The Nonlinear Schroedinger Equation (NLSE) with a random potential:Results and Puzzles

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Experimental Relevance

Nonlinear Optics
Bose Einstein Condensates (BECs)

Paradigm for competition between randomness and Nonlinearity a Fundamental Question

Outline

- Introduction
- Rigorous results
- Perturbation Theory
- Numerical results and Effective Noise Theories
- Scaling Theory
- Summary
- Open Questions

The Nonlinear Schroedinger (NLS) Equation

$$i\frac{\partial}{\partial t}\psi = \mathcal{H}_o\psi + \beta|\psi|^2\psi$$

1D lattice version

$$\mathcal{H}_0\psi(x) = -\left(\psi(x+1)+\psi(x-1)\right)+\varepsilon(x)\psi(x)$$

1D continuum version with no randomness, integrable

$$\mathcal{H}_0\psi(x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x)+\varepsilon(x)\psi(x)$$

$\mathcal{E}$ random $\longrightarrow$ $\mathcal{H}_0$ Anderson Model

$\beta = 0 \Rightarrow$ localization

Does Localization Survive the Nonlinearity???

Does Localization Survive the Nonlinearity???

1. Yes, if there is spreading the magnitude of the nonlinear term decreases and localization takes over.
2. No, may depend on realizations or on β found in numerical calculations.
3. No, the NLSE is a chaotic dynamical system. Will it remain chaotic for all densities??
4. No?, but localization asymptotically preserved beyond some front that is logarithmic in time

Point 4, conjectured by Wang and Zhang in the limit of strong disorder :

given $$\varepsilon = hopping + \beta \quad \delta > 0, A > 0$$

tail beyond j_0 of weight < δ

there exist $C, \varepsilon(A), K > A^2$

So that for all $t \le \left(\frac{\delta}{C}\right) \varepsilon^{-A}, \varepsilon < \varepsilon(A)$

$$\ln t \sim A_f$$

With probability $1 - \exp\left(-\frac{j_0}{K} e^{-2K^{1/CA}}\right)$

ail beyond $j_0 + K$ of weight < 2δ

conjecture **Logarithmic front** $j_0 + K_f \quad K_f > A^2$

Perturbation theory supports this conjecture for any disorder

Point 4 also in agreement with work of Basko (perturbation theory
Selecting a type of dominant excitations)

***Bourgain and Wang $\beta \Rightarrow \varepsilon |1+|x||^{-\alpha}$

Spreading slower than any power of t

Perturbation Theory

1. A perturbation expansion in β was developed
2. Secular terms were removed
3. A bound on the general term was derived
4. Perturbation theory was used to obtain a controlled numerical solution
5. A bound on the remainder was obtained, indicating that the series is asymptotic.
6. For limited time tending to infinity for small nonlinearity, front logarithmic in time $x_f \propto \ln t$
7. Improved for strong disorder
8. Posed problems in Linear Anderson Localization for example combinations of eigen-energies (in denominators)

Bound on the remainder

$\beta^N Q_n = \text{remainder}$ $\quad$ $n = \text{site}$ $\quad$ $x_n = \text{location}$

$\left|\beta^N Q_n\right| \le C t e^{-\gamma |x_n|}$ $\quad$ $N = \text{order}$

$C \propto \beta^N e^{cN^2}$

$\lim_{\beta \to 0} \left|\beta^N Q_n\right| / \beta^{N-1} = 0$ $\longrightarrow$ Asymptotic

Remainder to expansion in terms of exponentially localized
Functions negligible for $x > x_f \sim \frac{1}{\gamma} \ln t$

Defines front x_f

Main problem divergence of C with N

Numerical Simulations

- In regimes relevant for experiments looks that localization takes place
- Spreading for long time (**Shepelyansky**, Pikovsky, Mulansky, Molina,

  Flach, Kopidakis, Komineas, Krimer, Laptyeva, Bodyfelt)
- We do not know the relevant space and time scales
- All results in Split-Step
- No control (but may be correct in some range)
- Supported by various heuristic arguments

Pikovsky, Sheplyansky

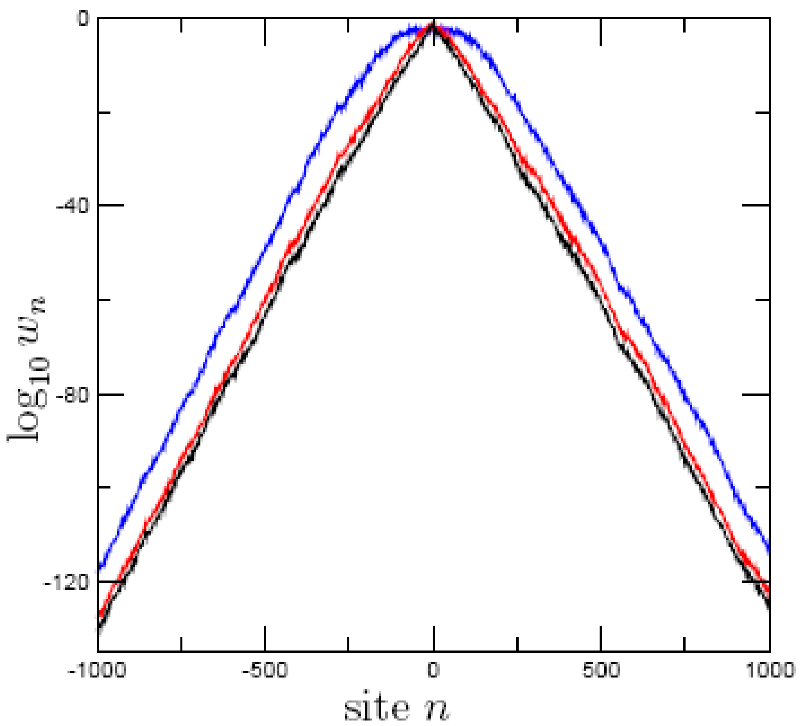


FIG. 2: (color online) Probability distribution w_n over lattice sites n at $W = 4$ for $\beta = 1$, $t = 10^8$ (top blue/solid curve) and $t = 10^5$ (middle red/gray curve); $\beta = 0, t = 10^5$ (bottom black curve; the order of the curves is given at $n = 500$). At $\beta = 0$ a fit $\ln w_n = -(\gamma|n| + \chi)$ gives $\gamma \approx 0.3$, $\chi \approx 4$. The values of $\log_{10} w_n$ are averaged over the same disorder realizations as in Fig. 1.

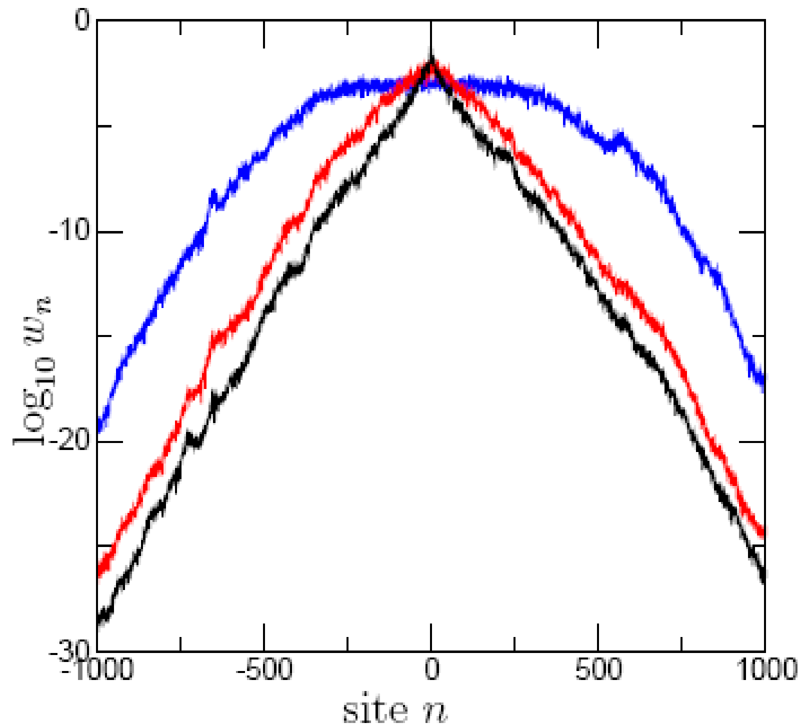


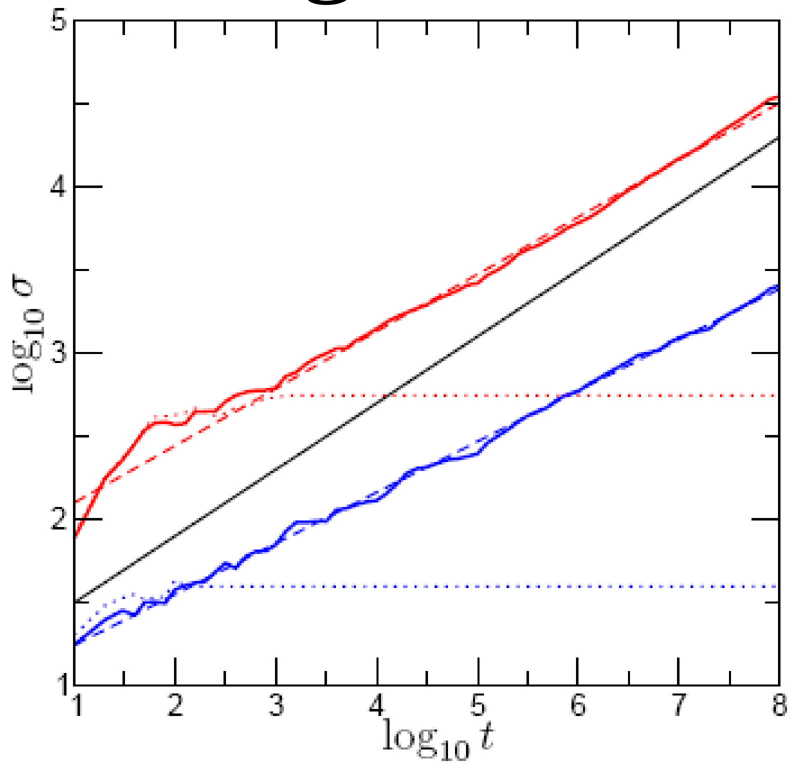
FIG. 3: (color online) Same as in Fig. 2 but with $W = 2$. At $\beta = 0$ a fit $\ln w_n = -(\gamma|n| + \chi)$ gives $\gamma \approx 0.06$, $\chi \approx -3$. The values of $\ln w_n$ are averaged over the same disorder realizations as in Fig. 1.

spreading sets on. A similar transition occurs for $W = 2$

Slope does not change (contrary to Fermi-Pasta-Ulam)

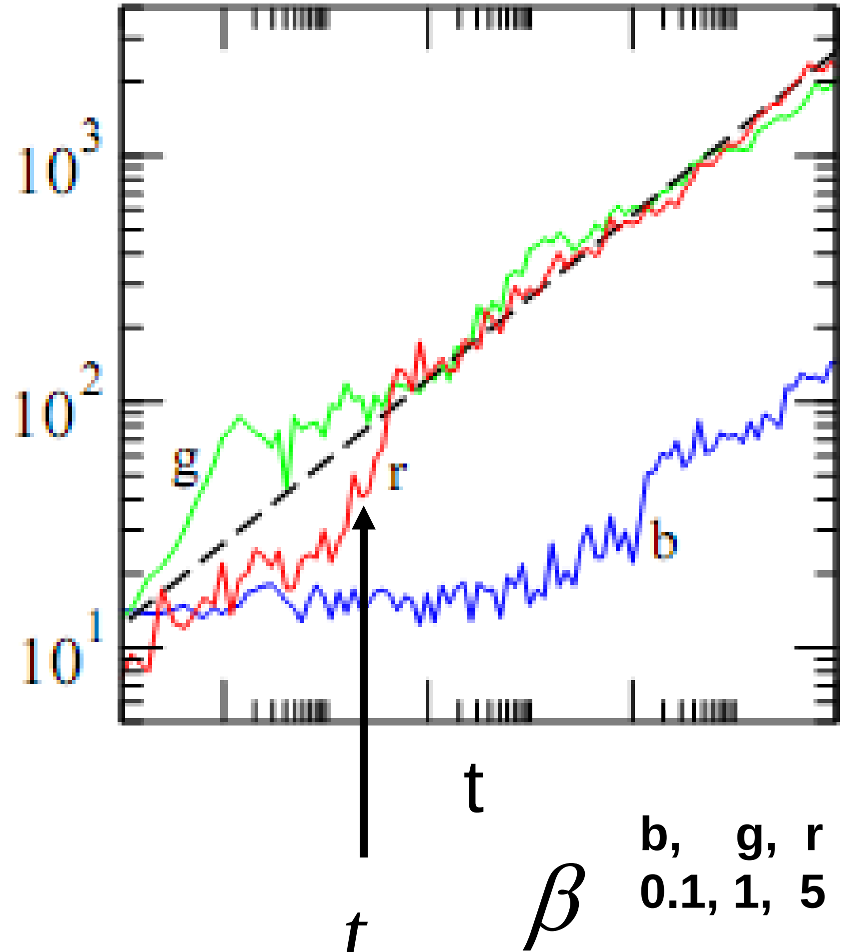
Pikovsky, Shepelyansky

$$\log < x^2 >$$



S.Flach, D.Krimer and S.Skokos

$$\log < x^2 >$$



β **b, g, r**
0.1, 1, 5

Effective Noise Theories

- D. Shepeyansky and A. Pikovsky
- S. Flach, Ch. Skokos, D.O. Krimer, S. Komineas
- E. Michaely and SF

$$\psi(x,t) = \sum_m c_m(t) e^{-iE_m t} u_m(x)$$

$$i\frac{\partial}{\partial t} c_n = \beta \sum_{m_1,m_2,m_3} V_n^{m_1,m_2,m_3} c_{m_1}^{\ *} c_{m_2} c_{m_3} e^{i(E_n + E_{m_1} - E_{m_2} - E_{m_3})t}$$

Overlap $$V_n^{m_1,m_2,m_3} = \sum_x u_n(x) u_{m_1}(x) u_{m_2}(x) u_{m_3}(x)$$

$$\left|V_n^{m_1 m_2 m_3}\right| \leq [const] e^{-\frac{1}{3}\gamma\left(\left|x_n - x_{m_1}\right| + \left|x_n - x_{m_2}\right| + \left|x_n - x_{m_3}\right|\right)}$$

of the range of the localization length ξ

$$i\frac{\partial}{\partial t}c_n = \beta \sum_{m_1,m_2,m_3} V_n^{m_1,m_2,m_3} c_{m_1}{}^* c_{m_2} c_{m_3} e^{i(E_n+E_{m_1}-E_{m_2}-E_{m_3})t}$$

Assume $\left|c_{m_1}^2\right| \approx \left|c_{m_2}^2\right| \approx \left|c_{m_3}^2\right| \approx \rho$ **initially** $\left|c_n^2\right| \ll \rho$

$$\frac{\partial}{\partial t}c_n \approx \beta\rho^{3/2} P f_n(t) = F_n(t)$$

$f_n(t)$ **Random uncorrelated**

$$P = C\beta\xi^{\alpha}\rho$$

Number of resonant states

leading to $\left\langle x^2 \right\rangle \sim t^{1/3}$

It was checked effective noise holds for realizations with sub-diffusion

Equilibrium $\left\langle \left| c_n^2 \right| \right\rangle = \rho$

Equilibration time T_{eq}

$$T_{eq} = \frac{1}{B(\beta,\xi)\,\xi^{-2}\rho^4}$$

$$D = C\frac{\xi^2}{T_{eq}} = CB\rho^4$$

For diffusion $< x^2 >= Dt$ but $< x^2 > \propto \frac{1}{\rho^2}$

$$\frac{1}{\rho^2} = CB\rho^4 t$$

$$< x^2 >= C_1\frac{1}{\rho^2} = C\left[\beta\xi^{\nu}t\right]^{1/3}$$

Consistent $T_{eq} \sim t^{2/3} \ll t$

Can it go on forever?

What happens when nearly no weight in localization volume?

$$P = C\beta\xi^{\alpha}\rho$$

Number of resonant states

numerically $1.85 < \alpha < 2.56$

For the theory to be valid require that the number of resonant terms larger then a number of order unity otherwise NO effective noise. Requires:

$$1 < P = (..)\xi^{\alpha}\rho \longrightarrow t < t_{*} \sim \xi^{\delta}$$

$$7.4 < \delta < 10.24$$

Model for the region of large amplitude

Competition

Spreading → **effective number of degrees of freedom increases**

chaos enhanced

Spreading → **amplitude decreases** → **regularity enhanced**

Who wins??

$$i\frac{d}{dt}\psi(x) = -J\left(\psi(x+1)+\psi(x-1)\right)+\varepsilon(x)\psi(x)+\left|\psi(x)\right|^2\psi(x)$$

x integer $1 \le x \le L$ $J = \frac{1}{1+W}$ $-JW \le \varepsilon \le JW$

$\psi(x)$ Are dynamical variables

Initial data, nearly homogeneous spreading in space

Growth of deviations

$$\delta\psi(t) \sim \delta\psi(t=0)e^{\lambda t}$$

λ Largest Lyapunov exponent

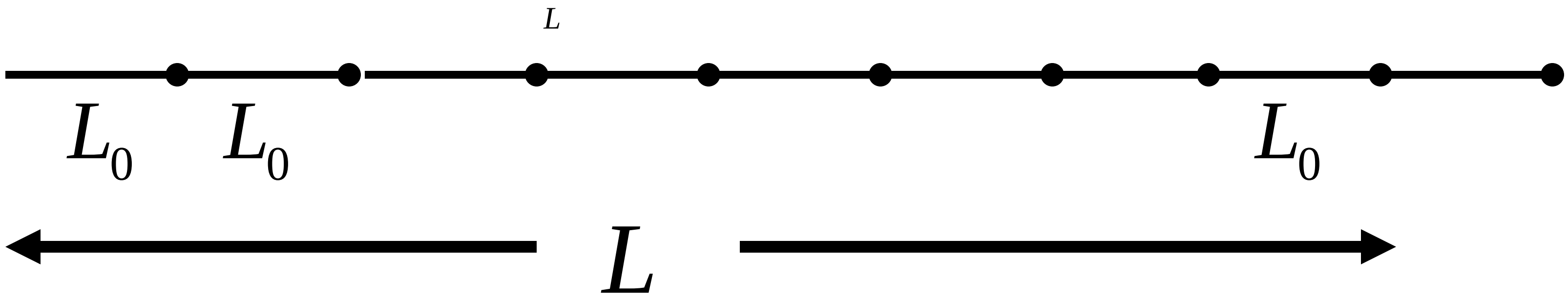
$\lambda > 0$ ⟶ Chaos

Is it possible that chaos disappears?

Divide chain into intervals of length L_0 Number of intervals $\frac{L}{L_0}$

Assuming independence, if intervals large enough $L_0 \gg \xi$

The probability to be regular:



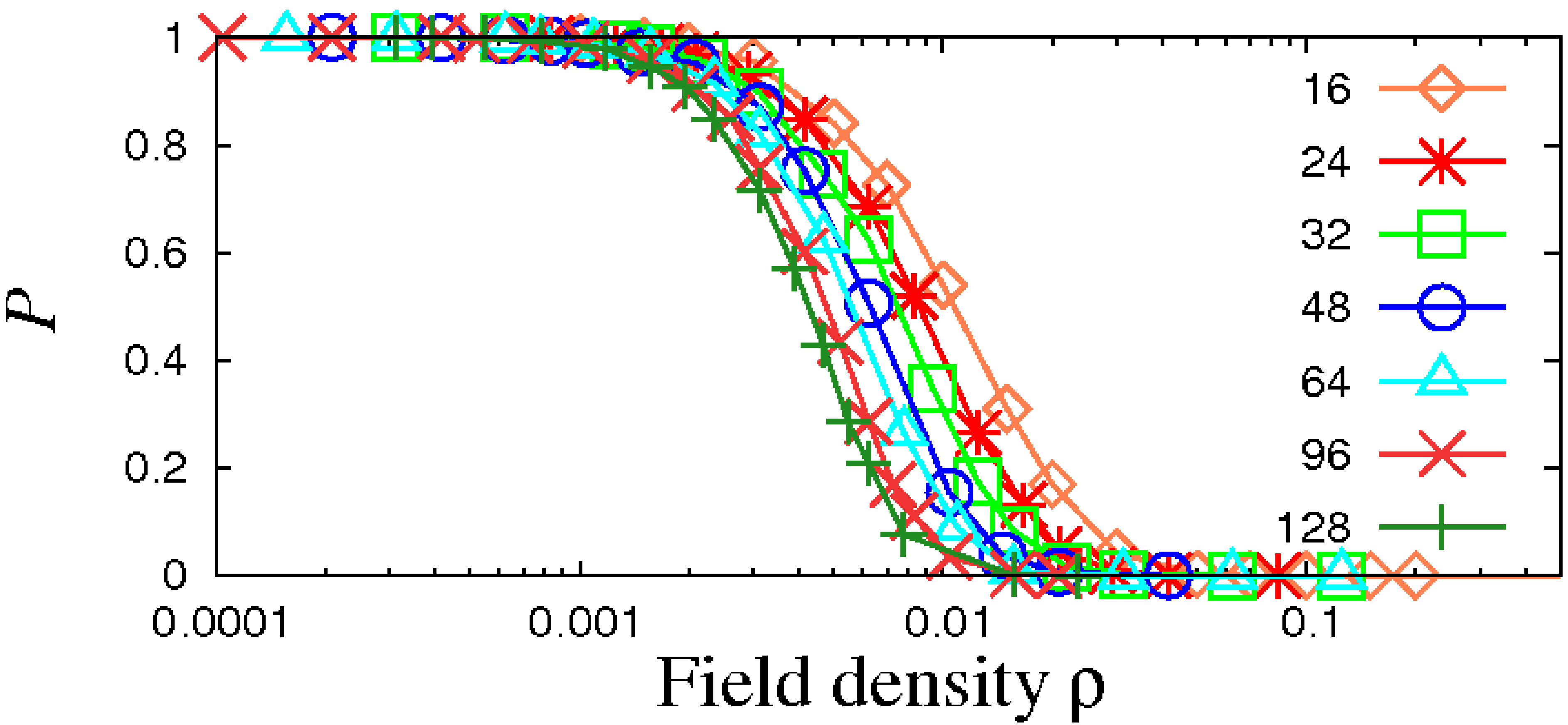
Regularity=all orbits regular

$$p_{reg}(\rho, W, L) = p_{reg}(\rho, W, L_0)^{L/L_0}$$

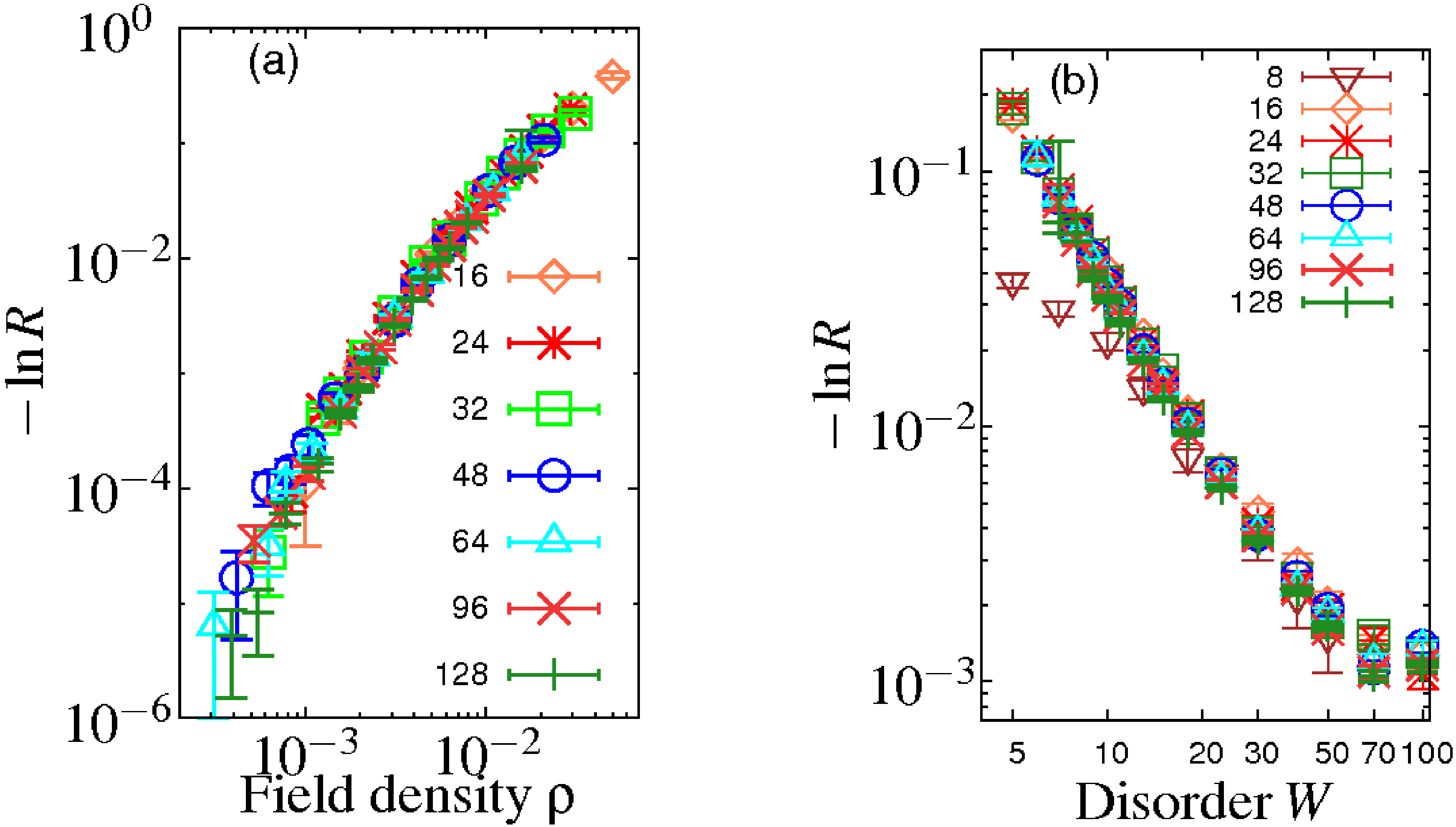
density $\rho = \frac{N}{L}$ $N = \sum_{x=1}^{L} |\psi(x)|^2$

$$R(\rho, W) \equiv p_{reg}(\rho, W, L)^{1/L} = p_{reg}(\rho, W, L_0)^{1/L_0}$$

independent of L



P
1
0.8
0.6
0.4
0.2
0
0.0001
0.001
0.01
0.1
Field density ρ
16
24
32
48
64
96
128


(a)
$-\ln R$
Field density ρ
16
24
32
48
64
96
128
(b)
$-\ln R$
Disorder W
8
16
24
32
48
64
96
128

Scaling

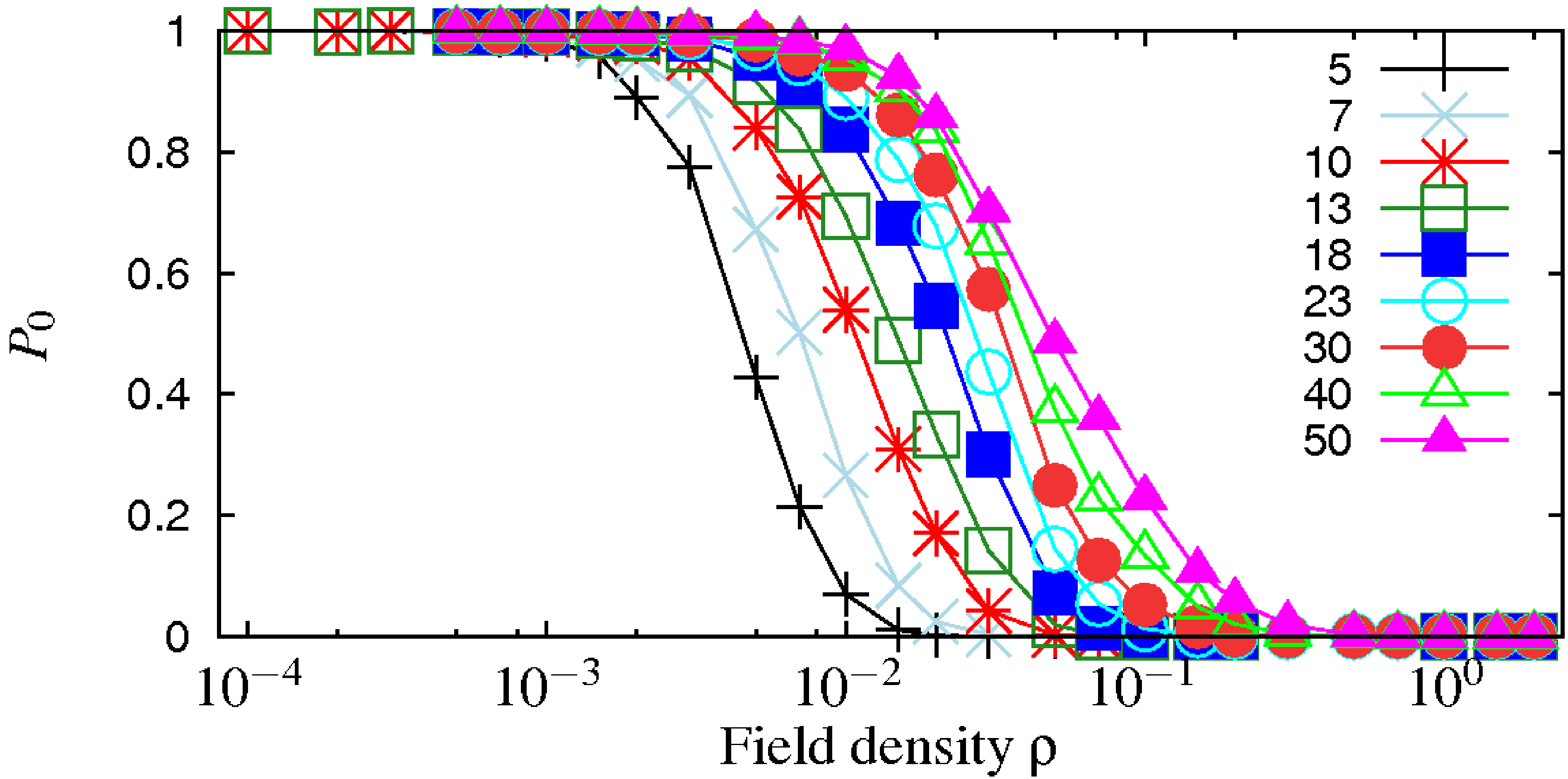
define $$P_0(\rho, W) \equiv P_{reg}(\rho, W, L_0)$$

define Q by $$P_0 = \frac{1}{1 + 1/Q}$$

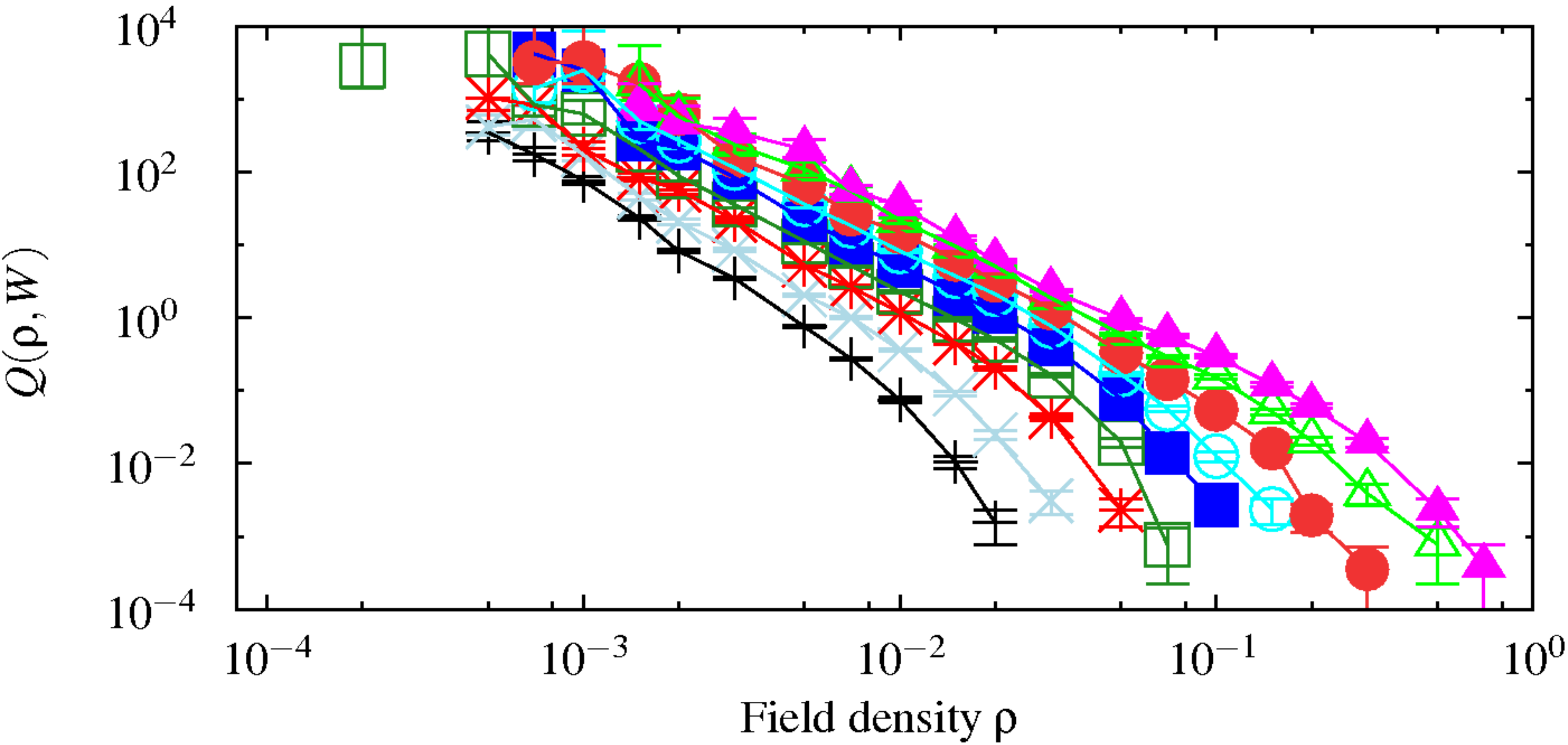
Regular limit $$P_{reg} \to 1, P_0 \to 1 \Leftrightarrow Q \to \infty$$

$$\rho \to 0 \qquad W \to 0$$ Singular limits

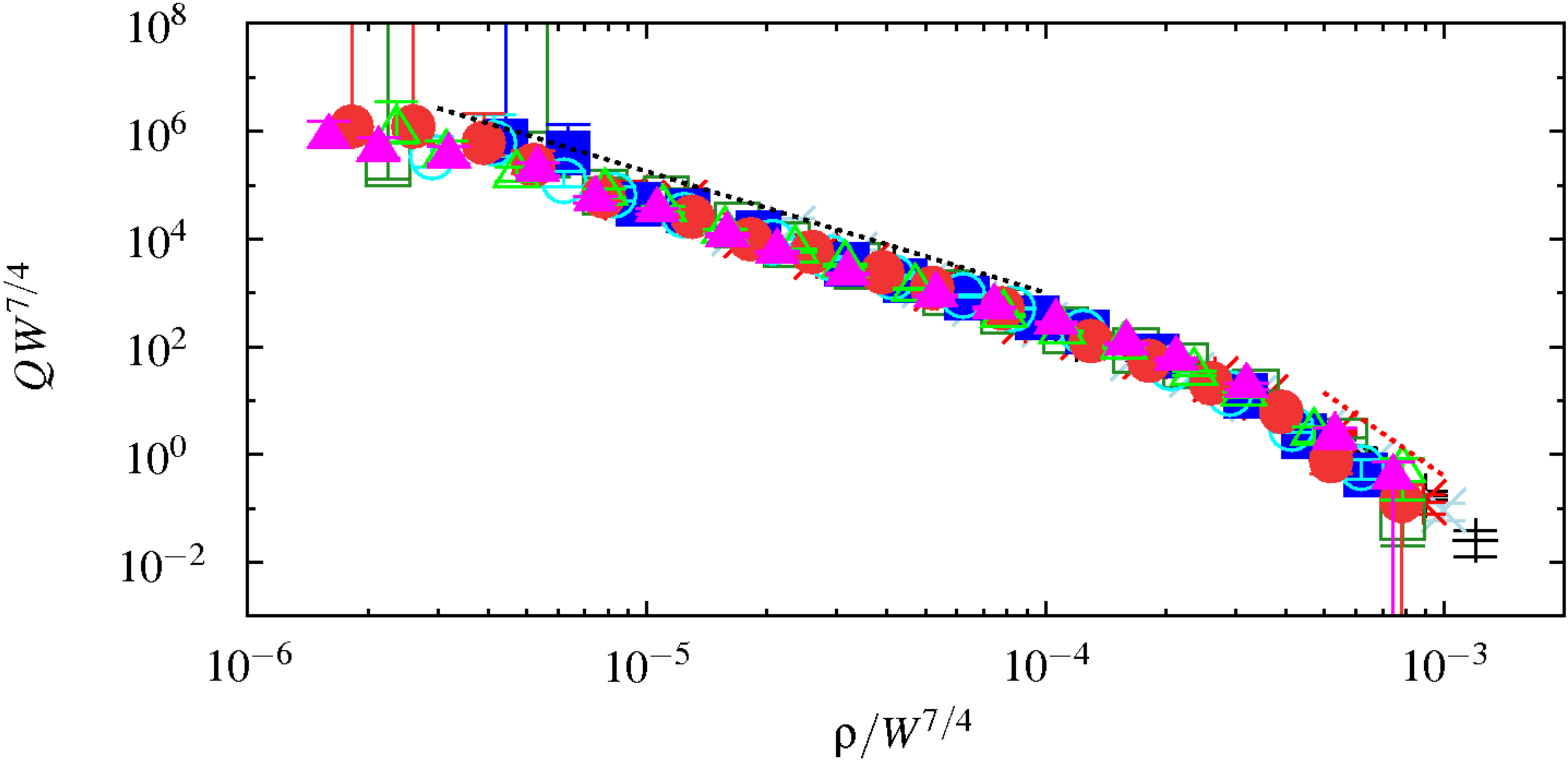
Assume scaling function $$Q(\rho, W) = \frac{1}{W^\alpha} q\left(\frac{\rho}{W^\beta}\right)$$



1
0.8
0.6
0.4
0.2
0
P_0
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
Field density ρ
5
7
10
13
18
23
30
40
50



10^4
10^2
10^0
10^{-2}
10^{-4}
$Q(\rho, W)$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
Field density ρ



10^{8}
10^{6}
10^{4}
10^{2}
10^{0}
10^{-2}
$QW^{7/4}$
10^{-6}
10^{-5}
10^{-4}
10^{-3}
$\rho/W^{7/4}$

Singular limits

small x $q \sim c_1 x^{-\varsigma}$

large x $q \sim c_2 x^{-\eta}$

Numerical fit

$\alpha = \beta \approx 1.75 = \frac{7}{4}$ $\varsigma \approx 2.25 = \frac{9}{4} > 1$ $\eta \approx 5.2$

$c_1 \approx 2.5 \cdot 10^{-7}$ $c_2 \approx 1.8 \cdot 10^{-18}$

very small !!!

What can one conclude from the scaling??

$$\rho \to 0 \Leftrightarrow x \to 0 \Rightarrow Q \to \infty$$

$$P_0 = \frac{1}{1+1/Q} \approx 1 - \frac{1}{Q} \to 1$$

$$p_{reg}(\rho, W, L) = p_0(\rho, W)^{L/L_0} = \left[1 - \frac{1}{Q}\right]^{\frac{L}{L_o}}$$

$$P_{ch} = 1 - P_{reg} \approx -\ln P_{reg} \approx \frac{L}{L_0}\frac{1}{Q} = \frac{L}{L_0} W^{\alpha} \frac{1}{q}$$

Q scaling function

$$P_{ch} = \frac{1}{c_1}\frac{L}{L_0} W^{\alpha(1-\varsigma)} \rho^{\varsigma}$$

density $\rho = \frac{N}{L}$ with $N = \sum_{x=1}^{L} |\psi(x)|^2$

$$P_{ch} = \frac{1}{c_1}\frac{L^{1-\varsigma}}{L_0} W^{\alpha(1-\varsigma)} N^{\varsigma}$$

Using exponents found here

$$P_{ch} \approx \frac{N^{9/4}}{c_1 L_0 W^{35/16}} L^{-5/4}$$

Large system limit

$$P_{ch} \approx \frac{N^{9/4}}{c_1 L_0 W^{35/16}} L^{-5/4} \to 0$$

$L \to \infty$ and N fixed

What the scale required to see it??

set minimal probability for chaos to be effective $P_{\min}$

$$L_{\max} = N^{\frac{\varsigma}{\varsigma-1}} W^{-\alpha} \left(P_{\min} L_0 c_1 \right)^{\frac{-1}{\varsigma-1}}$$

For $P_{\min} \approx 0.05$

and parameters used in numerical calculations $L_{\max} \approx 10^5$

If no additional singularity in Q

Spreading

↓

No chaos

↓

Localization ?

Emerging Picture

- For small nonlinearity initially no spreading
- For strong nonlinearity some part does not spread
- For some nonlinearity wide regime of sub-diffusion
- Asymptotic spreading at most logarithmic (shown for limited time):

a. perturbation theory

b. rigorous results in the limit of strong disorder

- Unlikely that sub-diffusion continues forever:

a. scaling theory showing that as result of spreading system becomes regular for an increasing fraction of realizations

b. Effective noise “theories” indicate that as a result of spreading noise Decays

c. More realizations quasiperiodic (Aubry)

Coherent picture for various regimes?   What is the relevant time scale for asymptotics?

What of this can be established rigorously??

Possible asymptotic behavior

- Localization
- Logarithmic spread
- Sub-diffsion
- Spread to some distance depending on realization of disorder