $\bigwedge^k V$, which is orthogonal but not normalized: the norm of e_I is $1/\sqrt{k!}$. To correct it, we modify the inner product that $\bigwedge^k V$ inherits from $V^{\otimes k}$ by a factor of $k!$ and set

$$(v_1\wedge\cdots\wedge v_k, v'_1\wedge\cdots\wedge v'_k)=\sum_{\sigma\in S_k} sgn(\sigma)\prod_{i=1}^k (v_i, v'_{\sigma(i)}). \tag{1.3}$$

In this inner product the e_I form an orthonormal basis. Moreover, the *volume* of the k-dimensional parallelopiped spanned by $v_1,\dots,v_k$ in V is nothing but the *norm* $|v_1\wedge\cdots\wedge v_k|$. To see it, choose an orthonormal basis e_i of V the first k vectors of which span $Span\,\{v_1,\dots,v_k\}$ (assuming the v_i are linearly independent) and observe that

$$v_1\wedge\cdots\wedge v_k=\det(a_{ij})e_1\wedge\cdots\wedge e_k \tag{1.4}$$

where $v_i=\sum a_{ij}e_j$.

1.2. **Length and norm in $L\otimes M$, and the rank 1 case of the theorem.** To begin the proof, note that

$$\mu(L\otimes M)=\mu(L)+\mu(M). \tag{1.5}$$

If α is a vector in $L\otimes M$ we denote by $l(\alpha)$ the *length* of α, which is the minimal number l such that

$$\alpha=\sum_{k=1}^l u_k\otimes u'_k. \tag{1.6}$$

(To avoid abuse of language, $|\alpha|$ will always be called the norm of α, and not its length.) In such a case, the u_k and the u'_k are linearly independent vectors in L and M respectively. Thus the length of a vector is at most $\min(rk(L), rk(M))$.

Let us write, for any lattice M

$$\mu_{\max}(M)=\max\{\mu(L);\ L\subseteq M\}. \tag{1.7}$$

The maximum is over all sublattices of all ranks (it is easily seen that the maximum is attained). A unimodular lattice M is semistable if and only if $\mu_{\max}(M) = 0$.

Proposition 1.1. *Let L and M be any two lattices, normalized so that $\mu_{\max}(L) = \mu_{\max}(M) = 0$ and $\alpha \in L \otimes M$. Then*

$$|\alpha| \geq \sqrt{l(\alpha)}. \tag{1.8}$$

Proof. Writing $\alpha = \sum_{k=1}^{l} u_k \otimes u'_k$, a direct computation gives

$$|\alpha|^2 = \sum_{k,m} (u_k, u_m)(u'_k, u'_m). \tag{1.9}$$

Consider the *Gramians* $A = ((u_k, u_m))$ and $A' = ((u'_k, u'_m))$. These are symmetric positive definite l by l matrices with $\det(A) \geq 1$ and $\det(A') \geq 1$ because we normalized L and M so that the volume of the parallelopiped formed by the u_k or the u'_k is at least 1.

We are indebted to Assaf Goldberger for pointing out the following fact.

Lemma 1.2. *Let A and A' be two symmetric positive definite l by l matrices. Then the spectrum of AA' is real and positive.*

The matrix AA', although not necessarily symmetric, is conjugate to

$$A^{1/2} A' A^{1/2} = (A^{1/2})^t A' A^{1/2}, \tag{1.10}$$

which is symmetric positive definite since A' is.

We now continue the proof of the proposition. By our assumption, the product of the eigenvalues of AA' is at least 1, and the lemma implies that they are real and positive. By the inequality between the geometric and arithmetic means, $Tr(AA') \geq l$. However,

$$Tr(AA') = |\alpha|^2, \tag{1.11}$$

so $|\alpha| \geq \sqrt{l}$ as needed. □

The proposition proves the case $\mathbf{rk(S) = 1}$ of the theorem, but to prove the cases $rk(S) = 2$ or 3 we shall need its full strength.

1.3. **The rank 2 case of the theorem.** Let now S be a sublattice of **rank** 2 in $L \otimes M$. Let α be a nonzero vector of shortest norm in S, and β another vector in S such that $S = \mathbb{Z}\alpha + \mathbb{Z}\beta$. Subtracting a multiple of α from β we may assume that the angle between α and β is between 60° and 120°. It follows that

$$|\alpha \wedge \beta| \geq \frac{\sqrt{3}}{2} |\alpha||\beta|. \tag{1.12}$$

If either of α or β is indecomposable (of length ≥ 2) we are done, since its norm is then at least $\sqrt{2}$ and the other's norm is at least 1, but $\sqrt{2}\sqrt{3} > 2$. We may therefore assume that

$$\alpha = v_1 \otimes w_1, \ \beta = v_2 \otimes w_2 \tag{1.13}$$

are both decomposable tensors. We now use the identity

$$\begin{aligned} |(v_1 \otimes w_1) \wedge (v_2 \otimes w_2)|^2 &= |v_1|^2 |v_2|^2 |w_1 \wedge w_2|^2 + |v_1 \wedge v_2|^2 |w_1|^2 |w_2|^2 \\ &\quad - |v_1 \wedge v_2|^2 |w_1 \wedge w_2|^2, \end{aligned} \tag{1.14}$$

which follows at once from the fact that

$$|\alpha\wedge\beta|^2=|\alpha|^2|\beta|^2-(\alpha,\beta)^2. \tag{1.15}$$

If v_1 and v_2 are proportional, then w_1 and w_2 are independent, and $|v_1|^2|v_2|^2|w_1\wedge w_2|^2\geq 1$, while the second and the third terms vanish. Otherwise, the first term is larger than the third, and the second is ≥ 1 by our assumption. This concludes the rank 2 case.

1.4. **The rank 3 case of the theorem.** The **rank 3** case is handled similarly, but the details are more complicated. Let $rk(S)=3$. Let α be a shortest nonzero vector in S and β a second shortest vector independent from α, and assume as before that the angle between α and β is between $60°$ and $120°$. Complete to a basis of S by a vector γ, $|\gamma|\geq|\beta|\geq|\alpha|\geq 1$. Using orthonormal coordinates x,y,z in the real subspace spanned by S, such that the x-axis is in the α-direction and the (x,y)-plane is the plane spanned by α and β, we may assume, subtracting from γ a suitable integral linear combination of α and β that

$$\gamma=(x,y,z) \tag{1.16}$$

with $|x|\leq|\alpha|/2$ and $|y|\leq|\beta|/2$. This is because the rectangle

$$\{|x|\leq|\alpha|/2,\ |y|\leq|\beta|\sin\theta/2\}, \tag{1.17}$$

where θ is the angle between α and β, is a fundamental domain for $\mathbb{Z}\alpha+\mathbb{Z}\beta$. We now have

$$z^2\geq|\gamma|^2-(|\alpha|^2+|\beta|^2)/4\geq|\gamma|^2/2 \tag{1.18}$$

and

$$|\alpha\wedge\beta|\geq 1 \tag{1.19}$$

(by the rank 2 case), so

$$|\alpha\wedge\beta\wedge\gamma|^2=|\alpha\wedge\beta|^2z^2\geq|\gamma|^2/2. \tag{1.20}$$

If $l(\alpha)$, $l(\beta)$ or $l(\gamma)\geq 2$, then by the proposition and the fact that $|\gamma|\geq|\beta|\geq|\alpha|$, we must have $|\gamma|^2\geq 2$, and we are done.

There remains the case $\alpha=v_1\otimes w_1$, $\beta=v_2\otimes w_2$ and $\gamma=v_3\otimes w_3$, in which the proposition does not help us. The key in this case is the following identity.

Lemma 1.3. *We have an 18-term relation*

$$\begin{aligned}
|\alpha\wedge\beta\wedge\gamma|^2 \quad = \quad & |v_1|^2|v_2|^2|v_3|^2|w_1\wedge w_2\wedge w_3|^2+|v_1\wedge v_2\wedge v_3|^2|w_1|^2|w_2|^2|w_3|^2\\
&-\frac{1}{2}\sum_{i=1}^{3}|v_i|^2|v_{i'}\wedge v_{i''}|^2|w_i|^2|w_{i'}\wedge w_{i''}|^2\\
&+\frac{1}{2}\sum_{i\neq j}|v_i|^2|v_{i'}\wedge v_{i''}|^2|w_j|^2|w_{j'}\wedge w_{j''}|^2\\
&-\frac{1}{2}\sum_{i=1}^{3}|v_i|^2|v_{i'}\wedge v_{i''}|^2|w_1\wedge w_2\wedge w_3|^2\\
&-\frac{1}{2}\sum_{i=1}^{3}|v_1\wedge v_2\wedge v_3|^2|w_i|^2|w_{i'}\wedge w_{i''}|^2\\
&+\frac{1}{2}|v_1\wedge v_2\wedge v_3|^2|w_1\wedge w_2\wedge w_3|^2,
\end{aligned} \tag{1.21}$$

where we have used the convention that $\{i', i''\}$ are the two indices complementary to i.

Proof. Expanding the grammian of three vectors in Euclidean space we get

$$
\begin{aligned}
|\alpha_1 \wedge \alpha_2 \wedge \alpha_3|^2 &= \prod_{i=1}^{3} |\alpha_i|^2 - \sum_{i=1}^{3} |\alpha_i|^2 (\alpha_{i'}, \alpha_{i''})^2 \\
&\quad +2(\alpha_1, \alpha_2)(\alpha_2, \alpha_3)(\alpha_3, \alpha_1).
\end{aligned} \tag{1.22}
$$

For our decomposable tensors we get the expression

$$
\begin{aligned}
|\alpha \wedge \beta \wedge \gamma|^2 &= \prod_{i=1}^{3} |v_i|^2 |w_i|^2 - \sum_{i=1}^{3} |v_i|^2 (v_{i'}, v_{i''})^2 |w_i|^2 (w_{i'}, w_{i''})^2 \\
&\quad +2(v_1, v_2)(v_2, v_3)(v_3, v_1)(w_1, w_2)(w_2, w_3)(w_3, w_1).
\end{aligned} \tag{1.23}
$$

We proceed to replace all the inner products by expressions involving only wedge-products (sines instead of cosines). The elements of the form $(x, y)^2$ can be replaced by $|x|^2|y|^2 - |x \wedge y|^2$ as in the case of rank 2. For $(x, y)(y, z)(z, x)$ use

$$
\begin{aligned}
2(x, y)(y, z)(z, x) &= |x \wedge y \wedge z|^2 - |x|^2|y|^2|z|^2 + |x|^2 (y, z)^2 + \cdots \\
&= |x \wedge y \wedge z|^2 - |x|^2|y|^2|z|^2 + |x|^2 (|y|^2|z|^2 - |y \wedge z|^2) + \cdots
\end{aligned} \tag{1.24}
$$

This is an 5-term expression and when we multiply the one for the $v_i's$ with the one for the $w_i's$ we get 25 terms, and a total of 38 terms for $|\alpha \wedge \beta \wedge \gamma|^2$. A direct computation (carried out by a computer) reveals that there are many cancellations, and the 18-term relation falls out. □

We continue the proof of the theorem, when α, β and γ are decomposable as above. Choose an orthonormal basis $e_1, e_2, e_3, ...$ of $L_{\mathbb{R}}$ so that $v_1 \in \langle e_1 \rangle$, $v_2 \in \langle e_1, e_2 \rangle$, $v_3 \in \langle e_1, e_2, e_3 \rangle$ and similarly an orthonormal basis $e'_1, e'_2, e'_3, ...$ of $M_{\mathbb{R}}$ putting the w_i in a triangular form. Assume first that the v_i are linearly independent. Expanding in the orthonormal basis $e_i \otimes e'_j$ of $L_{\mathbb{R}} \otimes M_{\mathbb{R}}$ and in the corresponding orthonormal basis of of $\bigwedge^3 (L_{\mathbb{R}} \otimes M_{\mathbb{R}})$ (recall the normalization from 1.1)

$$
\begin{aligned}
|\alpha \wedge \beta \wedge \gamma|^2 &\geq \sum_{i=1}^{2} \sum_{j=1}^{3} coef^2((e_1 \otimes e'_1) \wedge (e_2 \otimes e'_i) \wedge (e_3 \otimes e'_j)) \\
&= |v_1 \wedge v_2 \wedge v_3|^2 |w_1|^2 |w_2|^2 |w_3|^2 \geq 1.
\end{aligned} \tag{1.25}
$$

Assume next that both the v_i and the w_i are linearly dependent. If v_1 and v_2 are proportional then w_1 and w_2 can not be proportional, otherwise $rk(S) \leq 2$. In this case we may assume that $v_1, v_2 \in \langle e_1 \rangle$ and $v_3 \in \langle e_1, e_2 \rangle$, while $w_1 \in \langle e'_1 \rangle$ and $w_2, w_3 \in \langle e'_1, e'_2 \rangle$. We find

$$
\begin{aligned}
|\alpha \wedge \beta \wedge \gamma|^2 &= \sum_{i=1}^{2} coef^2((e_1 \otimes e'_1) \wedge (e_1 \otimes e'_2) \wedge (e_2 \otimes e'_i)) \\
&= |v_1||v_2||v_1 \wedge v_3||v_2 \wedge v_3||w_1 \wedge w_2|^2 |w_3|^2 \geq 1.
\end{aligned} \tag{1.26}
$$

There remains the case where the v_i and the w_i are linearly dependent, but no two vectors in each triplet are proportional.

Introduce the notation

$$
\lambda_i = |v_i||v_{i'} \wedge v_{i''}| \tag{1.27}
$$

and similarly $\mu_i = |w_i||w_{i'} \wedge w_{i''}|$. Note that

$$\lambda_i = |v_1||v_2||v_3||\sin\theta_i|$$

where θ_i is the angle between $v_{i'}$ and $v_{i''}$ in the plane spanned by the $v_i's$. We may assume that $\theta_i = \theta_{i'} + \theta_{i''}$. A direct computation shows then that

$$\lambda_i^2 - \lambda_{i'}^2 - \lambda_{i''}^2 = \pm 2\lambda_{i'}\lambda_{i''}\cos\theta_i. \tag{1.28}$$

Similarly denote by ω_i the angle between $w_{i'}$ and $w_{i''}$. We now observe that in the 18-term relation, nine terms drop out and the remaining nine give

$$\begin{aligned}
|\alpha\wedge\beta\wedge\gamma|^2 &= \frac{1}{2}\left\{\begin{array}{ccc} -\lambda_1^2\mu_1^2 & +\lambda_1^2\mu_2^2 & +\lambda_1^2\mu_3^2 \\ +\lambda_2^2\mu_1^2 & -\lambda_2^2\mu_2^2 & +\lambda_2^2\mu_3^2 \\ +\lambda_3^2\mu_1^2 & +\lambda_3^2\mu_2^2 & -\lambda_3^2\mu_3^2 \end{array}\right\} \\
&= \frac{1}{2}\left\{\begin{array}{c} \lambda_1^2(-\mu_1^2+\mu_2^2+\mu_3^2) + \lambda_2^2(\mu_1^2-\mu_2^2+\mu_3^2) \\ +(\lambda_1^2+\lambda_2^2 \pm 2\lambda_1\lambda_2\cos\theta_3)(\mu_1^2+\mu_2^2-\mu_3^2) \end{array}\right\} \\
&= \lambda_1^2\mu_2^2 + \lambda_2^2\mu_1^2 \pm \lambda_1\lambda_2\cos\theta_3(\mu_1^2+\mu_2^2-\mu_3^2) \\
&= \lambda_1^2\mu_2^2 + \lambda_2^2\mu_1^2 \pm 2\lambda_1\lambda_2\mu_1\mu_2\cos\theta_3\cos\omega_3 \\
&\geq \lambda_1^2\mu_2^2 + \lambda_2^2\mu_1^2 - \lambda_1^2\mu_2^2\cos^2\theta_3 - \lambda_2^2\mu_1^2\cos^2\omega_3 \\
&= \lambda_1^2\mu_2^2\sin^2\theta_3 + \lambda_2^2\mu_1^2\sin^2\omega_3 \\
&\geq \frac{\lambda_1^2\mu_2^2}{|v_1|^2|v_2|^2} + \frac{\lambda_2^2\mu_1^2}{|w_1|^2|w_2|^2} \\
&= \frac{|v_2\wedge v_3|^2|w_2|^2|w_1\wedge w_3|^2}{|v_2|^2} + \frac{|w_2\wedge w_3|^2|v_2|^2|v_1\wedge v_3|^2}{|w_2|^2} \\
&\geq \frac{|w_2|^2}{|v_2|^2} + \frac{|v_2|^2}{|w_2|^2} \geq 2. \qquad (1.29)
\end{aligned}$$

This concludes the proof of the theorem.

1.5. **An argument from duality.** Duality allows us to conclude more. Quite generally, if

$$0 \to P \to Q \to R \to 0 \tag{1.30}$$

is an exact sequence of Euclidean lattices (this means that Q/P is torsion-free and that the metrics on $P_{\mathbb{R}}$ and $R_{\mathbb{R}}$ are the ones induced from the metric on $Q_{\mathbb{R}}$) then

$$\mu(Q) = \frac{rkP}{rkQ}\mu(P) + \frac{rkR}{rkQ}\mu(R). \tag{1.31}$$

If $R^\vee$ is the dual lattice, $\mu(R^\vee) = -\mu(R)$. From these two observations it follows at once that if L is semistable, so is $L^\vee$.

Let L and M be two unimodular semistable lattices of ranks n and m. Then $L^\vee$ and $M^\vee$ are unimodular and semistable too. Suppose we know the desired results for all S of some rank s. Let P be a rank $nm-s$ (we shall say it has *corank s*) sublattice of $L\otimes M$ for which we want to prove that $\mu(P) \leq 0$. We may assume that $(L\otimes M)/P$ is torsion free. We equip it with the quotient metric and call it R. Then R is a rank s lattice and $R^\vee = P^\perp$ is a rank s sublattice of $L^\vee\otimes M^\vee$. By what we have seen already, $\mu(R^\vee) \leq 0$, but

$$\mu(P) = -\frac{s}{nm-s}\mu(R) \tag{1.32}$$

since $\mu(L \otimes M) = 0$, and $\mu(R) = -\mu(R^\vee)$. This shows that if we know the desired result for all rank s lattices, we also know it for all corank s lattices.

Corollary 1.4. *If L and M are semistable and they are both of rank 2, or one of them is of rank 2 and the other is of rank 3, then $L \otimes M$ is semistable.*

References

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