Exponential Decay of Eigenfunctions of Partial Differential Operators

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Introduction

In this talk I will consider the decay as $|x| \to \infty$ of solutions ψ to the eigenvalue problem

$$(H - \lambda)\psi = 0.$$

Here H is in a class of self-adjoint partial differential operators of the form

$$H = Q(-i\nabla) + V(x)$$

where Q is a real elliptic polynomial and V is bounded.
But first a review of some results for the Laplacian.

Laplacian - the one body problem

Consider solutions ψ to

$$(-\Delta + V(x) - \lambda)\psi = 0, \psi \in L^2(\mathbb{R}^d),$$

$V(x) \to 0$ as $|x| \to \infty, \lambda$ and V real. Well known results: define σ_c

$$\sigma_c = \sup\{\sigma : e^{\sigma r}\psi \in L^2(\mathbb{R}^d)\}$$

Theorem (Combes-Thomas, Bardos-Merigot, FH2HO)

i) *If $\lambda < 0$ and $V(x) = o(1)$ or $\lambda > 0$ and $V(x) = o(|x|^{-1})$, then $\sigma_c > 0$.*

ii) *If $V(x) = O(|x|^{-1/2})$ then either $\sigma_c < \infty$ or $\psi = 0$.*

iii) *If $V(x) = o(|x|^{-1/2})$ and $\sigma_c \in (0, \infty)$ then $\lambda < 0$ and $\sigma_c = \sqrt{-\lambda}$.*

OPEN PROBLEM: If V is bounded or even $V(x) = O(|x|^{-\delta}), \delta > 0$, is it true that $\sigma_c < \infty$ unless $\psi = 0$?

Laplacian - the N-body problem

Consider now the N-body problem, $H = -\Delta + V, (H - \lambda)\psi = 0$ where for simplicity V is a sum of fast decaying 2-body potentials plus a fast decaying N-particle interaction.

Let

$$\Sigma_c = \{\sqrt{\tau - \lambda} : \tau \in T(H), \tau > \lambda\}$$

$T(H)$ is the set of thresholds of H, a closed countable set independent of the N-particle interaction.

Theorem (Froese, H)

Suppose $H, \lambda, \psi\, (\neq 0)$ are as above and $\lambda \notin T(H)$. Then $\sigma_c \in \Sigma_c$.

The particular value of $\sigma_c \in \Sigma_c$ will depend on the N-particle interaction as well as λ.
The point of this slide is to emphasize that there are situations (unlike the one-body problem with the Laplacian) where there may be several possibilities for the decay rate for a given λ.

Results for $Q(-i\nabla)$

We consider $L^2(\mathbb{R}^d)$ solutions ψ to

$$(Q(-i\nabla) + V(x) - \lambda)\psi = 0$$

with Q a real elliptic polynomial and $\lim_{|x|\to\infty} V(x) = 0$. Consider the two conditions on a point $(\sigma, \xi, \omega) \in \mathbb{R}_+ \times \mathbb{R}^d \times S^{d-1}$

$$Q(\xi + i\sigma\omega) = \lambda \qquad (1)$$

$$P_\perp(\omega)\nabla_\xi Q(\xi + i\sigma\omega) = 0 \qquad (2)$$

where $P_\perp(\omega)$ is the projection onto the subspace of $\mathbb{R}^d$ perpendicular to ω.

Let

$$\Sigma_c = \{\sigma > 0 : \exists\,(\xi, \omega) \in \mathbb{R}^d \times S^{d-1} \text{ so that (1) and (2) are satisfied}\}$$

We will give conditions under which the decay rate σ_c is in Σ_c.
Let $V = V_1 + V_2$, both real and bounded.

Theorem (There is exponential decay)

Under either of the following two conditions we can conclude that $\sigma_c > 0$:

1. $\lambda \notin RanQ$ *and* $V(x) = o(1)$ *at infinity*
2. $\lambda \in RanQ$ *but* λ *is not a critical value of* Q *and in addition* $V_2(x) = o(|x|^{-1})$ *and* $\forall \alpha : \partial^\alpha V_1(x) = o(|x|^{-|\alpha|})$.

The next theorem eliminates the possibility of super-exponential decay (under strong conditions on the potential - too strong).
Let q be the degree of the polynomial Q.

Theorem (There is no super-exponential decay)

Suppose $V_2(x) = O(|x|^{-(q/2+\delta)})$, $\partial^\alpha V_1(x) = O(|x|^{-(\delta+|\alpha|+q)/2})$, *where* $1 \le |\alpha| \le q$ *and* $\delta > 0$. *Then* $\sigma_c < \infty$ *or* $\psi = 0$.

Given the last two theorems we know conditions on the potential so that $\sigma_c \in (0, \infty)$. In this case we can determine the decay rate "algebraically".

Theorem (Decay rate determined)

Assume $V_2(x) = o(|x|^{-1/2})$ *and* $\forall \alpha : \partial^\alpha V_1(x) = o(|x|^{-|\alpha|})$. *Suppose* $\sigma_c \in (0, \infty)$. *Then* $\sigma_c \in \Sigma_c$.

Example: $Q(\xi) = G(\xi^2)$

We take degree of $Q = q$. Then degree $G = q/2$. We assume $\sigma \in (0, \infty)$. With $z = (\xi + i\sigma\omega)^2$ the critical equations (1) and (2) which determine the possible decay rates σ reduce to

$$
\begin{aligned}
&G(z) = \lambda \\
&G'(z)P_{\perp}(\omega)\xi = 0
\end{aligned}
$$

Except for at most $(q-2)/2$ λ's (arising from non-simple roots of $G - \lambda$), these equations reduce to $G((|\xi| + i\sigma)^2) = \lambda$.

If $G(-\Delta) = \prod_{j=1}^{n}(\Delta^2 - \sigma_j^4), 0 < \sigma_1 < \cdots < \sigma_n$, all these σ's occur as decay rates for embedded eigenvalue 0 with different compact support potentials.

In fact for any real λ except possibly for the at most $(q-2)/2$ exceptional ones, any $\sigma \in \Sigma_c$ is a decay rate σ_c for some real compact support potential and some eigenfunction ψ.

Proof: decay rate σ_c determined when $\sigma_c \in (0,\infty)$

We only give some of the main ingredients of the proofs.

Condition (1) above, $Q(\xi + i\sigma\omega) = \lambda$, involves an energy estimate for the state $e^{\sigma r}\psi$ which we do not discuss here.

To understand Condition (2), $P_\perp(\omega)\nabla_\xi Q(\xi + i\sigma\omega) = 0$, we use commutator methods with a special *conjugate operator* dependent on a parameter σ constructed as follows. First decompose $Q(\xi + i\eta)$ into its real and imaginary parts:

$$Q(\xi + i\eta) = X(\xi,\eta) + iY(\xi,\eta)$$

Let

$$a(x,\xi) = rY(\xi, \sigma\omega(x))$$

Then the conjugate operator A is defined as the operator with Weyl symbol a:

$$A = Op^w(a)$$

.

Note that if $Q(p) = p^2$, then $A = \sigma(x \cdot p + p \cdot x)$, $p = -i\nabla$.
To get an idea where this comes from note that
$Q(p + i\sigma\omega) = e^{\sigma r} Q(p) e^{-\sigma r}$ and

$$i[Q(p), e^{\sigma r} A e^{\sigma r}] = e^{\sigma r}(i[\tilde{X}, A] + 2\mathrm{Re}\tilde{Y}A)e^{\sigma r},$$

where $\tilde{X} = \mathrm{Re}(e^{\sigma r} Q(p) e^{-\sigma r})$ and $\tilde{Y} = \mathrm{Im}(e^{\sigma r} Q(p) e^{-\sigma r})$.
To leading order the symbol of the operator between exponentials to the right has symbol

$$r\{X, Y\} + 2rY^2 + \{X, r\}Y. \tag{3}$$

We can calculate the Poisson bracket $\{X, Y\}$ using the Cauchy - Riemann equations:

$$\{X, Y\} = |P_\perp(\omega)\nabla_\xi Q(\xi + i\sigma\omega)|^2$$

The last term in (3) can be bounded by the middle term and something of lower order.

Absence of super-exponential decay, $\sigma_c < \infty$

Two key points:

◇ No pseudo-differential operators. Exact computations.

◇ $r = |x| \to r =< x > - < x >^{1-\epsilon} + 1$ (Rodnianski - Tao)

Let $a = p - i\sigma\omega$, $\omega = \nabla r$. The eigenvalue equation for $\psi_\sigma = e^{\sigma r}\psi$ is

$$(Q(a^*) + V - \lambda)\psi_\sigma = 0$$

Thus

$$< \psi_\sigma, ([Q(a), Q(a^*)] + |Q(a) + V_1 - \lambda|^2)\psi_\sigma >=$$
$$- < \psi_\sigma, (2\mathrm{Re}[Q(a), V_1] + |V_2|^2)\psi_\sigma >$$

Another key point: $P := [a, a^*] \geq c\sigma r^{-(1+\epsilon)}$

We extract positivity from $[Q(a), Q(a^*)]$. Write the commutator as a sum of Wick ordered operators. Let $J_m = (j_1, ..., j_m), K_m = (k_1, ..., k_m)$. Then

$$[Q(a), Q(a^*)] = F + E$$
$$F = \Sigma_{m=1}^{q} \Sigma_{J_m, K_m} \partial_{J_m} Q(a^*) P_{J_m, K_m} \partial_{K_m} Q(a)/m!$$

Here $\partial_{J_m} = \partial_{j_1} \cdots \partial_{j_m}, \partial_{K_m} = \partial_{k_1} \cdots \partial_{k_m}, P_{J_m, K_m} = P_{j_1, k_1} \cdots P_{j_m, k_m}$ and E is negligible for large σ.
Note for example that the term with $m = q$ is bounded below by

$$c\sigma^q r^{-q(1+\epsilon)}.$$

Improvement for the bi-Laplacian

Theorem (Improved super-exponential decay result)

Take $Q(-i\nabla) = (-\Delta)^{q/2}, q = 2, 4.$

Suppose

$$V_2(x) = O(|x|^{-q/4-\delta})$$
$$\partial^\alpha V_1(x) = O(|x|^{-(\delta+q/2+|\alpha|)/2}) \textit{ for } 1 \leq |\alpha| \leq q/2, \textit{ and } \delta > 0.$$

Then $\sigma_c < \infty$.

This theorem replaces q by $q/2$ in the assumptions on the potential for $Q(-i\nabla) = (-\Delta)^{q/2}, q = 2, 4$.