Matrix Valued and Trimmed Anderson Models

Alexander Elgart

Collaborators: Abel Klein, Daniels Schmidt, Mira Shamis and Sasha Sodin

Avronfest, Jerusalem, July 2013

Projecting a projection

Given a rank one orthogonal projection P, find the largest 1×1 principal sub matrix Q.

Projecting a projection

Given a rank one orthogonal projection P, find the largest 1×1 principal sub matrix Q.

$$\mathrm{P} = \frac{1}{36} \begin{bmatrix} 1 & -3 & -1 & 5 \\ -3 & 9 & 3 & -15 \\ -1 & 3 & 1 & -5 \\ 5 & -15 & -5 & 25 \end{bmatrix}.$$

Projecting a projection

Given a rank one orthogonal projection P, find the largest 1×1 principal sub matrix Q.

$$P = \frac{1}{36} \begin{bmatrix} 1 & -3 & -1 & 5 \\ -3 & 9 & 3 & -15 \\ -1 & 3 & 1 & -5 \\ 5 & -15 & -5 & 25 \end{bmatrix}.$$

In general, if $P = (k \times k)$ we look for the largest diagonal entry (i, i), with $Q \geq 1/k$.

Projecting a projection

Given a rank one orthogonal projection P, find the largest 1×1 principal sub matrix Q.

$$P = \frac{1}{36}\begin{bmatrix} 1 & -3 & -1 & 5 \\ -3 & 9 & 3 & -15 \\ -1 & 3 & 1 & -5 \\ 5 & -15 & -5 & 25 \end{bmatrix}.$$

In general, if $P = (k \times k)$ we look for the largest diagonal entry (i, i), with $Q \geq 1/k$.

Given a rank two orthogonal projection P, find the largest 2×2 principal sub matrix Q.

Projecting a projection

$$P = \frac{1}{52}\begin{bmatrix} 36 & 0 & -24 & 0 & 0 \\ 0 & 13 & 0 & 13 & 13 \\ -24 & 0 & 16 & 0 & 0 \\ 0 & 13 & 0 & 13 & 13 \\ 0 & 13 & 0 & 13 & 13 \end{bmatrix}.$$

Projecting a projection

$$P = \frac{1}{52}\begin{bmatrix} 36 & 0 & -24 & 0 & 0 \\ 0 & 13 & 0 & 13 & 13 \\ -24 & 0 & 16 & 0 & 0 \\ 0 & 13 & 0 & 13 & 13 \\ 0 & 13 & 0 & 13 & 13 \end{bmatrix}.$$

Choose the first index as in the rank one case. How to choose the second index?

Projecting a projection

$$P = \frac{1}{52}\begin{bmatrix} 36 & 0 & -24 & 0 & 0 \\ 0 & 13 & 0 & 13 & 13 \\ -24 & 0 & 16 & 0 & 0 \\ 0 & 13 & 0 & 13 & 13 \\ 0 & 13 & 0 & 13 & 13 \end{bmatrix}.$$

Choose the first index as in the rank one case. How to choose the second index?

Use Schur complement:

$$D = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}, \quad D/A := C - B^*A^{-1}B.$$

Projecting a projection

$$P = \frac{1}{52}\begin{bmatrix} 36 & 0 & -24 & 0 & 0 \\ 0 & 13 & 0 & 13 & 13 \\ -24 & 0 & 16 & 0 & 0 \\ 0 & 13 & 0 & 13 & 13 \\ 0 & 13 & 0 & 13 & 13 \end{bmatrix}.$$

Choose the first index as in the rank one case. How to choose the second index?

Use Schur complement:

$$D = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}, \quad D/A := C - B^*A^{-1}B.$$

To choose the second index, compute $P/(P_{ii})$ and <u>then</u> choose the largest diagonal entry.

Projecting a projection

Notation: For an $n \times n$ matrix A, let $A[\alpha, \beta]$ denote the submatrix of A with rows indexed by α and columns indexed by β. Let $A[\alpha] = A[\alpha, \alpha]$. If $A[\alpha]$ is nonsingular, the Schur complement of $A[\alpha]$ in A is

$$A/A[\alpha] = A[\alpha^c] - A[\alpha^c, \alpha]\,(A[\alpha])^{-1}\,A[\alpha, \alpha^c].$$

Projecting a projection

Notation: For an $n \times n$ matrix A, let $A[\alpha, \beta]$ denote the submatrix of A with rows indexed by α and columns indexed by β. Let $A[\alpha] = A[\alpha, \alpha]$. If $A[\alpha]$ is nonsingular, the Schur complement of $A[\alpha]$ in A is

$$A/A[\alpha] = A[\alpha^c] - A[\alpha^c, \alpha]\,(A[\alpha])^{-1}\,A[\alpha, \alpha^c].$$

For a Hermitian matrix A and a positive number κ let

$$\mathcal{B}_\kappa(A) := |\sigma(A) \cap [\kappa, \infty)|.$$

Projecting a projection

Notation: For an $n \times n$ matrix A, let $A[\alpha, \beta]$ denote the submatrix of A with rows indexed by α and columns indexed by β. Let $A[\alpha] = A[\alpha, \alpha]$. If $A[\alpha]$ is nonsingular, the Schur complement of $A[\alpha]$ in A is

$$A/A[\alpha] = A[\alpha^c] - A[\alpha^c, \alpha]\,(A[\alpha])^{-1}\,A[\alpha, \alpha^c].$$

For a Hermitian matrix A and a positive number κ let

$$\mathcal{B}_\kappa(A) := |\sigma(A) \cap [\kappa, \infty)|.$$

Theorem

Let A be an $N \times N$ positive definite matrix, and suppose that $\mathcal{B}_\kappa(A) = k$ for some $\kappa > 0$. Then there exists an index subset $\alpha_k = \{i_1, i_2, \ldots, i_k\}$ of $\{1, \ldots, N\}$ such that $A[\alpha_k] \geq \frac{\kappa}{k! 2^k N}$.

Counting eigenvalues in the interval

Let

$$\mathcal{C}_\epsilon(\mathrm{B}) := |\sigma(\mathrm{B}) \cap (-\epsilon, \epsilon)|$$

for the Hermitian matrix B. Then

$$\mathcal{C}_\epsilon(\mathrm{B}) \neq 0 \quad \Longleftrightarrow \quad \left\|\mathrm{B}^{-1}\right\|^{-1} \leq \epsilon.$$

Counting eigenvalues in the interval

Let

$$\mathcal{C}_\epsilon(\mathrm{B}) := |\sigma(\mathrm{B}) \cap (-\epsilon, \epsilon)|$$

for the Hermitian matrix B. Then

$$\mathcal{C}_\epsilon(\mathrm{B}) \neq 0 \quad \Longleftrightarrow \quad \left\|\mathrm{B}^{-1}\right\|^{-1} \leq \epsilon.$$

How to detect $\mathcal{C}_\epsilon(\mathrm{B})$ using the resolvent B^{-1}?

Counting eigenvalues in the interval

Let

$$\mathcal{C}_\epsilon(\mathrm{B}) := |\sigma(\mathrm{B}) \cap (-\epsilon, \epsilon)|$$

for the Hermitian matrix B. Then

$$\mathcal{C}_\epsilon(\mathrm{B}) \neq 0 \quad \Longleftrightarrow \quad \left\|\mathrm{B}^{-1}\right\|^{-1} \leq \epsilon.$$

How to detect $\mathcal{C}_\epsilon(\mathrm{B})$ using the resolvent B^{-1}?

Let's use the above result!

Counting eigenvalues in the interval

Let

$$\mathcal{C}_\epsilon(\mathrm{B}) := |\sigma(\mathrm{B}) \cap (-\epsilon, \epsilon)|$$

for the Hermitian matrix B. Then

$$\mathcal{C}_\epsilon(\mathrm{B}) \neq 0 \quad \Longleftrightarrow \quad \left\|\mathrm{B}^{-1}\right\|^{-1} \leq \epsilon.$$

How to detect $\mathcal{C}_\epsilon(\mathrm{B})$ using the resolvent B^{-1}?

Let's use the above result!

Remark: $\left(\mathrm{H}^{-1}[\beta]\right)^{-1} = \mathrm{H}/\mathrm{H}[\beta^c]$.

Counting eigenvalues in the interval

Theorem

Let H be the Hermitian $N \times N$ matrix. Consider the following two assertions:

(I) $\mathcal{C}_\epsilon(H) \geq m$;

(II) There exists an index subset α_{2m} such that

$$\mathcal{C}_{K\epsilon}\left(H/H[\alpha_{2m}^c]\right) \geq m.$$

Then (I) implies (II) with $K = C_m N$, $C_m = 2^{2m} m!$.

Conversely, (II) with $K = 1$ implies (I).

Matrix valued random models (with Daniel Schmidt)

Let $H = D_\omega + J$ be $kN \times kN$ Hermitian matrix with

$$D_\omega = \begin{bmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & A_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_N \end{bmatrix}.$$

Each A_i is $k \times k$ random matrix, and J is independent of the randomness in $\{A_i\}$.

Matrix valued random models (with Daniel Schmidt)

Let $H = D_\omega + J$ be $kN \times kN$ Hermitian matrix with

$$D_\omega = \begin{bmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & A_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_N \end{bmatrix}.$$

Each A_i is $k \times k$ random matrix, and J is independent of the randomness in $\{A_i\}$.

Examples:

Anderson tight-binding model: H_A acts on $\ell^2(\mathbb{Z}^d)$ by

$$(H_A\psi)(n) = \sum_{m\sim n} \psi(m) + g\, v(n)\psi(n).$$

Entries $v(n)$ of the potential are i.i.d. random variables.

Matrix valued random models

Anderson model with correlations: H_{A_c} acts on $\ell^2(\mathbb{Z}^d)$ by

$$(H_{A_c}\psi)(n) = \sum_{m\sim n} \psi(m) + g\, v(n)\psi(n),$$

with $v(n) = \sum_{m\in\Gamma} u(n-m)w(m)$ and $v(m)$ are i.i.d. random variables. Γ is a sublattice of $\mathbb{Z}^d$ and u is such that for all $n \in \mathbb{Z}^d$ the function $v(n)$ depends exactly on one random variable w.

Matrix valued random models

Anderson model with correlations: H_{A_c} acts on $\ell^2(\mathbb{Z}^d)$ by

$$(H_{A_c}\psi)(n) = \sum_{m\sim n} \psi(m) + g\, v(n)\psi(n),$$

with $v(n) = \sum_{m\in\Gamma} u(n-m)w(m)$ and $v(m)$ are i.i.d. random variables. Γ is a sublattice of $\mathbb{Z}^d$ and u is such that for all $n \in \mathbb{Z}^d$ the function $v(n)$ depends exactly on one random variable w.

Wegner k-orbital model: H_W acts on $\ell^2(\mathbb{Z}^d \otimes \mathbb{C}^k)$ (the space of square-summable functions $\psi : \mathbb{Z}^d \to \mathbb{C}^k$) by

$$(H_W\psi)(n) = \sum_{m\sim n} \psi(m) + g\, V(n)\psi(n).$$

$\{V(n)\}$ are $k \times k$ i.i.d. Wigner matrices with

$$\langle V(n)\rangle = 0, \quad \langle (V^2(n))_{ij}\rangle = 1/k.$$

Matrix valued random models

Random block operators: Bogoliubov-de Gennes Eq.

$$\mathbb{H} = \begin{bmatrix} \mathrm{H} & \mathrm{B} \\ \mathrm{B} & -\mathrm{H} \end{bmatrix}$$

The disordered s-wave superconductors are often described by an effective random multiplication operator B, while $\mathrm{H_A}$ is one possible effective description for H. After a suitable change of the coordinate basis, the BdG. model can be described as above with $\mathrm{A_i} = \sigma_{\mathrm{i}}$, where

$$\sigma_{\mathrm{i}} = \begin{bmatrix} \mathrm{u_i} & \mathrm{v_i} \\ \mathrm{v_i} & -\mathrm{u_i} \end{bmatrix},$$

and $\mathrm{u_i}, \mathrm{v_i}$ variables are i.i.d. random variables (in i index).

Spectral statistics

For H_A we have

Theorem (Combes, Germinet & Klein, 2009)

Assume

$$\mathbb{P}\left(|v(n)+j| \leq \epsilon\right) \leq K\epsilon^{\alpha} \qquad (A)$$

for all $j \in \mathbb{R}$ and any $\epsilon \in [0,1]$. Then

$$\mathbb{P}\left(\mathcal{C}_{\epsilon}(H_A^{(\Lambda)} - E) \geq m\right) \leq C_m (N\epsilon/g)^{m\alpha} \qquad (1)$$

for $|\Lambda| = N$, any $E \in \mathbb{R}$ and all $m \in \mathbb{N}$.

Spectral statistics

For H_A we have

Theorem (Combes, Germinet & Klein, 2009)

Assume

$$\mathbb{P}\left(|v(n)+j| \leq \epsilon\right) \leq K\epsilon^{\alpha} \qquad \text{(A)}$$

for all $j \in \mathbb{R}$ and any $\epsilon \in [0,1]$. Then

$$\mathbb{P}\left(\mathcal{C}_{\epsilon}(H_A^{(\Lambda)} - E) \geq m\right) \leq C_m (N\epsilon/g)^{m\alpha} \qquad (1)$$

for $|\Lambda| = N$, any $E \in \mathbb{R}$ and all $m \in \mathbb{N}$.

First result of this type for $m = 1$ case was argued by Wegner (1981) and for $m = 2$ by Minami (1996), so we will refer to (1) as the m-level Wegner estimate or the generalized Minami estimate.

Spectral statistics

For H_A we have

Theorem (Combes, Germinet & Klein, 2009)

Assume

$$\mathbb{P}\left(|v(n) + j| \le \epsilon\right) \le K\epsilon^{\alpha} \qquad \text{(A)}$$

for all $j \in \mathbb{R}$ and any $\epsilon \in [0, 1]$. Then

$$\mathbb{P}\left(\mathcal{C}_{\epsilon}(H_A^{(\Lambda)} - E) \ge m\right) \le C_m (N\epsilon/g)^{m\alpha} \qquad (1)$$

for $|\Lambda| = N$, any $E \in \mathbb{R}$ and all $m \in \mathbb{N}$.

First result of this type for $m = 1$ case was argued by Wegner (1981) and for $m = 2$ by Minami (1996), so we will refer to (1) as the m-level Wegner estimate or the generalized Minami estimate.

Graf and Vaghi (2007), Bellissard, Hislop, and Stolz (2007).

Spectral statistics

Can something like that be done for more general matrix valued random models?

Spectral statistics

Can something like that be done for more general matrix valued random models?

Easy to construct examples where the answer will be no for at least some choice of the background operator J.

Spectral statistics

Can something like that be done for more general matrix valued random models?

Easy to construct examples where the answer will be no for at least some choice of the background operator J.

Let's formulate the question differently: How much randomness in A_i is needed to have m-level Wegner estimate, or in other words, what kind of condition can replace (A)?

Spectral statistics

Can something like that be done for more general matrix valued random models?

Easy to construct examples where the answer will be no for at least some choice of the background operator J.

Let's formulate the question differently: How much randomness in A_i is needed to have m-level Wegner estimate, or in other words, what kind of condition can replace (A)?

(B) For an integer n, let S be a given set of $2nk$ distinct integers. There exists an $\alpha > 0$ such that, for any integer $a \in S$, any $\epsilon \in [0, 1]$ and arbitrary Hermitian $k \times k$ matrix J the bound

$$\mathbb{P}\left(\left|\det\left((A_i - a)^{-1} + (J + a)^{-1}\right)\right| \le \epsilon\right) \le K\epsilon^{\alpha}$$

holds.

Condition (B)

If $\mathcal{C}_\epsilon(\mathrm{A}+\mathrm{J})=\mathrm{m}$ then $\det(\mathrm{A}+\mathrm{J})=\mathrm{O}(\epsilon^{\mathrm{m}})$. Not quite right: J can have large eigenvalues (suppose that A does not). How to remove this obstacle?

Condition (B)

If $\mathcal{C}_\epsilon(A+J)=m$ then $\det(A+J)=O(\epsilon^m)$. Not quite right: J can have large eigenvalues (suppose that A does not). How to remove this obstacle?

Lemma

Let A, J be Hermitian $n \times n$ matrices that satisfy $\|A\| \leq 1$. Then assuming $\epsilon < \epsilon_0$ for some ϵ_0 which only depends on n, there exists an integer $a \in [-n-2, -2] \cup [2, n+2]$ (which depends on J but not on A) so that

$$\max\left(\left\|(A-a)^{-1}\right\|, \left\|(J+a)^{-1}\right\|\right) \leq 1\,;$$

$$\mathcal{C}_{\epsilon/9n^2}\left(\hat{D}\right) \leq \mathcal{C}_\epsilon(D) \leq \mathcal{C}_{9n^2\epsilon}\left(\hat{D}\right)\,,$$

where $D = A + J$ and $\hat{D} = (A-a)^{-1} + (J+a)^{-1}$.

Spectral statistics

Theorem

Assume (B), then

$$\mathbb{P}\left(\mathcal{C}_\epsilon(\mathrm{H_g}-\mathrm{E})\geq \mathrm{m}\right)\leq \mathrm{C}\left(-\ln(\mathrm{N}(\epsilon/\mathrm{g})^\alpha)\mathrm{N}(\epsilon/\mathrm{g})^\alpha\right)^{\mathrm{m}}$$

for any $\mathrm{E}\in\mathbb{R}$, for any $\epsilon\in[0,\min(2^{-\mathrm{k}},\mathrm{N}^{-1/\alpha})]$ and for all $\mathrm{m}\leq\mathrm{n}$. Here the constant C depends on k,α and m but not on N or ϵ. In the $\mathrm{m}=1$ case we can improve the above bound to

$$\mathbb{P}\left(\mathcal{C}_\epsilon(\mathrm{H_g}-\mathrm{E})\geq 1\right)\leq \mathrm{CN}(\epsilon/\mathrm{g})^\alpha,$$

provided $\mathrm{N}\leq\epsilon^{-\alpha/4}$.

Spectral statistics

Theorem

Assume (B), then

$$\mathbb{P}\left(\mathcal{C}_\epsilon(\mathrm{H_g} - \mathrm{E}) \geq \mathrm{m}\right) \leq \mathrm{C}\left(-\ln(\mathrm{N}(\epsilon/\mathrm{g})^\alpha)\mathrm{N}(\epsilon/\mathrm{g})^\alpha\right)^\mathrm{m}$$

for any $\mathrm{E} \in \mathbb{R}$, for any $\epsilon \in [0, \min(2^{-\mathrm{k}}, \mathrm{N}^{-1/\alpha})]$ and for all $\mathrm{m} \leq \mathrm{n}$. Here the constant C depends on k, α and m but not on N or ϵ. In the $\mathrm{m} = 1$ case we can improve the above bound to

$$\mathbb{P}\left(\mathcal{C}_\epsilon(\mathrm{H_g} - \mathrm{E}) \geq 1\right) \leq \mathrm{CN}(\epsilon/\mathrm{g})^\alpha,$$

provided $\mathrm{N} \leq \epsilon^{-\alpha/4}$.

Works (i.e. (B) can be verified) for random block operators and the Wegner k-orbital model (enough randomness).

Spectral statistics

Correlated Anderson model: $A_i = v_i A$ where A is Hermitian.

For the sign definite A, one can verify (B) but it is too weak to give the meaningful Minami estimate.

For sign indefinite A fails altogether.

Spectral statistics

Correlated Anderson model: $A_i = v_i A$ where A is Hermitian.

For the sign definite A, one can verify (B) but it is too weak to give the meaningful Minami estimate.

For sign indefinite A fails altogether.

To establish 1-level Wegner estimate (and localization) for the sign indefinite A one has to use the structure of the background operator J.

Matrix valued Schrödinger operators (with Mira Shamis and Sasha Sodin)

- Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with degree at most κ.

Matrix valued Schrödinger operators (with Mira Shamis and Sasha Sodin)

- Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with degree at most κ.
- Let $\mathrm{v} : \Omega \times \mathcal{V} \longrightarrow \mathbb{R}$ be a i.i.d random variables with the distribution μ of every $\mathrm{v}(\mathrm{x})$.

Matrix valued Schrödinger operators (with Mira Shamis and Sasha Sodin)

- Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with degree at most κ.
- Let $\mathrm{v} : \Omega \times \mathcal{V} \longrightarrow \mathbb{R}$ be a i.i.d random variables with the distribution μ of every $\mathrm{v(x)}$.
- Assumptions on μ:

Matrix valued Schrödinger operators (with Mira Shamis and Sasha Sodin)

- Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with degree at most κ.
- Let $\mathrm{v} : \Omega \times \mathcal{V} \longrightarrow \mathbb{R}$ be a i.i.d random variables with the distribution μ of every $\mathrm{v}(\mathrm{x})$.
- Assumptions on μ:

A1 is α-regular for some $\alpha > 0$, meaning that $\mu[\mathrm{t} - \epsilon, \mathrm{t} + \epsilon] \le \mathrm{C}_{\mathrm{A1}} \epsilon^{\alpha}$ for any $\epsilon > 0$ and $\mathrm{t} \in \mathbb{R}$;

Matrix valued Schrödinger operators (with Mira Shamis and Sasha Sodin)

- Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with degree at most κ.
- Let $v : \Omega \times \mathcal{V} \longrightarrow \mathbb{R}$ be a i.i.d random variables with the distribution μ of every $v(x)$.
- Assumptions on μ:

A1 is α-regular for some $\alpha > 0$, meaning that $\mu[t - \epsilon, t + \epsilon] \leq C_{A1}\epsilon^{\alpha}$ for any $\epsilon > 0$ and $t \in \mathbb{R}$;

A2 has a finite q-moment for some $q > 0$, meaning that $\int |x|^q d\mu(x) \leq C_{A2}$.

Matrix valued Schrödinger operators (with Mira Shamis and Sasha Sodin)

- Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with degree at most κ.
- Let $\mathrm{v} : \Omega \times \mathcal{V} \longrightarrow \mathbb{R}$ be a i.i.d random variables with the distribution μ of every $\mathrm{v}(\mathrm{x})$.
- Assumptions on μ:

A1 is α-regular for some $\alpha > 0$, meaning that $\mu[\mathrm{t} - \epsilon, \mathrm{t} + \epsilon] \leq \mathrm{C}_{\mathrm{A1}} \epsilon^{\alpha}$ for any $\epsilon > 0$ and $\mathrm{t} \in \mathbb{R}$;

A2 has a finite q-moment for some $\mathrm{q} > 0$, meaning that $\int |\mathrm{x}|^{\mathrm{q}} \mathrm{d}\mu(\mathrm{x}) \leq \mathrm{C}_{\mathrm{A2}}$.

- Gaussian distribution and the uniform distribution on a finite interval satisfy A1 with $\alpha = 1$ and A2 with any $\mathrm{q} > 0$.

Matrix valued Schrödinger operators (with Mira Shamis and Sasha Sodin)

- Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with degree at most κ.
- Let $\mathrm{v} : \Omega \times \mathcal{V} \longrightarrow \mathbb{R}$ be a i.i.d random variables with the distribution μ of every $\mathrm{v}(\mathrm{x})$.
- Assumptions on μ:

A1 is α-regular for some $\alpha > 0$, meaning that $\mu[\mathrm{t} - \epsilon, \mathrm{t} + \epsilon] \leq \mathrm{C}_{\mathrm{A1}}\epsilon^{\alpha}$ for any $\epsilon > 0$ and $\mathrm{t} \in \mathbb{R}$;

A2 has a finite q-moment for some $\mathrm{q} > 0$, meaning that $\int |\mathrm{x}|^{\mathrm{q}} \mathrm{d}\mu(\mathrm{x}) \leq \mathrm{C}_{\mathrm{A2}}$.

- Gaussian distribution and the uniform distribution on a finite interval satisfy A1 with $\alpha = 1$ and A2 with any $\mathrm{q} > 0$.
- Expectation: $\langle \cdot \rangle$.

Hamiltonian

- Single site (matrix) potential: For any $x \in \mathcal{V}$, let $V(x) = v(x)A(x) + B(x)$ where $A(x), B(x) \in M_{k,k}(\mathbb{C})$ are hermitian and satisfy

  B1 $\|A(x)\|, \|A(x)^{-1}\| \leq C_{B1}$;
  B2 $\|B(x)\| \leq C_{B2}$.

Hamiltonian

- Single site (matrix) potential: For any $x \in \mathcal{V}$, let $V(x) = v(x)A(x) + B(x)$ where $A(x), B(x) \in M_{k,k}(\mathbb{C})$ are hermitian and satisfy
  - B1 $\|A(x)\|, \|A(x)^{-1}\| \leq C_{B1}$;
  - B2 $\|B(x)\| \leq C_{B2}$.

- Hopping: For every ordered pair $(x, y) \in \mathcal{V} \times \mathcal{V}$ of adjacent sites (i.e. $(x, y) \in \mathcal{E}$) we introduce $K(x, y) \in M_{k,k}(\mathbb{C})$ so that
  - B3 $K(y, x) = K(x, y)^*$ and $\|K(x, y)\| \leq C_{B3}$.

Hamiltonian

- Single site (matrix) potential: For any $x \in \mathcal{V}$, let $V(x) = v(x)A(x) + B(x)$ where $A(x), B(x) \in M_{k,k}(\mathbb{C})$ are hermitian and satisfy
  - B1 $\|A(x)\|, \|A(x)^{-1}\| \leq C_{B1}$;
  - B2 $\|B(x)\| \leq C_{B2}$.
- Hopping: For every ordered pair $(x, y) \in \mathcal{V} \times \mathcal{V}$ of adjacent sites (i.e. $(x, y) \in \mathcal{E}$) we introduce $K(x, y) \in M_{k,k}(\mathbb{C})$ so that
  - B3 $K(y, x) = K(x, y)^*$ and $\|K(x, y)\| \leq C_{B3}$.
- Let H act on $\ell^2(\mathcal{V}) \otimes \mathbb{C}^k$ as

$$(H\psi)(x) = V(x)\psi(x) + g^{-1} \sum_{y \sim x} K(x, y)\psi(y)$$

Hamiltonian

- Single site (matrix) potential: For any $x \in \mathcal{V}$, let $V(x) = v(x)A(x) + B(x)$ where $A(x), B(x) \in M_{k,k}(\mathbb{C})$ are hermitian and satisfy

  B1 $\|A(x)\|, \|A(x)^{-1}\| \leq C_{B1}$;
  B2 $\|B(x)\| \leq C_{B2}$.

- Hopping: For every ordered pair $(x, y) \in \mathcal{V} \times \mathcal{V}$ of adjacent sites (i.e. $(x, y) \in \mathcal{E}$) we introduce $K(x, y) \in M_{k,k}(\mathbb{C})$ so that

  B3 $K(y, x) = K(x, y)^*$ and $\|K(x, y)\| \leq C_{B3}$.

- Let H act on $\ell^2(\mathcal{V}) \otimes \mathbb{C}^k$ as

$$(H\psi)(x) = V(x)\psi(x) + g^{-1} \sum_{y \sim x} K(x, y)\psi(y)$$

- $g > 0$ is a coupling constant.

Exponential decay

- Let $G_z = (H - z)^{-1}$ be the resolvent of H, $z \notin \mathbb{R}$.

Exponential decay

- Let $G_z = (H - z)^{-1}$ be the resolvent of H, $z \notin \mathbb{R}$.

Theorem (Exponential decay)

Let $0 < s \leq \frac{\alpha q}{2k\alpha + kq}$. There exists $C = C(\alpha, q, C_{A1}–C_{B3}, s) > 0$ such that for any $z \in \mathbb{R}$ and any $g \geq C\kappa^{1/s}/(1 + |z|)$

$$\langle \|G_{z+i0}(x, y)\|^s \rangle \leq \frac{C}{(1 + |z|)^s} \left(\frac{C\kappa}{g^s(1 + |z|)^s} \right)^{\mathrm{dist}(x,y)} .$$

Exponential decay

- Let $G_z = (H - z)^{-1}$ be the resolvent of H, $z \notin \mathbb{R}$.

Theorem (Exponential decay)

Let $0 < s \leq \frac{\alpha q}{2k\alpha + kq}$. There exists $C = C(\alpha, q, C_{A1}–C_{B3}, s) > 0$ such that for any $z \in \mathbb{R}$ and any $g \geq C\kappa^{1/s}/(1 + |z|)$

$$\langle \|G_{z+i0}(x, y)\|^s \rangle \leq \frac{C}{(1 + |z|)^s} \left(\frac{C\kappa}{g^s(1 + |z|)^s} \right)^{\mathrm{dist}(x,y)} .$$

- Corollary for the homogeneous setting: assume that $\mathcal{G} = \mathbb{Z}^d$ and that

  C $A(m) \equiv A$, $B(m) \equiv B$, $K(m, n) \equiv K(m - n)$.

Localization

Theorem (Localization)

Assume C. Let I be a finite interval of energies, and let

$$\mathrm{g} \geq \frac{\mathrm{C}\mathrm{d}^{1/\mathrm{s}}}{1+\min_{\mathrm{z}\in\mathrm{I}}|\mathrm{z}|}\ .$$

Then, for any $\mathrm{m} \neq \mathrm{n} \in \mathbb{Z}^{\mathrm{d}}$,

$$\langle \sup_{\mathrm{t}\geq 0} \left|\mathrm{e}^{\mathrm{i}\mathrm{t}\mathrm{H}_{\mathrm{I}}}(\mathrm{m},\mathrm{n})\right|\rangle \leq \mathrm{C}\mathrm{dist}(\mathrm{m},\mathrm{n})^{2\mathrm{d}}\left(\frac{\mathrm{C}\mathrm{d}}{\mathrm{g}^{\mathrm{s}}(1+|\mathrm{z}|)^{\mathrm{s}}}\right)^{\frac{\mathrm{s}\,\mathrm{dist}_{(\mathrm{m},\mathrm{n})}}{8}},$$

where $\mathrm{H}_{\mathrm{I}} = \mathrm{P}_{\mathrm{I}}\mathrm{H}\mathrm{P}_{\mathrm{I}}$, P_{I} is the spectral projector corresponding to I. Therefore the spectrum of H in I is almost surely pure point.

Trimmed Anderson models (with Abel Klein)

A Γ-trimmed Anderson model is a discrete random Schrödinger operator on on $\ell^2(\mathbb{Z}^d)$ of the form

$$H_T := H_0 + gV_\omega.$$

Here $H_0 = -\Delta + V^{(0)}$, with $V^{(0)}$ a bounded (background) potential, and V_ω is the random potential given by

$$V_\omega = \sum_{\zeta\in\Gamma} \omega_\zeta \chi_\zeta,$$

where Γ is a subset of $\mathbb{Z}^d$ and $\{\omega_\zeta\}_{\zeta\in\Gamma}$ is a family of independent random variables.

We will consider relatively dense subsets Γ of $\mathbb{Z}^d$. Namely, let $\Lambda_L(x) = \{y \in \mathbb{Z}^d : \ |y - x|_\infty < L/2\}$.

Trimmed Anderson models

A set $\Gamma \subset \mathbb{Z}^d$ is (K, Q)-relatively dense, where $K, Q \in \mathbb{N}$, if

$$|\Gamma \cap \Lambda_K(\zeta)| \geq Q \text{ for all } \zeta \in K\mathbb{Z}^d.$$

The Γ-trimming of H is the restriction H_Γ of $\chi_{\Gamma^c} H \chi_{\Gamma^c}$ to $\ell^2(\Gamma^c)$.

We consider $E_\Gamma(H) = \inf \sigma(H_\Gamma)$, the ground state energy of the trimmed operator H_Γ. (Note that $H = H_\emptyset$ and $E_\emptyset(H) = \inf \sigma(H)$.) Trimming lifts the bottom of the spectrum: $E_\Gamma(H) \geq E_\emptyset(H)$. Let $\delta_\Gamma(H) = E_\Gamma(H) - E_\emptyset(H)$.

Trimmed Anderson model

Theorem

Let $\Gamma \subsetneq \mathbb{Z}^d$ be (K, Q)-relatively dense, and let $H = -\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where V is a bounded potential. Then

$$\delta_\Gamma(H) \geq \frac{Q}{(2dK-1)Y_{d,V}^{2dK-1}} > 0,$$

where $Y_{d,V} = 2d + 1 + \sup_{x\in\mathbb{Z}^d} V(x) - \inf_{x\in\mathbb{Z}^d} V(x)$.
In the special case $H = -\Delta$ we can improve the previous bound to

$$\delta_\Gamma(-\Delta) = E_\Gamma(-\Delta) \geq \frac{1}{4d(K+1)^{2d}}.$$

Happy birthday, Yosi!

The outline of the proof

Key proposition

For any $s \le \frac{\alpha q}{2k\alpha + kq}$ there exists $C > 0$ (depending on s and the constants in the assumptions) such that for any $z \notin \mathbb{R}$

$$\langle \|G_z(x,y)\|^s \rangle \le \frac{C}{2(1+|z|)^s} \left\{ g^{-s} \sum_{z \sim y} \langle \|G_z(x,z)\|^s \rangle + \delta_{xy} \right\} ,$$

where

$$\delta_{xy} = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$$

is the Kronecker δ.

Maximum is attained on the diagonal

Corollary: Maximum is attained on the diagonal

For any $s \leq \frac{\alpha q}{2k\alpha + kq}$, we have

$$\max_{y} \langle \|G_z(x, y)\|^s \rangle = \langle \|G_z(x, x)\|^s \rangle \ ,$$

provided $g^s \geq C\kappa/(1 + |z|)^s$.

Maximum is attained on the diagonal

Corollary: Maximum is attained on the diagonal

For any $s \leq \frac{\alpha q}{2k\alpha + kq}$, we have

$$\max_y \langle \|G_z(x,y)\|^s \rangle = \langle \|G_z(x,x)\|^s \rangle \ ,$$

provided $g^s \geq C\kappa/(1+|z|)^s$.

Proof.

Suppose the maximum M is attained at $y \neq x$. Then

$$M = \langle \|G_z(x,y)\|^s \rangle \leq \frac{C}{2g^s(1+|z|)^s} \sum_{z \sim y} \langle \|G_z(x,z)\|^s \rangle$$

$$\leq \frac{C\kappa M}{2g^s(1+|z|)^s} \leq \frac{CM}{2C} = \frac{M}{2} \ ,$$

a contradiction. □

A-priori bound

Corollary: A-priori bound

For any $s \leq \frac{\alpha q}{2k\alpha + kq}$ and $g^s \geq C\kappa/(1+|z|)^s$

$$\langle \| G_z(x,x) \|^s \rangle \leq \frac{C}{(1+|z|)^s} \; .$$

A-priori bound

Corollary: A-priori bound

For any $s \le \frac{\alpha q}{2k\alpha + kq}$ and $g^s \ge C\kappa/(1+|z|)^s$

$$\langle \|G_z(x,x)\|^s \rangle \le \frac{C}{(1+|z|)^s} \ .$$

Proof.

By the proposition above with $y = x$ and the previous corollary,

$$\begin{aligned} \langle \|G_z(x,x)\|^s \rangle &\le \frac{C}{2(1+|z|)^s} \left\{ g^{-s}\kappa \langle \|G_z(x,x)\|^s \rangle + 1 \right\} \\ &\le \frac{1}{2} \langle \|G_z(x,x)\|^s \rangle + \frac{C}{2(1+|z|)^s} \ , \end{aligned}$$

therefore $\langle \|G_z(x,x)\|^s \rangle \le \frac{C}{(1+|z|)^s}$. □

End game

Proof of Theorem 1.

For $x = y$ the inequality follows from the second corollary. For $x \neq y$ apply the proposition $\mathrm{dist}(x, y)$ times, and then use the two corollaries to estimate every term. □

End game

Proof of Theorem 1.

For $x = y$ the inequality follows from the second corollary. For $x \neq y$ apply the proposition $\mathrm{dist}(x, y)$ times, and then use the two corollaries to estimate every term. □

Happy birthday, Yosi!

Alloy-type models

- Consider the scalar operator H on $\ell^2(\mathbb{Z}^d)$ with potential $V(n)$ at a site $n \in \mathbb{Z}^d$ is obtained from i.i.d. $v(m)$ as

$$V(n) = \sum_{k \in \Gamma} a_{n-k} v(k) \ ,$$

where the index k takes values in some sub-lattice Γ of $\mathbb{Z}^d$.

Alloy-type models

- Consider the scalar operator H on $\ell^2(\mathbb{Z}^d)$ with potential $V(n)$ at a site $n \in \mathbb{Z}^d$ is obtained from i.i.d. $v(m)$ as
$$V(n) = \sum_{k \in \Gamma} a_{n-k} v(k) \ ,$$
where the index k takes values in some sub-lattice Γ of $\mathbb{Z}^d$.
- Let $\mathcal{B}_n$ be the set of $v(m)$ for which $a_{n-m} \neq 0$.

Alloy-type models

- Consider the scalar operator H on $\ell^2(\mathbb{Z}^d)$ with potential $V(n)$ at a site $n \in \mathbb{Z}^d$ is obtained from i.i.d. $v(m)$ as
$$V(n) = \sum_{k \in \Gamma} a_{n-k} v(k) \ ,$$
where the index k takes values in some sub-lattice Γ of $\mathbb{Z}^d$.
- Let $\mathcal{B}_n$ be the set of $v(m)$ for which $a_{n-m} \neq 0$.
- Assumptions:
  1. the set $\mathcal{B}_n$ is non empty for all n;
  2. the cardinality $k = \#\{m \,|\, a_m \neq 0\} < \infty$;
  3. the distribution of $v(m)$ satisfies A1 and A2.

Localization for alloy-type models

Localization for alloy-type models

Let $0 < s < \frac{\alpha q}{2k\alpha + kq}$. There exists $C > 0$ such that for any $z \in \mathbb{R}$ and any $g \geq Cd^{1/s}/(1+|z|)$

$$\langle |G_{z+i0}(m,n)|^s \rangle \leq \frac{C}{(1+|z|)^s} \left(\frac{Cd}{g^s(1+|z|)^s} \right)^{\mathrm{dist}(m,n)} .$$

Similarly to the previous setting, one can deduce the localization in the context of the alloy-type models.

Localization for alloy-type models

Localization for alloy-type models

Let $0 < s < \frac{\alpha q}{2k\alpha + kq}$. There exists $C > 0$ such that for any $z \in \mathbb{R}$ and any $g \geq Cd^{1/s}/(1 + |z|)$

$$\langle |G_{z+i0}(m,n)|^s \rangle \leq \frac{C}{(1+|z|)^s} \left(\frac{Cd}{g^s (1+|z|)^s} \right)^{\mathrm{dist}(m,n)} .$$

Similarly to the previous setting, one can deduce the localization in the context of the alloy-type models.

- Strong disorder regime result. Outside $\sigma(\Delta)$ one can use Klopp's trick to reduce problem to the monotone one.
- For $\Gamma = \mathbb{Z}^d$ the dynamical localization was established by Krüger.