Bulk-edge duality for topological insulators

Gian Michele Graf
ETH Zurich

Quantum Spectra and Transport
A conference in honor of Yosi Avron
June 30 - July 4, 2013
The Hebrew University of Jerusalem, Israel

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joint work with Marcello Porta
thanks to Yosi Avron

Topological insulators: first impressions

- Insulator in the Bulk: Excitation gap
  For independent electrons: band gap at Fermi energy

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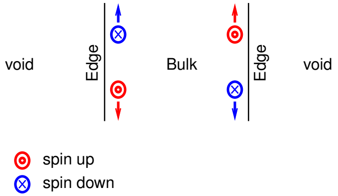

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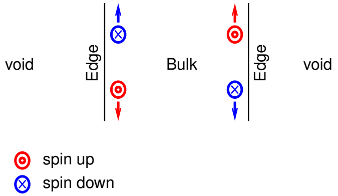

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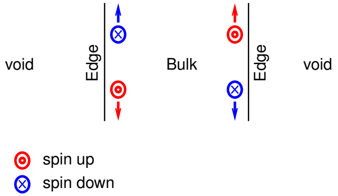

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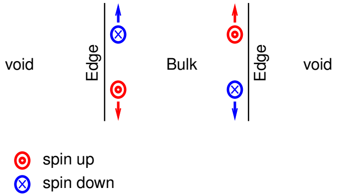

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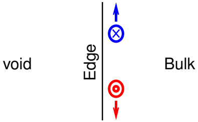
  Analogy: torus $\neq$ sphere (differ by genus).

Contributors to the field: Kane, Mele, Zhang, Moore; Fröhlich

Bulk-edge correspondence



In a nutshell: Termination of bulk of a topological insulator implies edge states

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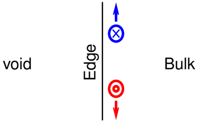

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More precisely:

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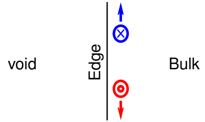

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- Express that property as an Index relating to the Bulk, resp. to the Edge.

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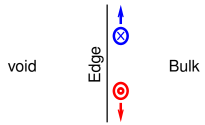

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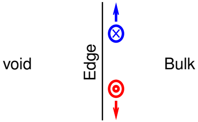
Bulk-edge correspondence. Done?



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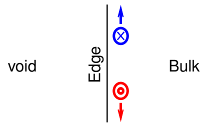
- State the (intrinsic) topological property distinguishing different classes of insulators.

More precisely:

- Express that property as an Index relating to the Bulk, resp. to the Edge. Yes, e.g. Kane and Mele.
- Bulk-edge duality: Can it be shown that the two indices agree? Schulz-Baldes et al.; Essin & Gurarie

Bulk-edge correspondence. Today



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Rueda de casino. Time 0′15″

Rueda de casino. Time $0'45''$

Rueda de casino. Time 3′23″

Rules of the dance

Dancers

- start in pairs, anywhere
- end in pairs, anywhere (possibly elseways & elsewhere)
- are free in between
- must never step on center of the floor

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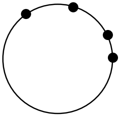
There are dances which can not be deformed into one another.

Which is the index that makes the difference?

The index of a Rueda

A snapshot of the dance



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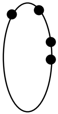
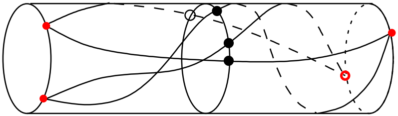


Dance D as a whole



D

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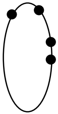

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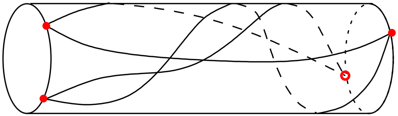

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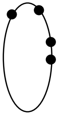

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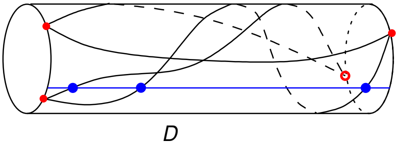

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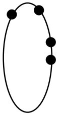

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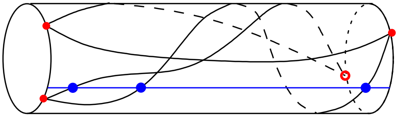
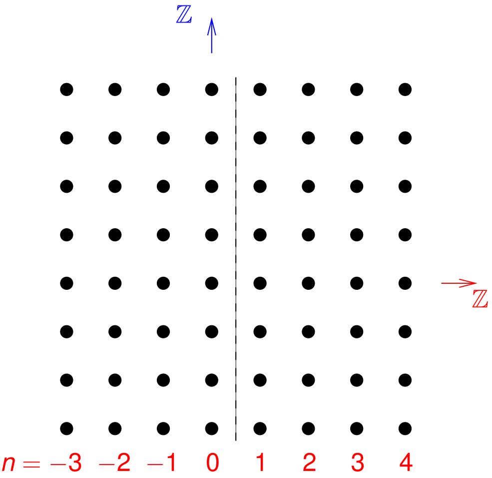


D

$\mathcal{I}(D) =$ parity of number of crossings of fiducial line

Bulk Hamiltonian

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)



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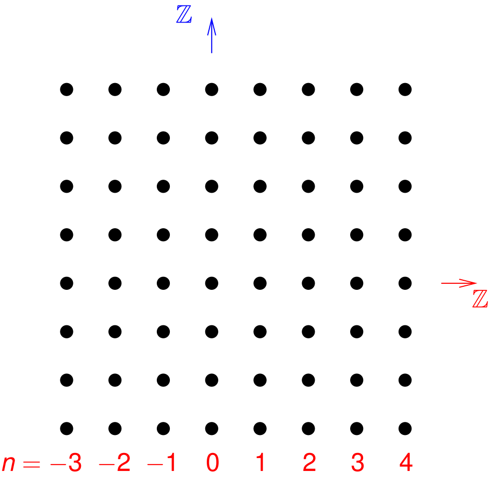

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End up with wave-functions $\psi = (\psi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}; \mathbb{C}^N)$ and Bulk Hamiltonian

$$\big(H(k)\psi\big)_n = A(k)\psi_{n-1} + A(k)^*\psi_{n+1} + V_n(k)\psi_n$$

with
$V_n(k) = V_n(k)^* \in M_N(\mathbb{C})$ (potential)
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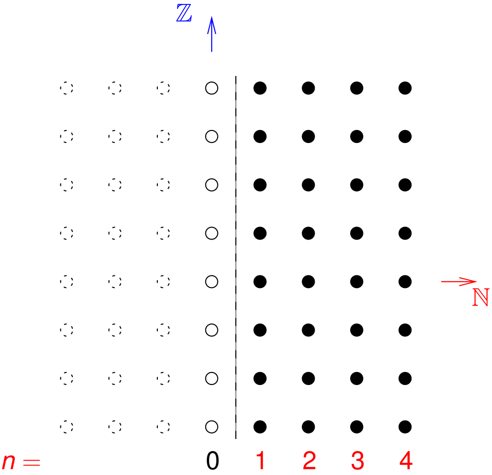
$A(k) \in \mathrm{GL}(N)$ (hopping): Schrödinger eq. is the 2nd order difference equation

Edge Hamiltonian

Hamiltonian on the lattice $\mathbb{N} \times \mathbb{Z}$ (half-plane) with $\mathbb{N} = \{1, 2, \ldots\}$



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Note: $\sigma_{\text{ess}}(H^\sharp(k)) \subset \sigma_{\text{ess}}(H(k))$, but typically $\sigma_{\text{disc}}(H^\sharp(k)) \not\subset \sigma_{\text{disc}}(H(k))$

General assumptions

- Gap assumption: Fermi energy μ lies in a gap for all $k \in S^1$:

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- Fermionic time-reversal symmetry: $\Theta : \mathbb{C}^N \to \mathbb{C}^N$
  - Θ is anti-unitary and $\Theta^2 = -1$;
  - For all $k \in S^1$,
  $$H(-k) = \Theta H(k) \Theta^{-1}$$
  where Θ also denotes the map induced on $\ell^2(\mathbb{Z}; \mathbb{C}^N)$. Likewise for $H^\sharp(k)$

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  $$H\psi = E\psi \implies H(\Theta\psi) = E(\Theta\psi)$$

  and $\Theta\psi = \lambda\psi$, $(\lambda \in \mathbb{C})$ is impossible:

  $$-\psi = \Theta^2\psi = \bar{\lambda}\Theta\psi = \bar{\lambda}\lambda\psi \qquad (\Rightarrow\Leftarrow)$$

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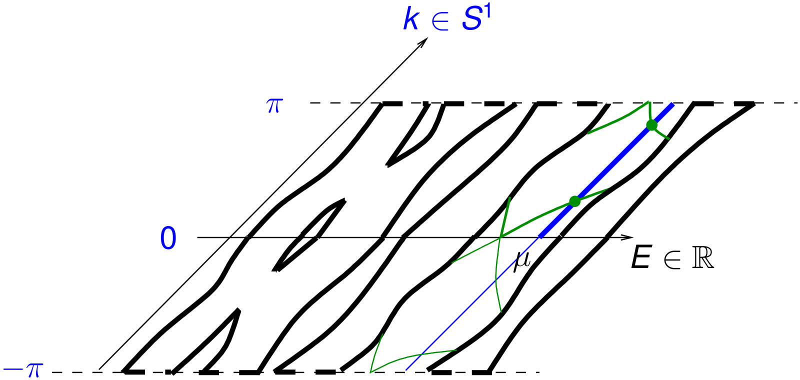
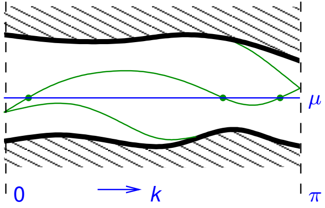


Bands, Fermi line (one half **fat**), edge states

The edge index

The spectrum of $H^{\sharp}(k)$

symmetric on $-\pi \leq k \leq 0$



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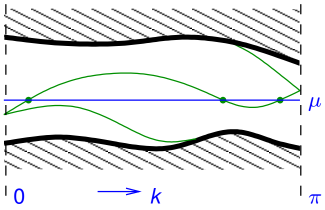

Bands, Fermi line, edge states

Definition: Edge Index

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At fixed k, map gap to $S^1 \setminus \{1\}$ and bands to $1 \in S^1$:
Edge Index is index of a rueda.

Towards the bulk index

Let $z \in \mathbb{C}$. The Schrödinger equation

$$(H(k) - z)\psi = 0$$

(as a 2nd order difference equation) has $2N$ solutions $\psi = (\psi_n)_{n\in\mathbb{Z}}$, $\psi_n \in \mathbb{C}^N$.

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Let $z \notin \sigma(H(k))$. Then

$$E_{z,k} = \{\psi \mid \psi \text{ solution}, \psi_n \to 0, \ (n \to +\infty)\}$$

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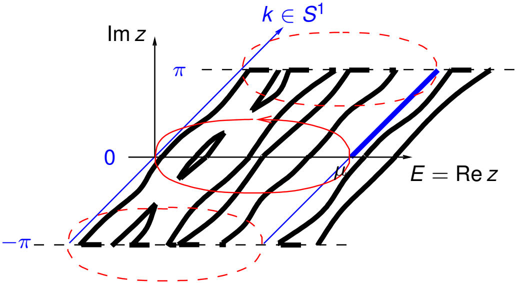
- $E_{\bar{z},-k} = \Theta E_{z,k}$

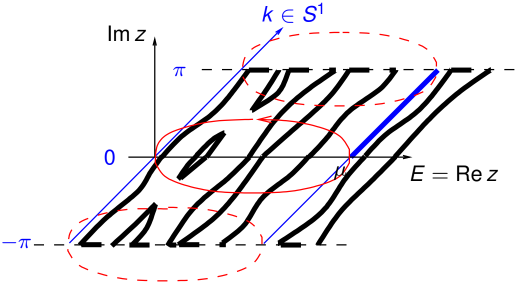
The bulk index



Loop γ and torus $\mathbb{T} = \gamma \times S^1$

Vector bundle E with base $\mathbb{T} \ni (z, k)$, fibers $E_{z,k}$, and involution Θ.

The bulk index



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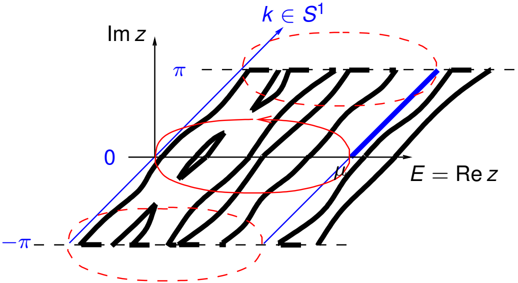
Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$ (besides of $N = \dim E$)

The bulk index



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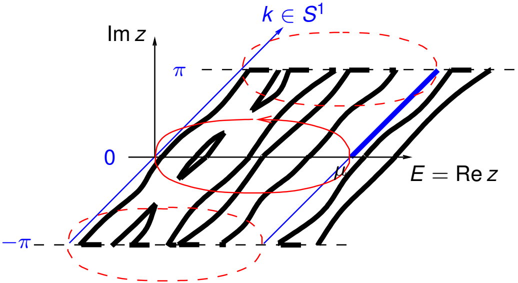

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Definition: Bulk Index

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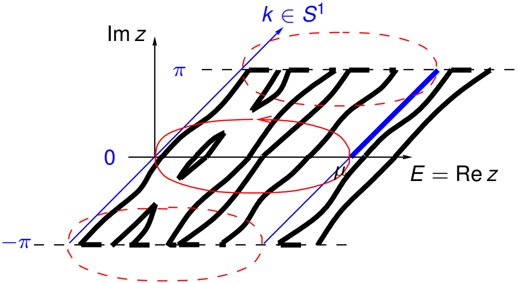
What's behind the theorem? How is $\mathcal{I}(E)$ defined?

The bulk index



Loop γ and torus $\mathbb{T} = \gamma \times S^1$

Vector bundle E with base $\mathbb{T} \ni (z, k)$, fibers $E_{z,k}$, and involution Θ.

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$ (besides of $N = \dim E$)

Definition: Bulk Index

$$\mathcal{I} = \mathcal{I}(E)$$

What's behind the theorem? How is $\mathcal{I}(E)$ defined? Aside ...

Time-reversal invariant bundles on the torus

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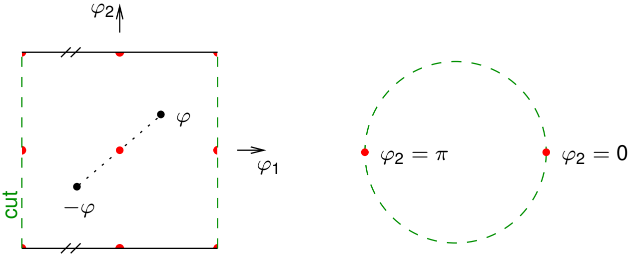

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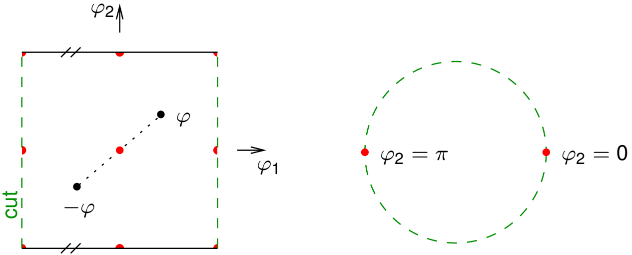

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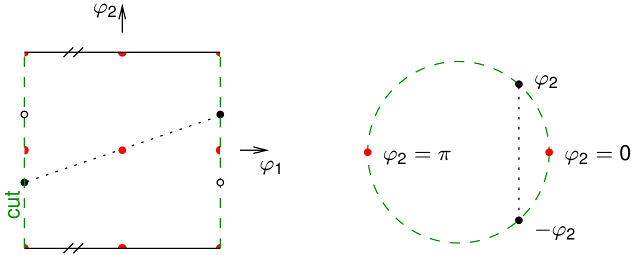

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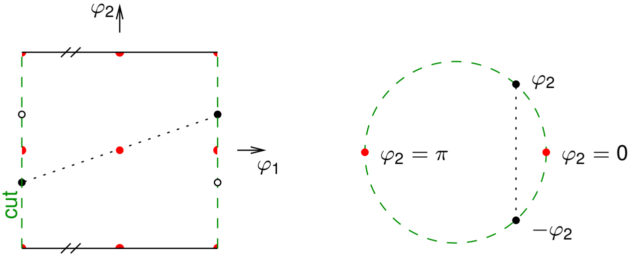

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$$\Theta_0 T(\varphi_2) = T^{-1}(-\varphi_2)\Theta_0 \, , \qquad (\varphi_2 \in S^1)$$

with $\Theta_0 : \mathbb{C}^N \to \mathbb{C}^N$ antilinear, $\Theta_0^2 = -1$

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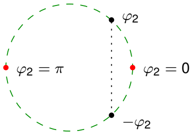

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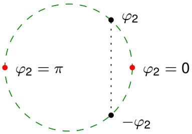

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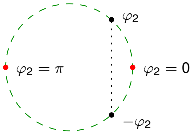

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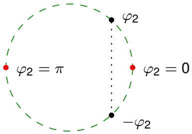

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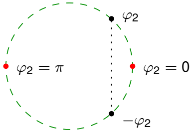

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Definition (Index): $\mathcal{I}(E) := \mathcal{I}(D)$

Remark: $\mathcal{I}(E)$ agrees (in value) with the Pfaffian index of Kane and Mele.

. . . aside ends here.

Main result

Theorem Bulk and edge indices agree:

$$\mathcal{I} = \mathcal{I}^\sharp$$

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$\mathcal{I} = +1$: ordinary insulator
$\mathcal{I} = -1$: topological insulator

Proof of Theorem (preliminary remark)

- For this slide only: $N = 1$.

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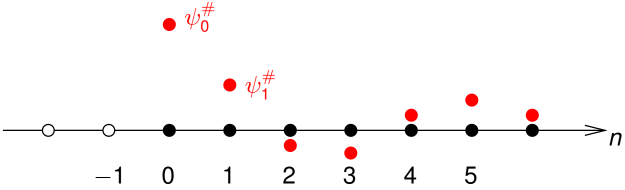

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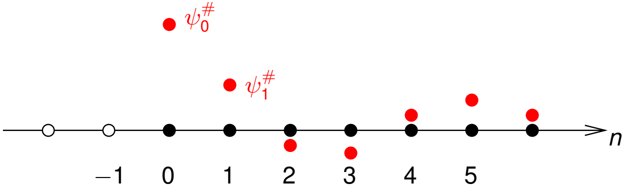

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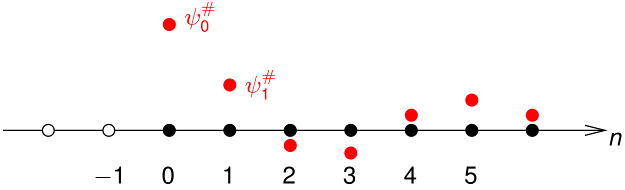

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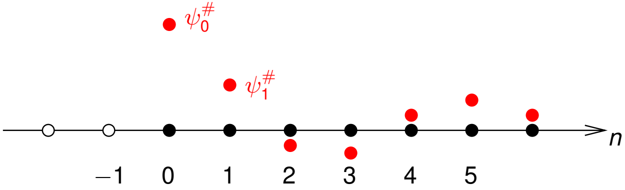

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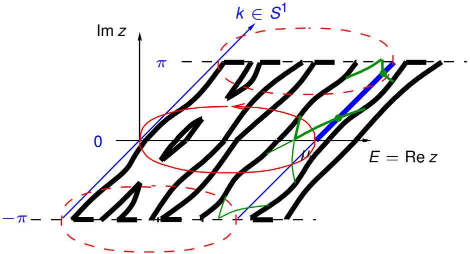
Proof of Theorem (sketch)



Fermi line (one half **fat)**
edge states
torus

- ψ, $\psi^\sharp$ solutions (bulk, edge) at z, k decaying at $n \to +\infty$
- Bijective map $\psi \mapsto \psi^\sharp$, so that $\psi_n = \psi_n^\sharp$ $(n > n_0)$
- $\exists \psi^\sharp \neq 0 \mid \psi^\sharp_{n=0} = 0 \Leftrightarrow z \in \sigma(H^\sharp(k))$
- There is a section of the frame bundle $F(E)$, global on $\mathbb{T}$, except at edge eigenvalue crossings
- Cut the torus along the Fermi line; let $T(k)$ be the transition matrix
- There $T(k) = \mathbb{I}_N$, except near eigenvalue crossings
- As k traverses one of them, $T(k)$ has eigenvalues 1 (multiplicity $N - 1$) and $\lambda(k)$ making one turn of S^1

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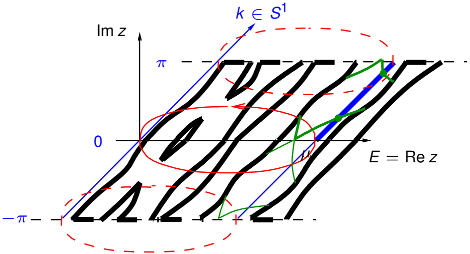

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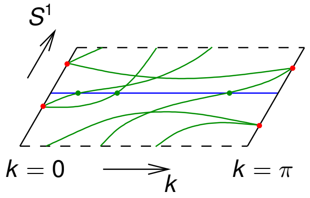
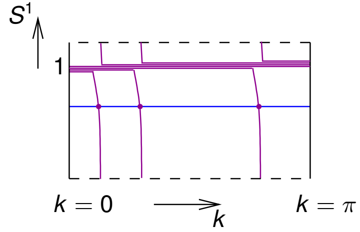
Proof of Theorem: Dual ruedas



Edge rueda: edge eigenvalues



Bulk rueda: eigenvalues of $T(k)$

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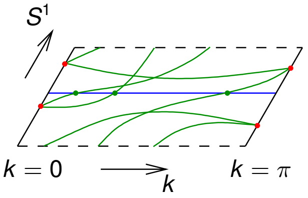

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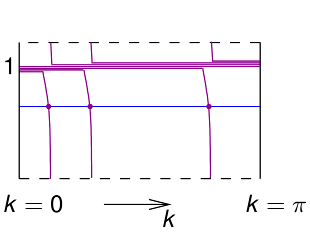

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Ruedas share intersection points.

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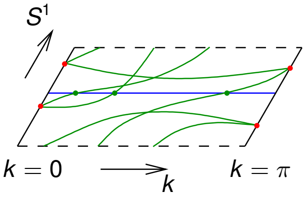

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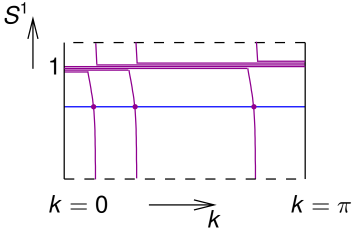

Bulk rueda: eigenvalues of $T(k)$

Ruedas share intersection points. Hence indices are equal $\square$

Final remarks

Further results:

- In case the Bulk Hamiltonian is doubly periodic:

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with product over filled pairs
- A direct link between indices of Bloch bundles and the edge index via Levinson's theorem.
- 3d topological insulators (weak and strong indices: 3+1)

Open questions:

- No periodicity (disordered case)?

Summary

Bulk = Edge

$$\mathcal{I} = \mathcal{I}^\sharp$$