Cumulants of the current and large deviations in the Symmetric Simple Exclusion Process (SSEP) on graphs

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Conference on quantum spectra and transport
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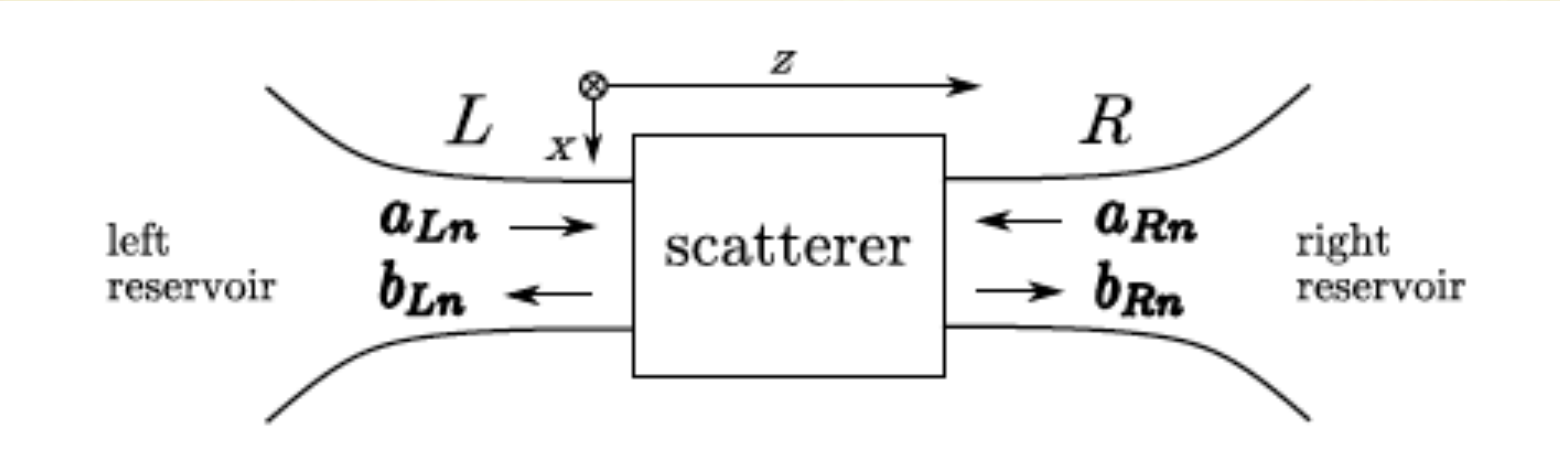
PHYSICAL MOTIVATION OF THIS WORK

Charge fluctuations in quantum mesoscopic conductors

Current that flows in an electric conductor fluctuates due to the stochastic nature of electron emission and transport

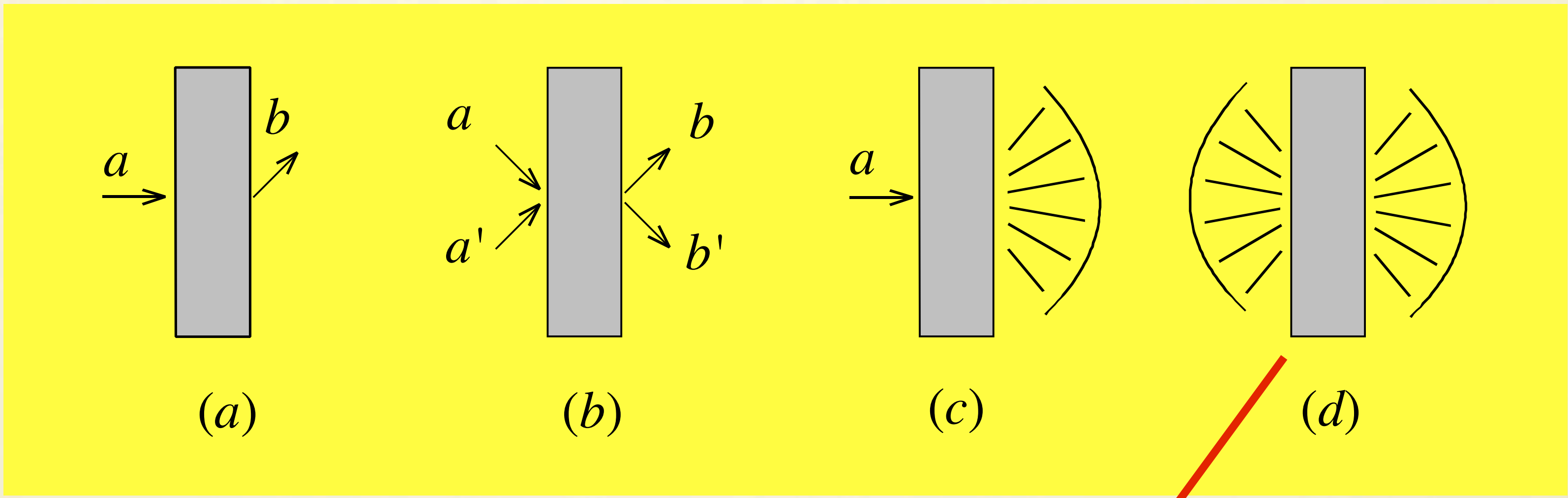


Study of Transport, Noise and Full Counting Statistics allow to characterize basic physical mechanisms at work.

Quantum conductance and shot noise

Two-terminal conductors

$$T_{ab} = |t_{ab}|^2$$



Electric conductance (Landauer)

$$G = \frac{e^2}{h} Tr\, t\, t^{\dagger}$$

Noise power is given by the current-current correlation function

$$S(\omega,V) = \int dt\, e^{i\omega t} \left\langle \delta\hat{I}(t)\delta\hat{I}(0) + \delta\hat{I}(0)\delta\hat{I}(t) \right\rangle$$

where $\delta\hat{I}(t) = \hat{I}(t) - \langle I \rangle$ are electronic current operators

Equilibrium noise (V=0)

$$S(\omega,0) = 2G\omega \coth\left(\frac{\omega}{2T}\right)$$

(Nyquist fluctuation-dissipation)

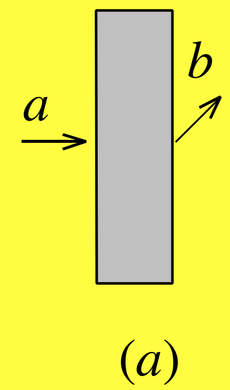
Non-equilibrium noise $V \neq 0$ at $T = 0$

$$S(0,V) - S(0,0) = \frac{e^2}{h} |2eV| Tr\ tt^{\dagger}\left(1 - tt^{\dagger}\right)$$

Excess noise measures the second cumulant of charge fluctuations :

$$S(0,V) - S(0,0) \propto \langle Q_t^2 \rangle - \langle Q_t \rangle^2$$

Fano factor



$$Tr\ tt^{\dagger}\left(1-tt^{\dagger}\right)$$

$$F=\frac{S(0,V)-S(0,0)}{e\,I}=\frac{\sum_{ab}T_{ab}(1-T_{ab})}{\sum_{ab}T_{ab}}$$

$$G=\frac{e^2}{h}Tr\,t\,t^{\dagger}$$

T_{ab} is the transmission coefficient along the channel ab

Fano factor

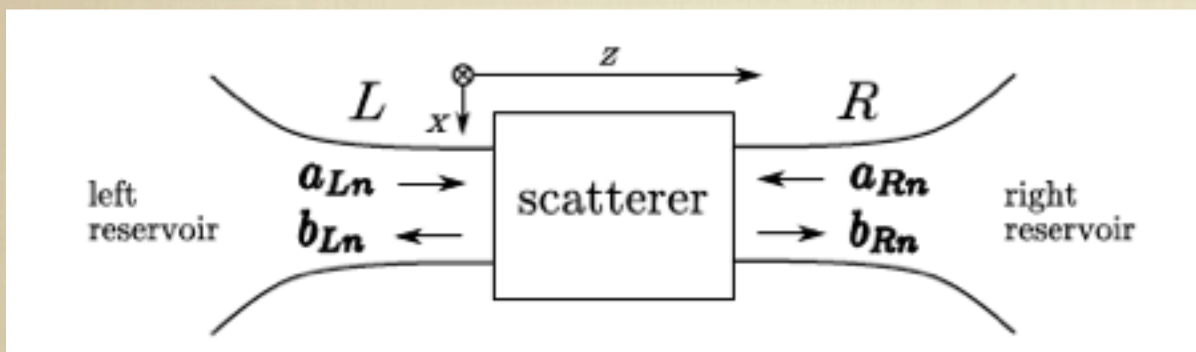
$$F = \frac{S(0,V) - S(0,0)}{e\,I} = \frac{\sum_{ab} T_{ab}(1-T_{ab})}{\sum_{ab} T_{ab}}$$

F has a universal value 1/3 for weakly disordered "one-dimensional" metals

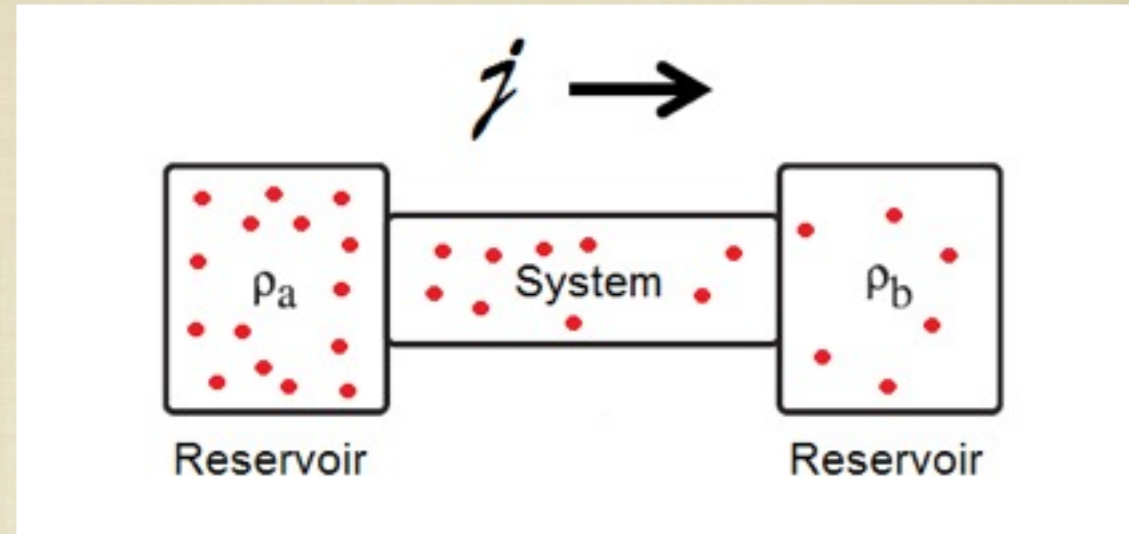
Is this result universal ?

nature of disorder, geometry, space dimensionality, extends to higher order cumulants,...

Classical version of the quantum conductor



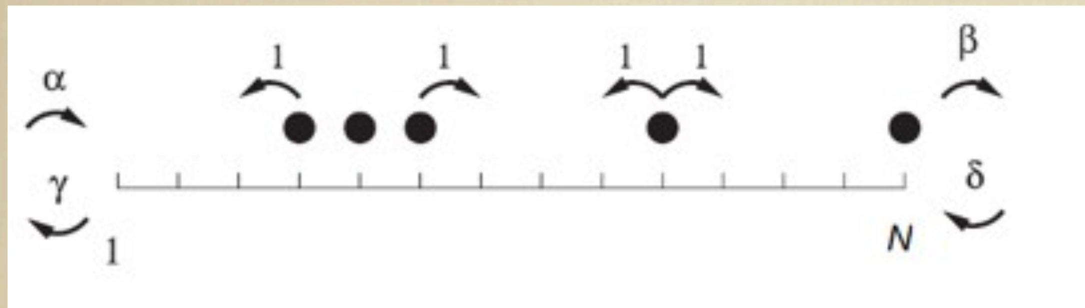
$\Leftrightarrow$



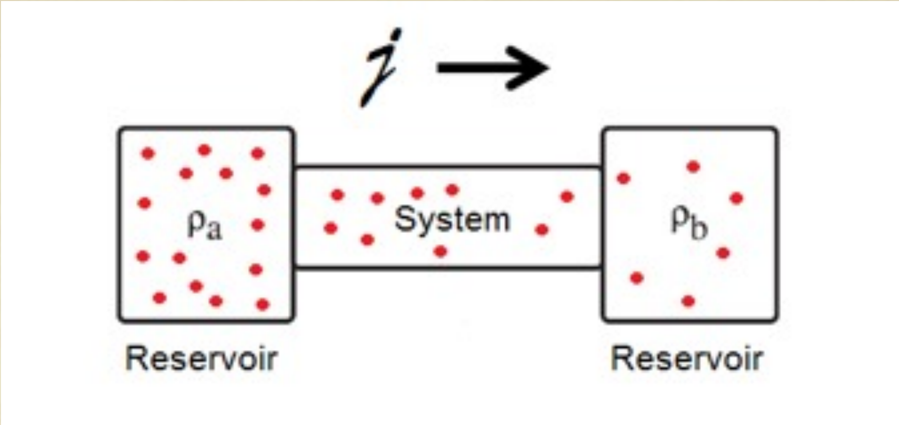
<u>Same physical content :</u> particles cannot pile up on the same site (Pauli principle or quantum crossings in quantum mesoscopic physics)

Defines the classical **Symmetric Simple exclusion process** (SSEP)

The SSEP model



$\Leftrightarrow$



For $N \gg 1$, $\rho_a = \frac{\alpha}{\alpha+\gamma}, \rho_b = \frac{\delta}{\beta+\delta}$

For large enough time, the system is in a **steady state**.

Define the probability $P(Q_t)$ of observing Q_t particles flowing through the system during a time interval t and for 2 reservoirs at densities ρ_a and ρ_b

All the cumulants are known for arbitrary densities ρ_a and ρ_b

The generating function

$$\lim_{N\to\infty}\lim_{t\to\infty}\frac{N}{t}\log\left\langle e^{\lambda Q_t}\right\rangle=\left(\sinh^{-1}\left(\sqrt{\omega}\right)\right)^2$$

depends on a single scaling variable

$$\omega=\rho_a\left(e^{\lambda}-1\right)+\rho_b\left(e^{-\lambda}-1\right)-\rho_a\left(e^{\lambda}-1\right)\rho_b\left(e^{-\lambda}-1\right)$$

And the Fano factor is

$$\lim_{N\to\infty}\lim_{t\to\infty}\frac{\langle Q_t^2\rangle-\langle Q_t\rangle^2}{\langle Q_t\rangle}=\frac{1}{3}$$

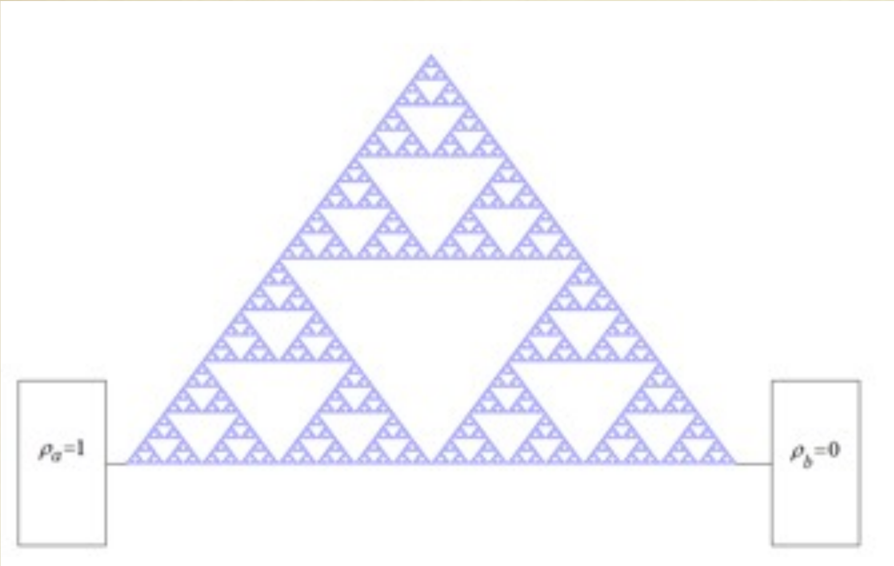
The Fano factor and all other cumulants are identical to those calculated in the quantum mesoscopic case.

How these results generalize to higher space dimensions ?

Numerical results on a Sierpinski gasket fractal network suggests a Fano factor $F = \frac{1}{3}$

(Groth et al. PRL 2008)

How these results generalize to higher space dimensions ?



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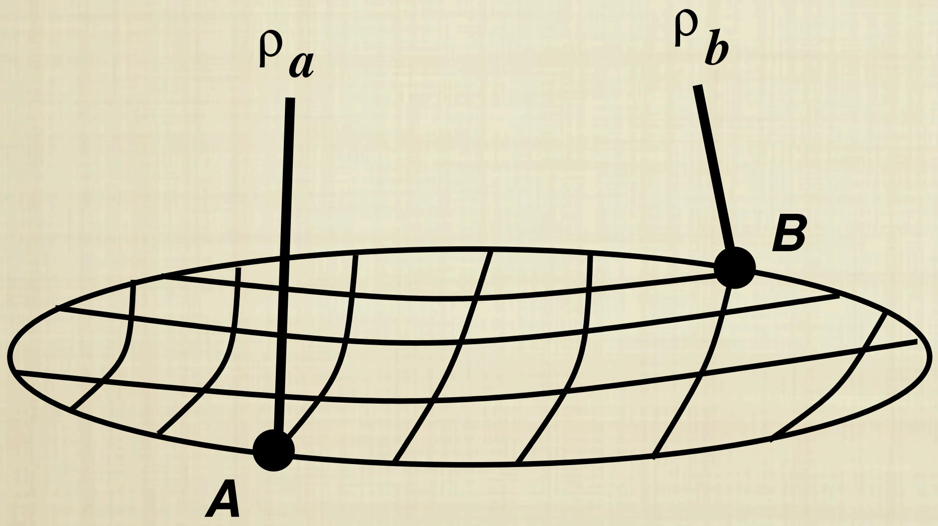
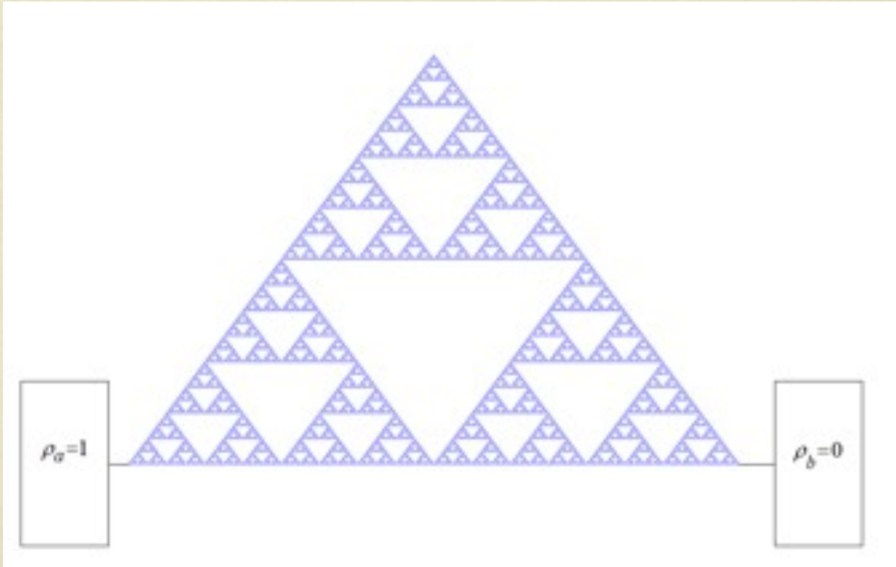
Our Result:

(T. Bodineau, B. Derrida, O. Shpielberg, E.A, 2013)

1. Large class of graphs (including fractals) can be characterized by an effective length L_e





2. For large values of L_e, the generating function of the cumulants of the current of the **SSEP** is the same as for a linear chain, up to a multiplicative function

$$\lim_{t\to\infty}\frac{1}{t}\log\left\langle e^{\lambda Q_t}\right\rangle = \kappa(L_e)\left(\sinh^{-1}\left(\sqrt{\omega}\right)\right)^2$$

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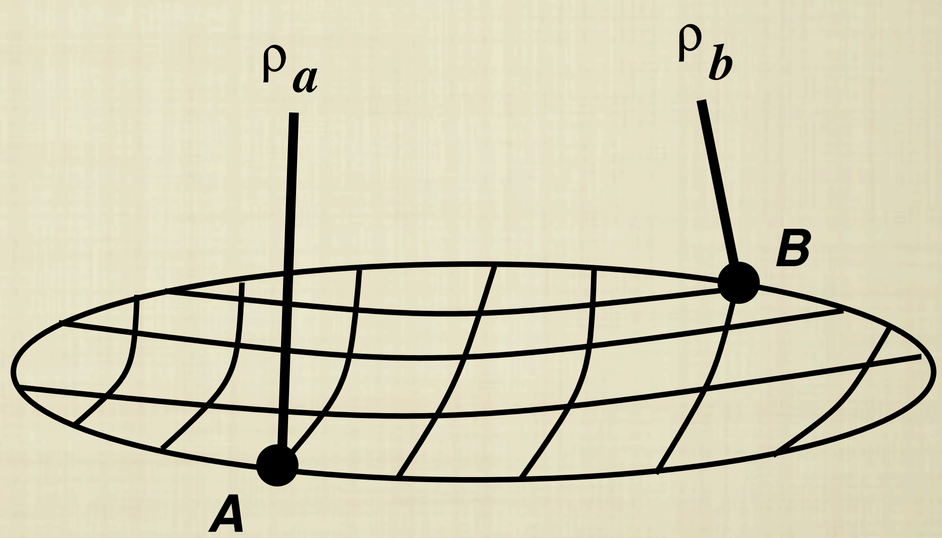
$$\lim_{t\to\infty} \frac{1}{t} \log\left\langle e^{\lambda Q_t} \right\rangle = \kappa(L_e)\left(\sinh^{-1}\left(\sqrt{\omega}\right)\right)^2$$

$$\kappa(L_e) \equiv L_e^{d-2} \int d\vec{r} \left(\vec{\nabla} v(\vec{r})\right)^2$$

$$\Delta v(\vec{r})=0,\ v(\partial A)=1, v(\partial B)=0$$



2. For large values of L_e, the generating function of the cumulants of the current of the **SSEP** is the same as for a linear chain, up to a multiplicative function

$$\lim_{t\to\infty}\frac{1}{t}\log\left\langle e^{\lambda Q_t}\right\rangle = \kappa(L_e)\left(\sinh^{-1}\left(\sqrt{\omega}\right)\right)^2$$

Thus, the ratio between any pair of cumulants of Q_t is the same as for the linear chain. Then,

$$F=\frac{1}{3}$$

ELEMENTS OF THE PROOF

- Use the macroscopic fluctuation theory of Bertini et al. and the additivity principle.
- Alternative description based on Energy/Dirichlet forms: allows to characterize the SSEP and to provide a derivation of the additivity principle.

The macroscopic fluctuation theory

Basic definitions and results

(Bertini, De Sole, Gabrielli, Jona-Lasinio, and Landim)

Q_t = Number of particles flowing through the system during t

$$\left\langle e^{\lambda Q_t} \right\rangle = e^{\mu(\lambda)t} \qquad \text{for large t}$$

The <u>large deviation function</u> F_L is defined from the probability

$$P_L\left(Q_t = jt, \rho_a, \rho_b\right) \equiv e^{t F_L(j, \rho_a, \rho_b)}$$

It is the <u>Legendre transform</u> of $\mu(\lambda)$

$$\mu(\lambda) = \max_j \left(\lambda j + F_L\left(j(\lambda)\right)\right)$$

Scaling - Electrical conductance

The large deviation function is a scaling function :

$$F_L(j) = \frac{1}{L^{2-d}} G\left(jL^{2-d}\right)$$

so that F_L scales like an electrical conductance.

(Bodineau,Derrida,Lebowitz - Thouless - Montambaux, E.A.)

Additivity principle and large deviation function

General diffusive system (e.g. SSEP) s.t., $\rho_a = \rho \quad , \rho_b = \rho + \Delta\rho, \quad \Delta\rho \ll \rho$

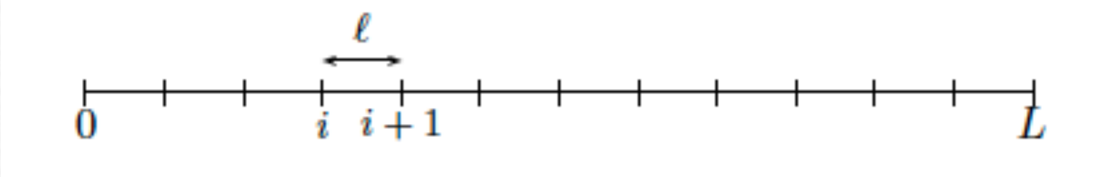
Weak current through the system : use Fick's law $\dfrac{\langle Q_t \rangle}{t} = \dfrac{D(\rho)}{L} \Delta\rho$

+ fluctuations : $\dfrac{\langle Q_t^2 \rangle}{t} = \dfrac{\sigma(\rho)}{L}$

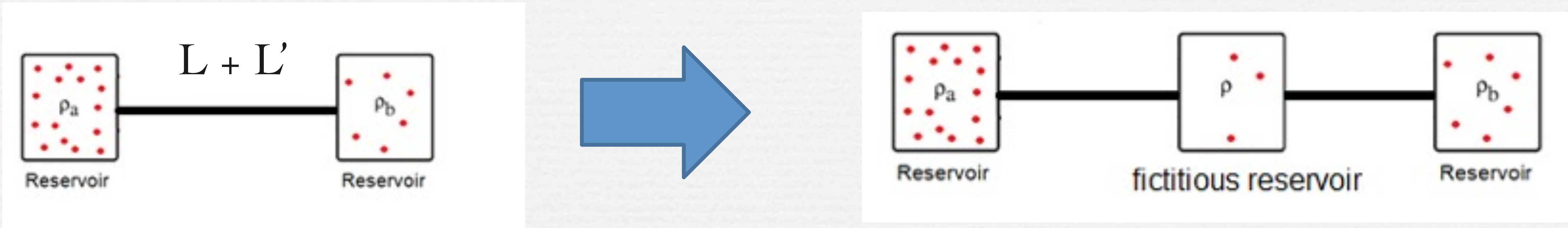
For SSEP, $D(\rho) = 1, \quad \sigma(\rho) = 2\rho(1-\rho)$

$F_L(j)$ has its maximum for $j = \dfrac{\langle Q_t \rangle}{t}$. Close to equilibrium : Gaussian distribution for the probability,

$$F_L(j) = -\frac{\left(j - \langle Q_t \rangle / t\right)^2}{2 \langle Q_t^2 \rangle / t} = -\frac{\left(j - \frac{\rho_i - \rho_{i+1}}{l} D(\rho_i)\right)^2}{2 \sigma(\rho_i) / l}$$

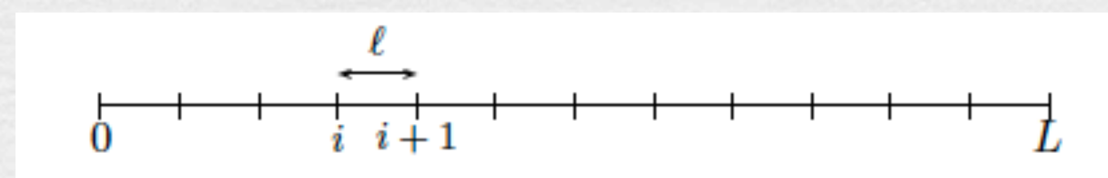


Additivity principle + scaling



$$F_{L+L'}\left(j,\rho_a,\rho_b\right) = \max_{\rho}\left\{F_L\left(j,\rho_a,\rho\right) + F_{L'}\left(j,\rho,\rho_b\right)\right\}$$

$$F_L(j) = -\frac{\left(j - \langle Q_t\rangle / t\right)^2}{2\langle Q_t^2\rangle / t} = -\frac{\left(j - \frac{\rho_i - \rho_{i+1}}{l} D(\rho_i)\right)^2}{2\sigma(\rho_i)/l}$$



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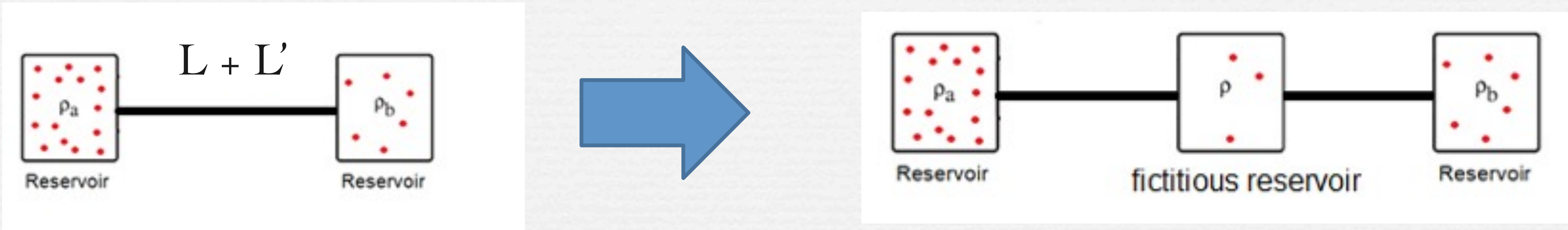


$$F_{L+L'}(j,\rho_a,\rho_b)=\max_{\rho}\{F_L(j,\rho_a,\rho)+F_{L'}(j,\rho,\rho_b)\}$$

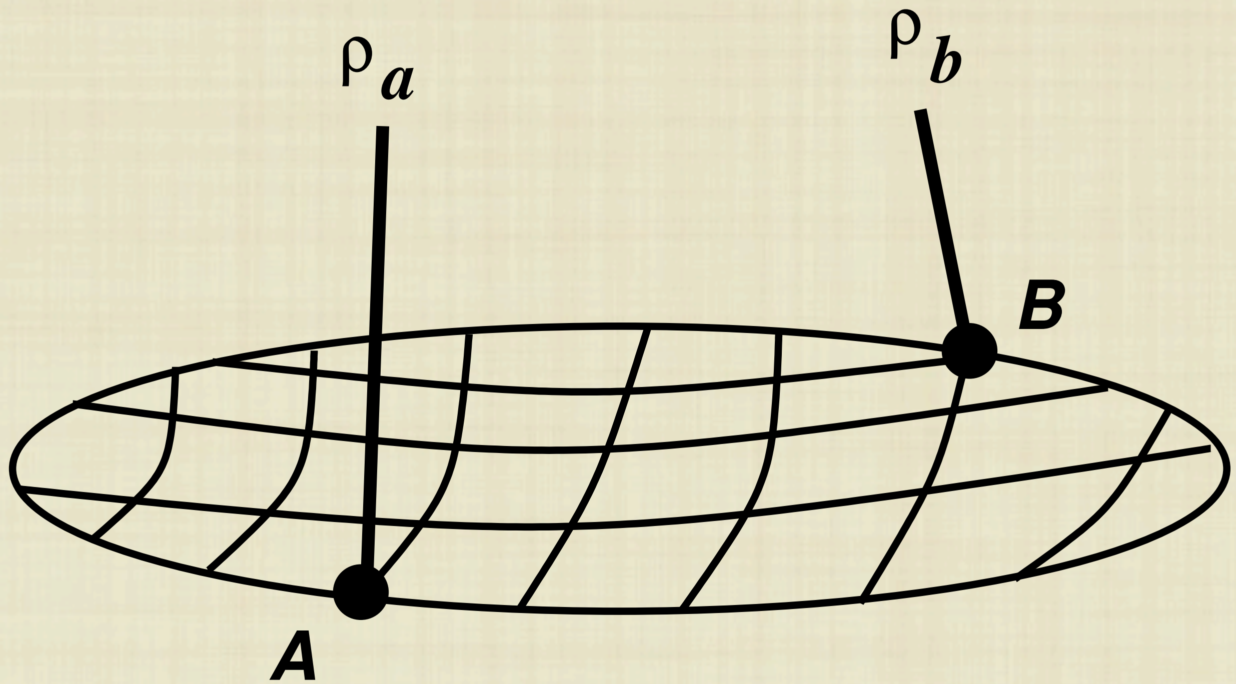
so that,

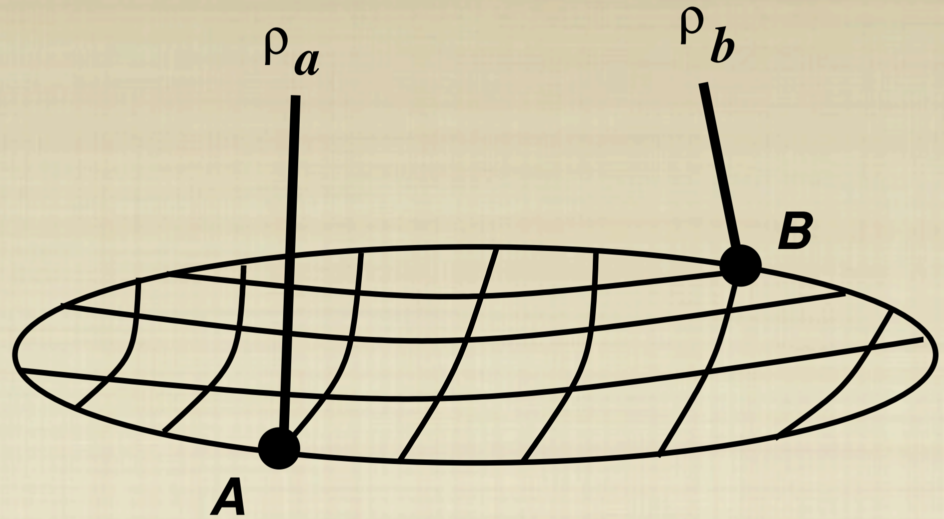
$$F_L(j,\rho_a,\rho_b)=\max_{\rho(x)}\left\{-\int_0^1 dx\,\frac{\left(jL+D(\rho(x))\rho'(x)\right)^2}{2\sigma(\rho(x))}\right\}$$

and

$$\mu(\lambda)=\max_{j}\left(\lambda j+F_L(j(\lambda))\right)$$

Macroscopic fluctuation theory for SSEP on a d-dimensional domain





DEFINE THE NUMBER Q_t OF PARTICLES FLOWING BETWEEN THE 2 RESERVOIRS :

$$Q_t = \frac{1}{2}\sum_{i,j}\left(V_i - V_j\right) q_{i,j}(t)$$

WHERE $q_{i,j}(t)$ IS THE NUMBER OF PARTICLES TRANSFERRED FROM i TO j DURING t AND V_i IS AN ARBITRARY FUNCTION ON SITE i EXCEPT FOR $V_A = 1\,, V_B = 0$

Nothing depends on the choice of the V_i's. We take it a solution of the Laplace eq. $\Delta V_i \equiv \sum_{j\sim i} V_j - V_i = 0$

Continuous version: $$Q_t = -L^d \int_0^{t/L^2} d\tau \int d\vec{r}\, \vec{\nabla} v(\vec{r}) \cdot \vec{j}(\vec{r})$$

where $$\Delta v(\vec{r}) = 0,\ v(\partial A) = 1, v(\partial B) = 0$$

The minimization in the generating function

$$\mu(\lambda) = -L^{d-2} \min_{\{\vec{j},\rho\}} \int d\vec{r} \left(\lambda \vec{\nabla} v(\vec{r}) \cdot \vec{j}(\vec{r}) + \frac{\left[\vec{j}(\vec{r}) + D(\rho(\vec{r}))\, \vec{\nabla}\rho(\vec{r})\right]^2}{2\sigma(\rho(\vec{r}))} \right)$$

leads to

$$\vec{\nabla} \cdot \left(D(\rho(\vec{r}))\, \vec{\nabla}\rho(\vec{r}) \right) = \vec{\nabla} \cdot \left(\sigma(\rho(\vec{r})) \vec{\nabla} H(\vec{r}) \right)$$

$$D(\rho(\vec{r}))\, \Delta H(\vec{r}) = -\frac{\sigma'(\rho(\vec{r}))}{2} \left(\vec{\nabla} H(\vec{r}) \right)^2$$

where $H(\vec{r})$ is a Lagrange multiplier field associated to current conservation.

The link between $d=1$ and higher dimensions:

If one knows the solution of

$$\vec{\nabla}\cdot\left(D(\rho(\vec{r}))\,\vec{\nabla}\rho(\vec{r})\right)=\vec{\nabla}\cdot\left(\sigma(\rho(\vec{r}))\vec{\nabla}H(\vec{r})\right)$$

$$D(\rho(\vec{r}))\,\Delta H(\vec{r})=-\frac{\sigma'(\rho(\vec{r}))}{2}\left(\vec{\nabla}H(\vec{r})\right)^2 \qquad (1)$$

in $d=1$ (chain of length L), then we know the solution in any dimension and for any domain !

This results from $\Delta v(\vec{r})=0\,,\ v(\partial A)=1, v(\partial B)=0$

so that $H(\vec{r})=H_{d=1}(v(\vec{r}))\,,\rho(\vec{r})=\rho_{d=1}(v(\vec{r}))$

solve (1)

THE GENERATING FUNCTION IN d DIMENSIONS IS

$$\mu(\lambda) = L^{d-2} \int d\vec{r} \left(\left(\vec{\nabla} v(\vec{r}) \right)^2 \Phi\left(v(\vec{r}) \right) \right)$$

WHERE $\Phi\left(v(\vec{r})\right)$ IS SUCH THAT

$$\int_0^1 dx\, \Phi\left(v(x)\right) = L\, \mu_{d=1}(\lambda)$$

WE HAVE THE FOLLOWING REMARKABLE IDENTITY:

$$\int d\vec{r}\, \Phi\left(v(\vec{r})\right) \left(\vec{\nabla} v(\vec{r})\right)^2 = \int_0^1 dx\, \Phi\left(v(x)\right) \times \int d\vec{r} \left(\vec{\nabla} v(\vec{r})\right)^2$$

SO THAT

$$\mu(\lambda) = \left[L^{d-2} \int d\vec{r} \left(\vec{\nabla} v(\vec{r})\right)^2 \right] \times \left[L\, \mu_{d=1}(\lambda) \right]$$

$$\mu(\lambda) = \left[L^{d-2} \int d\vec{r} \left(\vec{\nabla} v(\vec{r}) \right)^2 \right] \times \left[L \mu_{d=1}(\lambda) \right]$$

SINCE

$$L \mu_{d=1}(\lambda) = \lim_{t \to \infty} \frac{L_e}{t} \log \left\langle e^{\lambda Q_t} \right\rangle \Big|_{d=1} = \left(\sinh^{-1}\left(\sqrt{\omega} \right) \right)^2$$

THEN,

$$\mu(\lambda) = \lim_{t \to \infty} \frac{1}{t} \log \left\langle e^{\lambda Q_t} \right\rangle = \kappa(L_e) \times \left(\sinh^{-1}\left(\sqrt{\omega} \right) \right)^2$$

WITH

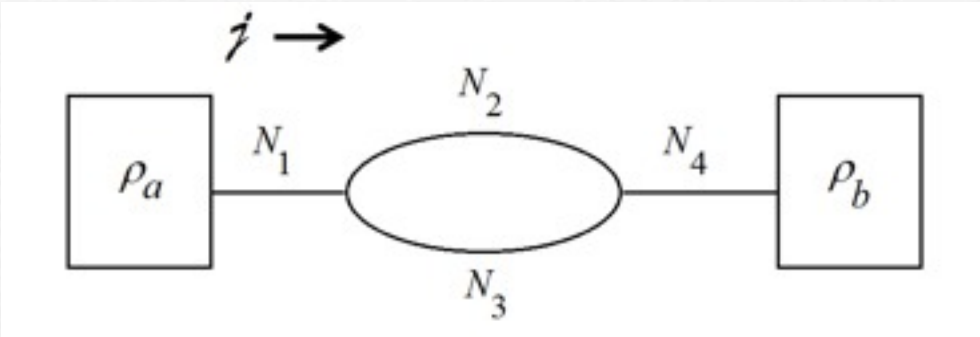
$$\kappa(L_e) \equiv L_e^{d-2} \int d\vec{r} \left(\vec{\nabla} v(\vec{r}) \right)^2$$

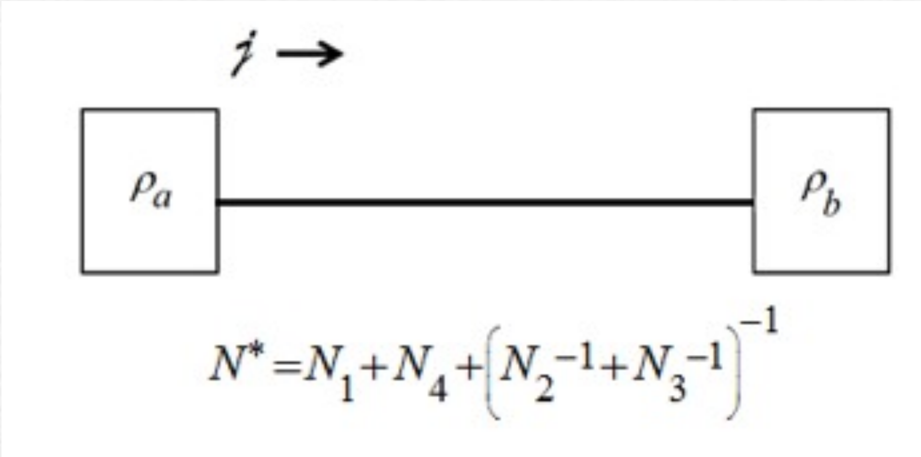
The generating function $\mu(\lambda)$ for an arbitrary domain in d-dimensions is the same as the $d=1$ generating function $\mu_{d=1}(\lambda)$ for the effective length L_e up to a multiplicative function independent of $(\lambda, \rho_a, \rho_b)$

Therefore, for any d-dimensional domain, the ratio of any pair of cumulants is the same as in $d=1$.

Analogies between SSEP on a graph and resistor networks

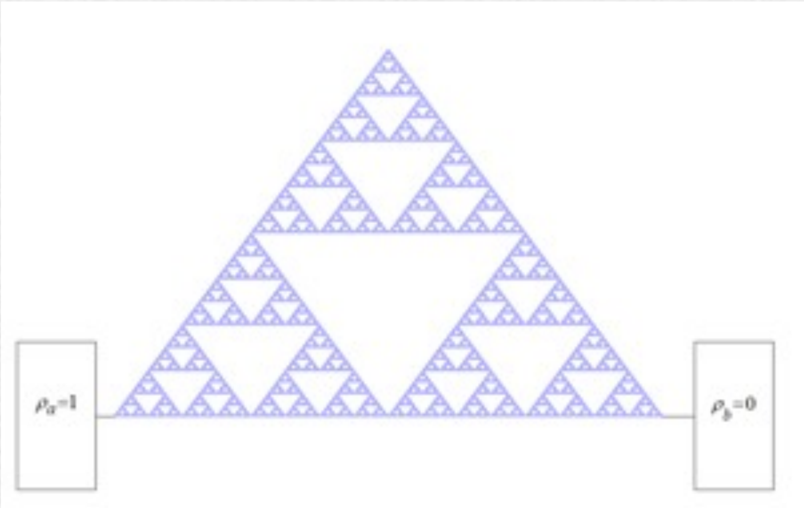
Kirchhoff's rules – Addition in series and in parallel

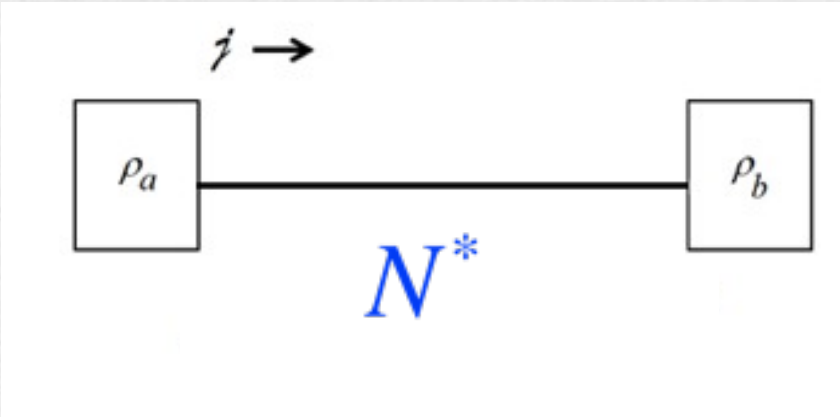




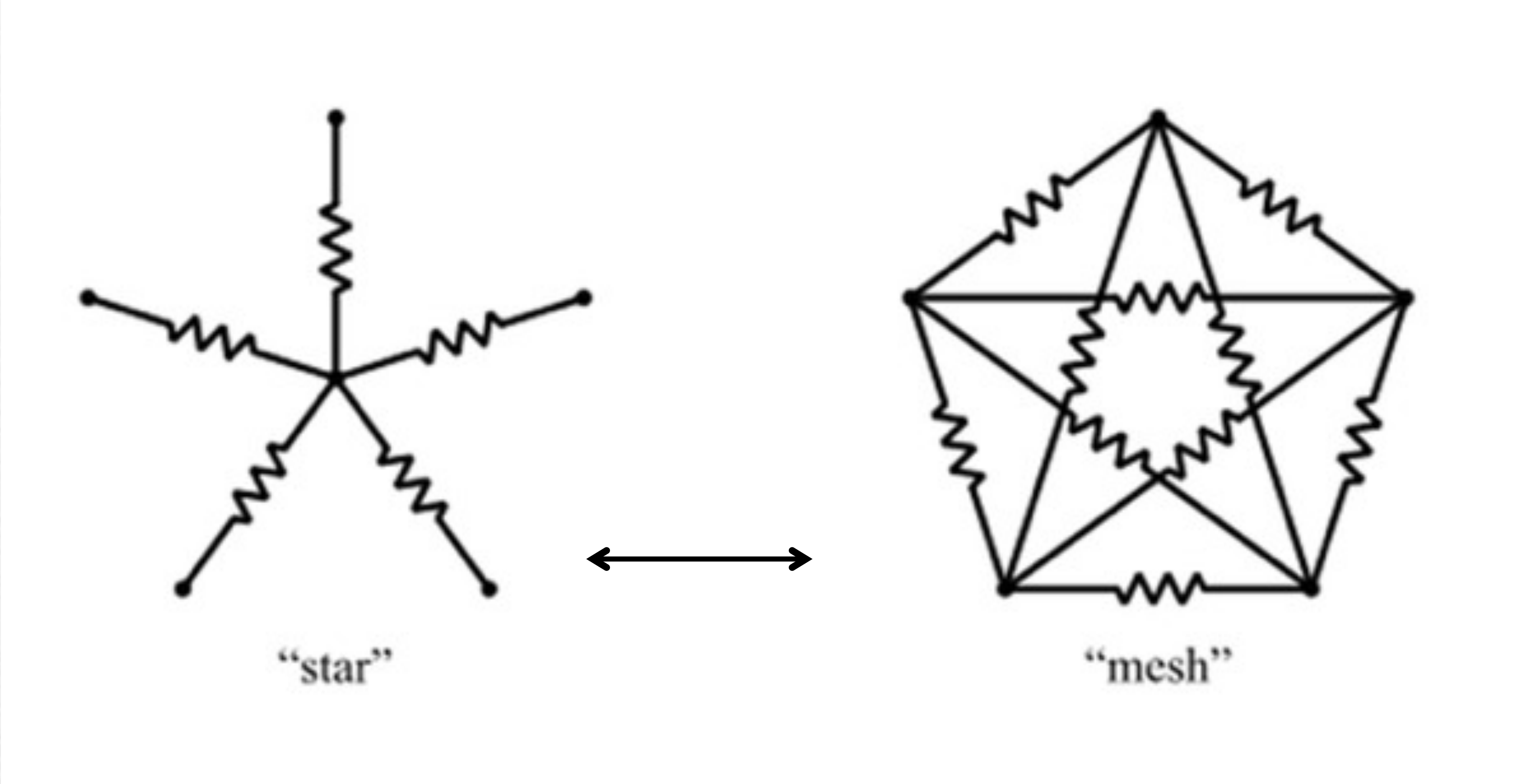
(Derrida, Bodineau PRL 2004)

More generally, using the Δ–Y transform



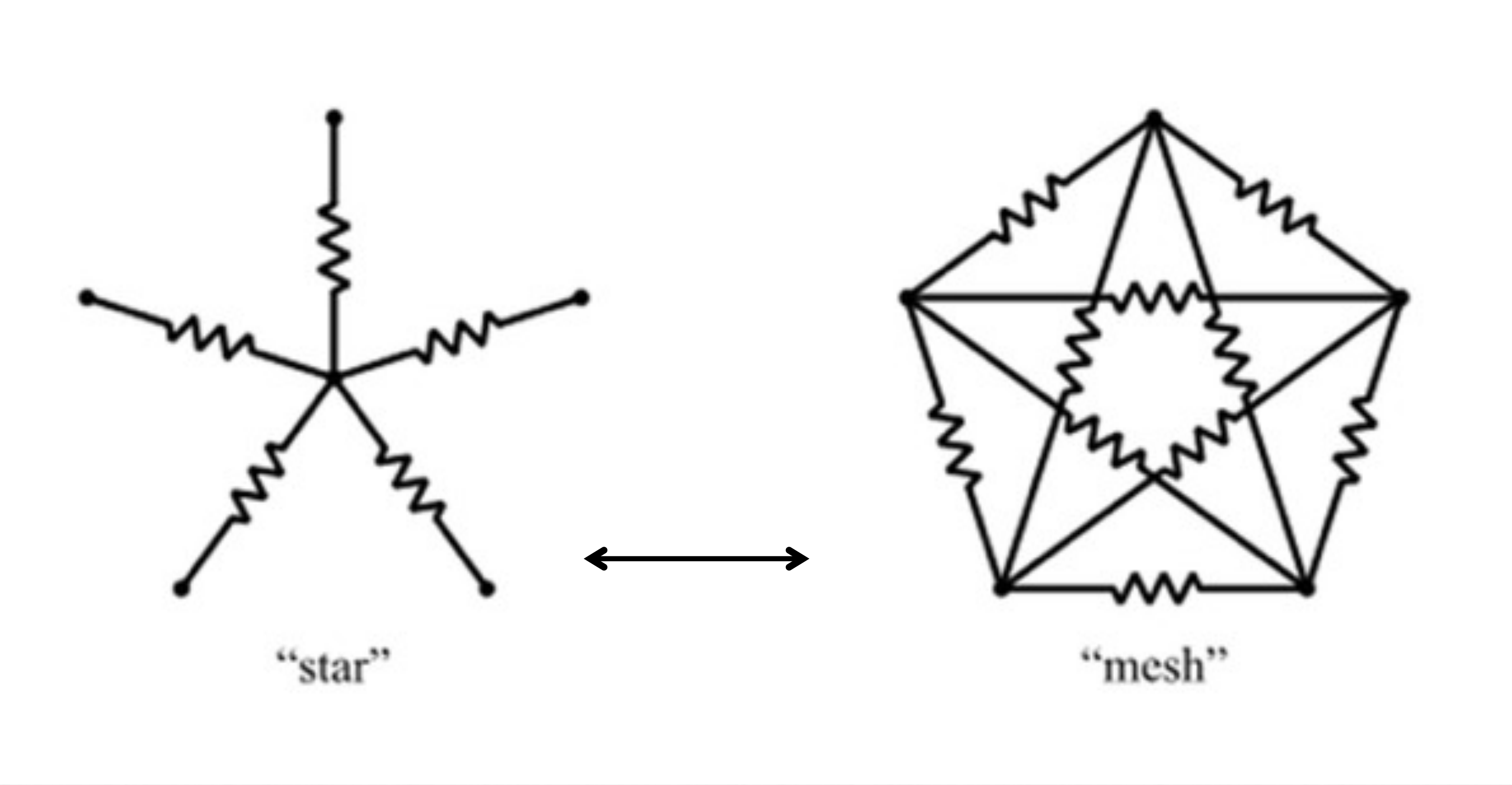


Star – mesh transform



- A Two-terminal resistor network always has an equivalent resistor (Helmholtz, Thevenin).
- The equivalent resistor can be obtained through repeated use of the star-mesh transform.

Star - mesh transform



- A Two-terminal resistor network always has an equivalent resistor (Helmholtz, Thevenin).
- The equivalent resistor can be obtained through repeated use of the star-mesh transform.

The same applies to any SSEP graph

The SSEP resistor theorem

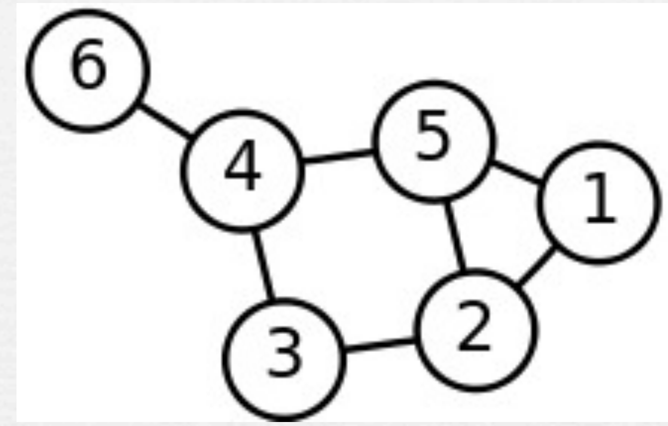
- For any graph G, $F_G(j,\rho_a,\rho_b) = F_{N^*}(j,\rho_a,\rho_b)$
- N^* can be obtained by Kirchhoff's resistor rules
- The theorem applies for any non-eq. process given that

1. The additivity principle applies
2. The scaling assumption applies
3. There is a steady state

Energy/Dirichlet forms

Energy forms

A graph with sites and bonds



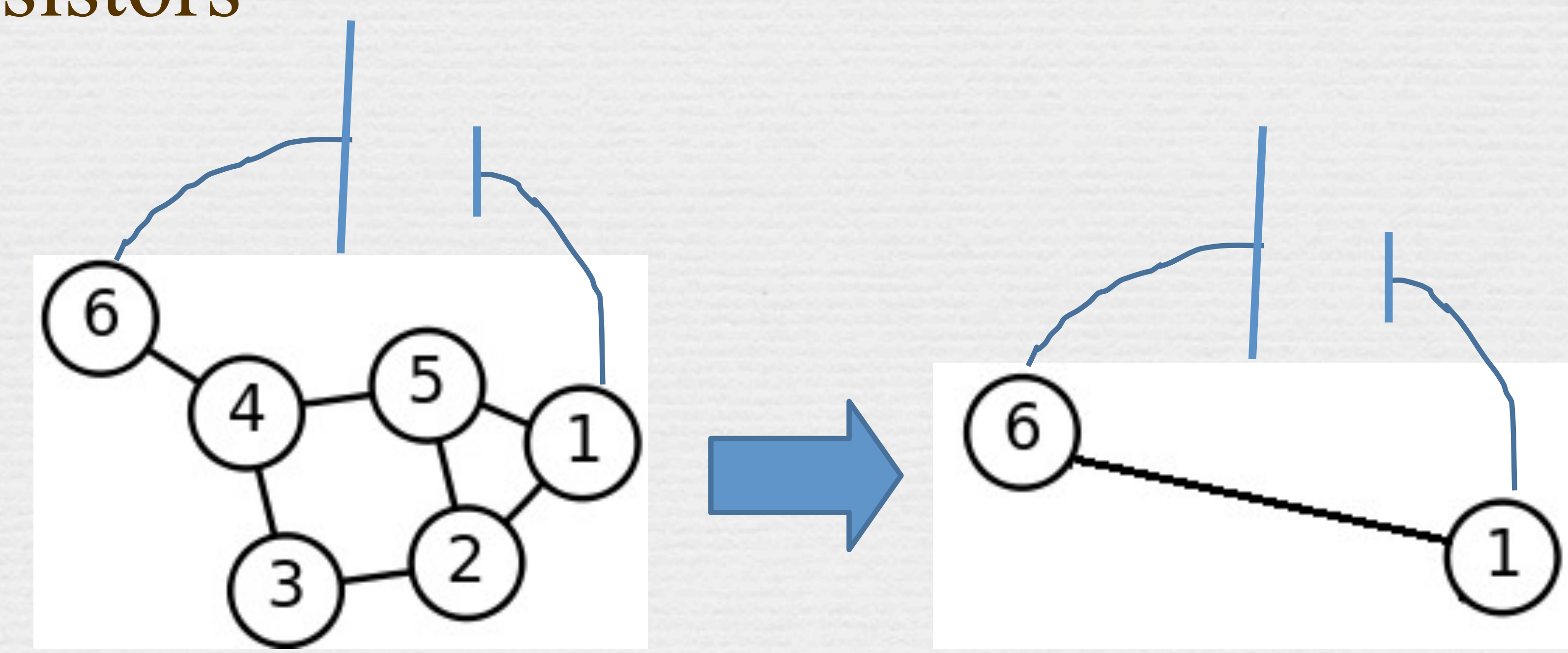
Each bonds carries a weight – r_{xy}

We define the energy function $E_G(u) = \sum_{x \sim y} \frac{1}{r_{xy}} \left[u(x) - u(y) \right]^2$

Connect the network of resistors to a battery

$$E_G(h) = \inf_u E_G(u)$$

h - harmonic function



Well known exact mapping between electric networks of resistances and random walk on a lattice

(Doyle & Snell)

Useful theorem by Beurling and Deny which extends these results to the equivalence between energy forms and symmetric Markov processes.

This theorem allows to describe the SSEP as an effective conductance network whose electric energy is the large deviation function.

Moreover, it guarantees the additivity principle (through the concavity property of the minimum energy)

Energy forms and SSEP

Consider the energy form $$E_L(u,u) = \sum_{x,y} \frac{\left[u(x) - u(y)\right]^2}{r}$$

with $$u(x) = \frac{\kappa(x) r + D(\rho(x))\rho(x)}{\left[2\sigma(\rho(x))\right]^{1/2}}$$

Boundary conditions

$\rho(0) = \rho_a$

$\rho(1) = \rho_b$

$\kappa(x_{i+1}) - \kappa(x_i) = j$

$$E_L(h(j,\rho_a,\rho_b)) = \max_{\{\rho_i\}} \sum_i \frac{\left[jn + D(\rho_i)\Delta\rho_i\right]^2}{2n\sigma(\rho_i)}$$

Energy forms and SSEP

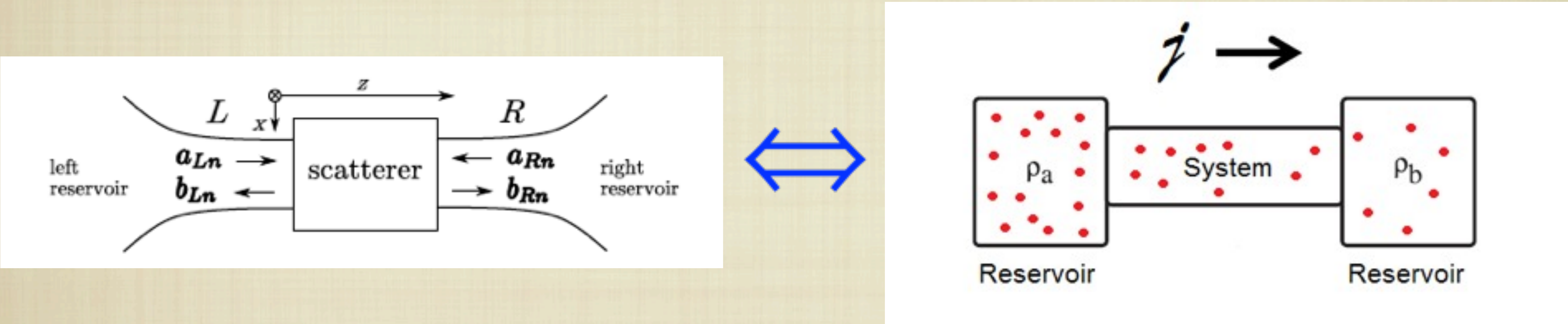
$$E_L\left(h\left(j,\rho_a,\rho_b\right)\right)=\max_{\{\rho_i\}}\sum_i \frac{\left[jn+D\left(\rho_i\right)\Delta\rho_i\right]^2}{2n\sigma\left(\rho_i\right)}$$

The Large Deviation Function is the minimum of an energy form – it is a conductance

$$E_L\left(h\left(j,\rho_a,\rho_b\right)\right)=F_L\left(j,\rho_a,\rho_b\right)$$

Summary - further issues

- Full counting statistics of quantum mesoscopic conductors is well described by means of the classical 1D SSEP model:



- SSEP - resistor theorem : analogy between electric networks and non-equilibrium stochastic processes.
- Energy forms provide a useful framework to derive the large deviation function of symmetric Markov processes.
- The additivity principle results from the energy form description.
- Extension to more complicated stochastic processes (ASEP) - with phase transitions.
- More than 2 reservoirs ?
- Random graphs
- Back to the quantum case : semi-classical description (A. Pilgram, Sukhorukov).