Statistical Mechanics of Large Dense Random Graphs

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Joint work with Charles Radin

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- A Large Random Graph means taking the $n \to \infty$ limit.
- A Large Dense Random Graph means non-trivial edge density.

$$
\begin{aligned}
e &:= \frac{\#\text{ of edges}}{n^2/2} \not\to 0. \\
t &:= \frac{\#\text{ of triangles}}{n^3/6}.
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  $\left| \frac{\#\text{ edges}}{n^2/2} - e \right| < \delta$ and $\left| \frac{\#\text{ triangles}}{n^3/6} - t \right| < \delta$.

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$$s(e, t) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\ln Z_{e,t}^{\delta,n}}{n^2}.$$

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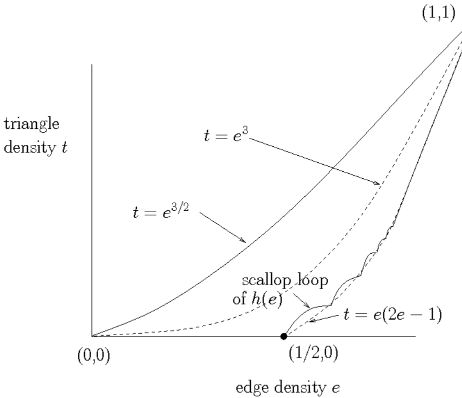
Like microcanonical ensemble with

edges	$\leftrightarrow$	particles
triangles	$\leftrightarrow$	interaction energy

The shape of (e, t) space



Instead of using the microcanonical ensemble, most work has picked parameters β_1, β_2 and set

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Exponential Random Graphs

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Bad idea! (We'll see why soon)

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5. Analytic evidence that this last formula works for all $e \leq 1/2$ below the ER curve.
6. Numerical evidence of other phase transitions, some 1st order and some 2nd.

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- A graph G can be viewed as a checkerboard graphon. Break $[0,1]$ into n equal intervals, each corresponding to a vertex. This breaks $[0,1]^2$ into n^2 squares, one for each possible edge. Let $g(x,y) = 1$ if the edge exists and 0 if it doesn't.

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- Topology of space of graphons is technical. Buzzword is "Cut metric". Also need to mod out by measure-preserving transformations of $[0,1]$. (Unnecessary for today's talk.)

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The expected edge and triangle densities are:

$$e(g) = \iint g(x, y) dx\, dy, \quad t(g) = \iiint g(x, y) g(y, z) g(z, x) dxdydz.$$

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- Entropy of random process described by graphon is $-n^2 I(g)$
- Entropy density is $-I(g)$.

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The entropy density $s(e,t)$ is well-defined, and equals $-\inf I(g)$, where $\inf$ is over graphons g with $\iint g(x,y)dx\,dy = e$ and $\iiint g(x,y)g(y,z)g(x,z)dx\,dy\,dz = t$.

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This reduces all questions to minimizing $I(g)$ for fixed e and t.
Goodbye statistical mechanics. Hello functional analysis.

Two examples

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This is the maximum entropy for fixed e and variable t.
How smooth is the maximum? (Answer: not very)

2) On bottom boundary, minimizing graphon is

$$g(x,y) = \begin{cases} 2e & x < \frac{1}{2} < y \text{ or } y < \frac{1}{2} < x \\ 0 & \text{otherwise.} \end{cases}$$

Entropy is $s(e, 0) = -\frac{1}{2} I_0(2e)$.

The symmetric bipartite phase

Consider the graphon

$$g(x,y) = \begin{cases} e+b & x < \frac{1}{2} < y \text{ or } y < \frac{1}{2} < x \\ e-b & \text{otherwise.} \end{cases}$$

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Appears to minimize $I(g)$. When $b \approx e$, 2nd variation w.r.t. L^∞-small changes is positive operator. But L^1-small changes needn't be L^∞-small.

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Rigorous proof that this minimizes $I(g)$ for $e = 1/2$, $t \leq e^3$. (Later slide)

Phase transition on ER curve.

Theorem (Radin-S)

There exist positive constants c_1, c_2 *such that:*
If $t < e^3$, $s(e,t) \leq s(e,e^3) - c_1(e^3 - t)^{2/3}$,
If $t > e^3$, $s(e,t) \leq s(e,e^3) - c_2(t - e^3)$.

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$s(e,t)$ has at most a 1-sided derivative w.r.t. t at $t = e^3$.
$s(e,t)$ is concave up for t slightly less than e^3. Invisible to Legendre transform.

A proof (finally)

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Last integral is $O(\delta g^3)$, so if $t < e^3$, $e^3 - t < c\|\delta g\|_{L^2}^{3/2}$.

Proof, continued

$$s(e, e^3) - s(e, t) = \iint (I_0(e) - I_0(g(x, y))dx\, dy = O(\delta g^2).$$

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When $e = 1/2$ and $\delta t < 0$, all estimates are saturated by symmetric bipartite graphon, so we know it's a minimizer.

Second phase

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Phase II: Asymmetric bipartite. Like symmetric bipartite, but two subintervals have unequal size and unequal probabilities of internal edges. There is a constant $c \neq 1/2$ such that

$$g(x,y) = \begin{cases} a & x,y < c \\ b & x,y > c \\ d & x < c < y \text{ or } y < c < x \end{cases}$$

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Transition from Phase I to Phase II is 2nd-order.

Third phase

Phase III: Tripartite. Like bipartite, but interval $[c, 1]$ divided into equal sub-intervals. When $x, y > c$, $g(x, y)$ depends on whether x and y are in same or different sub-intervals.

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Applies near first "scallop".
Transition from Phase II to Phase III is first-order. (Partial derivatives of s diverge).

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Bottom line: Lots of phase transitions, and lots of interesting behavior, but only in the microcanonical ensemble.

Canonical ensemble (e fixed, t variable) misses most of the fun.

Grand canonical ensemble misses almost all of the fun.

Happy Birthday

Happy Birthday, Yosi!

Cake

Ein Hagigah Bli Uga