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GROUPS AND EXPANDERS

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ABSTRACT. Most explicit examples of expanding graphs are families of Cayley graphs of finite groups. In this note we consider the question of what is it that makes a family of such Cayley graphs expanders- is it a property of the groups alone or does it also depend upon the set of generators? We also give some conditions which prevent an infinite family of groups from being expanders, discuss some examples and raise some further problems and conjectures.

1. INTRODUCTION

Expanding graphs have received a great deal of attention in recent years, mainly because of their importance in computer science and in particular their usefulness in constructing communication networks. This interest led to various explicit constructions of families of expanders ([M1], [LPS], [M2], [L1], [C], [Mo]). These constructions usually take the following from. Starting with an infinite group Γ and finite set of generators Σ, let $\pi_i : \Gamma \to G_i$ be an infinite family of finite quotients. The Cayley graphs $\{X(G_i, \pi_i(\Sigma))\}$ will be the family of expanders if the right conditions prevail. The basic reasons vary, one uses property T of Γ, Selberg's theorem $\lambda_1 \geq \frac{3}{16}$, the Ramanujan conjecture or one of their modifications. We give a brief survey in §2.

In §3, we give some general methods of constructing families of bounded degree Cayley graphs that are not expanders. For example: theorem 3.1 states that if Γ is a finitely generated amenable group, $\Gamma = \langle\Sigma\rangle$, then the Cayley graphs $\{X_i = X(\pi_i(\Gamma), \pi_i(\Sigma))\}$ are non expanders for any infinite family of finite quotients. That means that for every $\varepsilon > o$, there is some i such that X_i is not an ε-expander.

As a corollary we deduce in 3.2 that if $\{G_i\}$ is an infinite family of finite solvable groups of bounded derived length, and $G_i = \langle\Sigma_i\rangle$ with the cardinality of the Σ_i bounded, then $\{X(G_i, \Sigma_i)\}$ is a non expander family. In contrast to this we give an example in 3.3 of an infinite family of solvable groups which are expanders thus answering a question raised in [AB].

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The results of §2 and §3 suggest a very basic question: does the expander - non expander property of a family of groups depend on the groups alone or is it also a function of the generating sets. More precisely we formulate

Problem 1.1. Let $\{G_i\}$ be a family of finite groups, $\langle\Sigma_i\rangle = \langle\Sigma_i'\rangle = G_i$ and $|\Sigma_i|$, $|\Sigma_i'| \leq k$, for all i. Does the fact that $\{X(G_i, \Sigma_i)\}$ is an expander family imply the same for $\{X(G_i, \Sigma_i')\}$?

The known methods of proving that a family is an expander certainly depend on the choice of generators. For example the groups $SL_n(p)$ for fixed n and p a prime are known to be expanders with the generators chosen globally (from $SL_n(\mathbb{Z})$). Computations by Laffery and Rockmore [LR] suggest (for $n = 2$) that randomly chosen generators, or possible even "worst case" generators for these groups also give expanders.

A positive answer to problem 1.1 would in fact be surprising. Some evidence for such an answer is the following. The natural strategy to 1.1 is to find a profinite group K with two finitely generated dense subgroups, A and B, with A amenable and B having property T. Then the finite quotients $\{G_i\}$ of K would give a negative answer to 1.1. We conjecture that this is impossible.

Conjecture 1.2. Let K be a compact group. If A and B are both finitely generated subgroups that are dense in K with A amenable and B having property T then K is finite.

The latter part of the paper is devoted mainly to some evidence favoring this conjecture. We show in 5.5. that this conjecture is valid if K is a linear group over some field.

We also observe that while for $n \geq 3$

$$K_n = \prod_p SL_n(p)$$

has a dense subgroup with property T it doesn't have a dense finitely generated amenable group. In contrast with this we consider in §4

$$K^p = \prod_n SL_n(p)$$

and show that it does contain a finitely generated dense amenable group. Some standard conjectures on the congruence subgroup problem imply that all known examples of groups with property T **cannot** be densely embedded in K^p. Thus both K_n and K^p seem to support our conjecture for very different reasons.

2. Expander groups

A finite k-regular graph X is called an **ε-expander** ($\varepsilon > 0$) if for all subsets $A \subset X$ with $|A| \leq \frac{1}{2}|X|$ we have $|\partial A| \geq \varepsilon|A|$ where

$$\partial A = \{y \in X : \text{ distance } (y, A) = 1\}.$$

We will say that a family of groups $Y = \{G_i\}$ is an **expander** family if for some k and $\varepsilon > 0$ there are generating sets Σ_i for G_i of size at most k such that all the Cayley graphs $X(G_i, \Sigma_i)$ are ε-expanders. The family Y will be called a **non-expander** family if for some k there are generating sets Σ_i for G_i of size at most k, and for all positive ε at least one of the Cayley graphs $X_i = X(G_i, \Sigma_i)$ is not an ε-expander.

The major problem that motivated the work presented here is whether or not a family of groups can be both an expander family and a non-expander family. In this section we will survey the known methods of constructing, of finding, expander families, while the next section will be devoted to non-expander families.

The known expander families of groups have the following common structure. Start with an infinite group Γ that is finitely generated and fix a set of generators Σ. The family $\{G_i\}$ is a family of homomorphic images of Γ, $\pi_i(\Gamma)$ with $\Sigma_i = \pi_i(\Sigma)$. Here are the main expander families. (The reader is referred to [L1] for more details.)

I: Let Γ be a countable group with property T (see [M3] or [L1] for definitions) Σ any finite generating set, and take for G_i all finite quotient groups of Γ. This result is due to Margulis in [M1]. If Γ is a lattice in a semi simple Lie group all of whose factors H_j have rank ≥ 2 then Γ has property T (the H_j can be Lie groups over an arbitrary local field).
A weaker property, called τ, suffices for the above to hold, see [LZ]. A group Γ has property τ if all of the finite dimensional unitary representations of Γ are bounded away from the trivial representation. For this it suffices that Γ be an irreducible lattice in $\prod_j H_j$, with the H_j simple Lie groups and at least **one** of the H_j's having rank ≥ 2.

II: Let M be a compact Riemannian manifold, $\Gamma = \pi_1(M)$ and N_i a family of finite index normal subgroups of Γ such that the corresponding finite covers M_i of M satisfy $\lambda_1(M_i) \geq c > 0$ where $\lambda_1(M_i)$ is the smallest positive eigenvalue of the Laplacian on M_i. Assume Σ is a finite set of generators for Γ and take for π_i the canonical projections $\pi_i : \Gamma \to \Gamma/N_i$.
Instead of compactness one can assume $M = X/\Gamma$ with X a symmetric space and Γ any lattice in $\mathrm{Aut}(X)$. An example of this situation is when M is the modular surface, $\Gamma = SL_2(\mathbb{Z})$ and N_i the congruence subgroups of Γ, i.e. the G_i are $SL_2(\mathbb{Z}/i\mathbb{Z})$. The fact that one can take here $c = 3/16$ is a well known result of Selberg [S]. The Jacquet-Langlands correspondence extends this to uniform arithmetic lattices of $SL_2(\mathbb{R})$ as well.
Selberg's result was extended to all arithmetic groups in $SO(n,1)$ ([EGM], [LPSS]) and $SU(n,1)$ ([Li]). From these results it can be shown that one gets an expander family for every characteristic zero S-arithmetic group (with N_i the congruence subgroups) provided that Γ is an ineducable lattice in $\prod H_j$ and at least one of the H_j's is a real simple Lie group of rank≥ 1.

III: Let Γ be an arithmetic lattice in $SL_2(Q_p)$ and N_i the family of congu-vence subgroups, Σ any finite set of generators. The fact that we get a family of expanders follows now from the Ramanujan conjecture proved by Eichler and Deligne. This is shown in [LPS] and [M2]. Here Q_p may be replaced by any non-Archimedean local field of arbitrary characteristic. For positive characteristic this follows from the work of Drinfeld who proved the analogue of the Ramanujan conjecture there. This was shown in [Mo], (as explained there the Jacauet-Langlands correspondence is needed once again).

Given all of the above the following conjecture suggests itself:

Conjecture 2.1. Let k be a global field, S a finite set of primes (i.e. valuations) of k that contains all Archimedean ones. Let G be a simple, simply connected, (connected) algebraic group defined over k and $\Gamma = G(O_S)$ where O_S is the ring of S-integers of k, i.e.

$$O_S = \{x \in k : \nu(x) \geq 0 \text{ for all } \nu \notin S\}.$$

Assume that Γ has a finite generating set Σ and that N_i are normal congruence subgroups. Then with $\pi_i : \Gamma \to \Gamma/N_i$ the canonical projections the Cayley graphs $X(\pi_i(\Gamma), \pi_i(\Sigma))$ are ε-expanders for some fixed $\varepsilon > 0$.

Remark 2.2.

(i): In almost all cases Γ is indeed finitely generated, the only exception is when $\mathrm{char}(k) = p > 0$ and $\sum_{\nu \in S} \mathrm{rank}(G(k_\nu)) = 1$ (see [Be] or [L2])

(ii): In most cases the conjecture is known to hold. Indeed Γ is an irreducible lattice in $\prod_{\nu \in S} G(k_\nu)$, where k_ν is the completion of k with respect to ν. If for some ν, $\mathrm{rank}(G(k_\nu))$ is at least 2 then the conjecture holds as explained in (I). If for all ν, $\mathrm{rank}(G(k_\nu)) \leq 1$ but one of these k_ν is $\mathbb{R}$ or $\mathbb{C}$ then the conjecture holds by (II) above. If at least one of the $G(k_\nu)$ is $SL_2(k_\nu)$ the conjecture holds by (III). In order to prove it in full generality it would suffice to work out the analogues of [EGM] (or [LPSS]) and [Li] for rank one groups (not SL_2) over non-Archimedean fields.

3. Non expander groups

In the preceding section we surveyed the methods of finding expander families of groups. Here we show how to find non expander families. In many ways the opposite of groups with property T are amenable groups. As we shall see this is also the case with respect to the expander property. Recall that a group Γ is amenable if $\ell^\infty(\Gamma)$ has a Γ-invariant mean. By a well known result of Følner (cf. [L1]) and the references there) a discrete group is amenable if and only if for every finite set Σ and every $\delta > 0$, there is a finite set $A \subset \Gamma$ satisfying

$$|\Sigma A \Delta A| < \varepsilon |A|$$

where $\Sigma A = \{sa : s \in \Sigma, a \in A\}$ and Δ denotes the symmetric difference. In words, A is "almost invariant" with respect to left multiplication by Σ.

Theorem 3.1. *Let Γ be a finitely generated amenable group and $Y = \{G_i\}$ an infinite family of quotient groups, then Y is a non expander family.*

Remark 3.2. If $\pi_i : \Gamma \to G_i$ are the canonical projections and Σ is **any** generating set for Γ we shall show that the Cayley graphs $X(G_i, \pi_i(\Sigma))$ are not an ε-expander family for any $\varepsilon > 0$. This is a little more than the assertion that Y is a non expander family.

Proof. Fix a generating set Σ, and a positive $\varepsilon > 0$. Find a finite set A in Γ such that $|sA\Delta A| < \varepsilon|A|$ for all $s \in \Sigma \cup \Sigma^{-1}$. If $\pi : \Gamma \to G_i$ is onto one of our quotient groups and $L_i = \ker(\pi)$, define a function on G_i by

$$\varphi(g) = \sum_{h\in\pi^{-1}(g)} 1_A(h).$$

Calculating

$$\begin{aligned} ||\pi(s)\varphi - \varphi||_{\ell_1} &= \sum_g |\varphi(\pi(s)g) - \varphi(g)| \\ &= \sum_g |\sum_{h\in\pi^{-1}(g)} 1_A(sh) - \sum_{h\in\pi^{-1}(g)} 1_A(h)| \\ &\le \sum_{h\in\Gamma} |1_A(sh) - 1_A(h)| \\ &= |A\Delta sA| + |s^{-1}A\Delta A| < 2\varepsilon|A| \\ &= 2\varepsilon\varepsilon||\varphi||_{\ell_1}. \end{aligned} \tag{1}$$

Thus φ is an approximately invariant ℓ_1 function on G_i. Furthermore, since there are infinitely many G_i's it is clear that we may assume that φ is far from being a constant function, indeed it's support is at most of size $|A|$ and we may assume $|G_i| > 2|A|$.

To conclude the proof we repeat the standard argument that an approximately invariant function gives rise to approximately invariant sets which will preclude the possibility that $\{G_i\}$ is an $\sqrt{2\varepsilon}$-expander family.

Define

$$B_j = \{g \in G_i : \varphi(g) \ge j\}.$$

Clearly, if 1_{B_j} are the indicator functions of the B_j's we have $\varphi = \sum_{j=1}^{\infty} 1_{B_j}$, and one can also easily check that

$$||\pi(s)\varphi - \varphi||_{\ell_1} = \sum_{j=1}^{\infty} ||\pi(s)1_{B_j} - 1_{B_j}||_{\ell_1} \tag{2}$$

Denote by

$$J_s = \{j : ||\pi(s)1_{B_j} - 1_{B_j}||_{\ell_1} > \sqrt{2\varepsilon}||1_{B_j}||_{\ell_1}\}$$

and compute

$$\begin{aligned} \sqrt{2\varepsilon}\sum_{j\in J_s} ||1_{B_j}||_{\ell_1} < \sum_{j\in J_s} ||\pi(s)1_{B_j} - 1_{B_j}||_{\ell_1} &\le ||\pi(s)\varphi - \varphi||_{\ell_1} \\ &\le 2\varepsilon||\varphi||_{\ell_1} \end{aligned}$$

by (1) and (2), hence

$$\sum_{j\in J_s} ||1_{B_j}||_{\ell_1} \le \sqrt{2\varepsilon}||\varphi||_{\ell_1} \tag{3}$$

Since $||\varphi||_{\ell_1} = \sum_1^{\infty} ||1_{B_j}||_{\ell_1}$, if $|\Sigma| \cdot \sqrt{2\varepsilon} < 1$ there are indices j_0 that are not in any of the J_s's and any such B_{j_0} is clearly $\sqrt{2\varepsilon}$-invariant under all the $\pi(s)$, $s \in \Sigma$. Since $\varepsilon > 0$ was arbitrary we have shown that $\{G_i\}$ is a non-expander family. □

For a group G, denote $G_1 = G$ and $G_{i+1} = [G_i, G_i]$. Recall that G is solvable of derived length ℓ if $G_\ell \neq \{e\}$ but $G_{\ell+1} = \{e\}$.

Corollary 3.3. *Let $Y = \{G_i\}$ be an infinite family of finite solvable groups that are all of derived length at most ℓ and are also generated by k elements, say Σ_i generates G_i, $|\Sigma_i| = k$ for all i. Then the Cayley graphs $X(G_i, \Sigma_i)$ are not a family of expanders.*

Proof. Let $F = F(x_1, \ldots, x_k)$ be the free group on k-generators, $\Sigma = \{x_1, \ldots, x_k\}$. The group $\Gamma = F/F_{\ell+1}$ is a solvable group and the map π_i taking Σ to Σ_i extends to a homomorphism from Γ onto G_i. Since Γ is amenable the corollary follows at once from the theorem. □

The bound on the derived length is essential as the following example of an infinite expander family of solvable groups shows:

Example 3.4. Let $\Gamma = SL_n(\mathbb{Z})$ and for a fixed prime p set

$$\Gamma(p^m) = \ker(SL_n(\mathbb{Z}) \to SL(\mathbb{Z}/p^m\mathbb{Z})).$$

As is well known, Γ is finitely generated, for example if $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$, and T, S are defined by

$$Te_i = \begin{cases} e_1 + e_2 & i = 1 \\ e_i & i \neq 1 \end{cases}$$

$$Se_i = \begin{cases} e_{i+1} & i < n \\ (-1)^{n-1}e_1 & i = n \end{cases}$$

then T and S generate Γ. Since $\Gamma(p)$ is of finite index in Γ it too is finitely generated, say by a finite set Σ. Now the Cayley graphs $X_m = X(\Gamma(p)/\Gamma(p^m), \Sigma)$ satisfy:

- **(i):** $\Gamma(p)/\Gamma(p^m)$ is a p-group and hence solvable (it is even nilpotent) for all m.
- **(ii):** For some $\varepsilon > 0$, the X_m are all ε-expanders.

To check (i) one verifies easily that the order of $\Gamma(p)/\Gamma(p^m)$ is $p^{(m-1)(n^2-1)}$. For (ii) we distinguish between $n \geq 3$ and $n = 2$. In the first case $\Gamma(p)$ is a lattice in $SL_n(\mathbb{R})$ and thus has property (T) and by I. of §2 we have (ii). The result for the second case follows from Selberg's theorem as we have already indicated in II of §2.

This example answers a question raised in [AB].

We turn now to another method for showing that a family of groups cannot be made into a family of expanders with respect to any (bounded) set of generators. First a definition;

Definition 3.5. Let G be a group together with a finite generating set Σ. For a fixed unitary representation ρ denote by

$$K(G, \Sigma, \rho) = \inf_{||v||=1} \max_{s \in \Sigma} ||\rho(s)v - v||^2$$

and let

$$K(G, \Sigma) = \inf\{K(G, \Sigma, \rho) : \ \rho \text{ a representation that contain no non zero fixed vectors}\}$$

This will be called the Kazhdan constant of (G, Σ), and a group has property T if and only if this constant is positive for some (and hence all) finite generating set.

Theorem 3.6. *For any $k > 0$ and positive d, there is a constant $c = c(k, d)$ such that if G is any group with generators Σ satisfying*

$$(a)\ |\Sigma| \leq d; \quad (b)\ K(G, \Sigma) \geq k$$

then for any subgroup $G_0 < G$ with $[G : G_0] \leq n$

$$|G_0/[G_0, G_0]| < c^n.$$

Before proving this rather technical theorem here is a corollary that fulfills the promise that we just made about more non expander families.

Corollary 3.7. *1. If $X_i = X(G_i, \Sigma_i)$ is a family of Cayley graphs with $|\Sigma_i|$ bounded that is an expander, then there is a constant $c > 0$ such that for all i, and any subgroup M of index n in G_i*

$$|M/[M, M]| \leq c^n.$$

2. Let M_i be any family of finite groups, $\{p_i\}$ an infinite sequence of primes and let

$$G_i = C_{p_i}\, wr M_i$$

denote the wreath product of the cyclic group with p_i elements and M_i, i.e.

$$G_i = F_{p_i}[M_i] \rtimes M_i$$

when $F_{p_i}[M_i]$ is the additive group of the group algebra of M_i over the field F_{p_i} and M_i acts on it by multiplication. Then $X(G_i, \Sigma_i)$ is not an expander family with respect to any bounded set of generators Σ_i.

Proof. 1. It is shown in [L1,Chap. 4] (cf. also [LZ]) that $X(G_i, \Sigma_i)$ are an expander family if and only if $K(G_i, \Sigma_i)$ is bounded away from zero. Thus this follows at once from theorem 3.5.

2. This follows from 1. since G_i contains an abelian subgroup $A_i = F_{p_i}[M_i]$ which is of index $|M_i|$ but of size $p_i^{|M_i|}$ and hence for all c, if p_i is sufficiently large $|A_i/[A_i, A_i]| < c^{[G_i:A_i]}$ cannot hold. □

We do not know that the corollary continues to hold if the p_i are a fixed prime, although we do expect it to. An interesting special case is $F_2[S_n] \rtimes S_n$ where S_n is the symmetric group on n letters.

Remark 3.8. If Γ has property T then it cannot have finite index group with infinite abelian quotients. Our theorem 3.5 may be viewed as a quantitative version of this giving for any such Γ an estimate

$$|H/[H, H]| \leq c^{[\Gamma:H]} \tag{4}$$

for some constant c, since as is well known groups with property T are finitely generated. A polynomial bound in (4), suggested by the known examples of groups with property T, would have interesting applications.

For the proof of theorem 3.5 we need the following:

Proposition 3.9. *Let G be a group generated by Σ, $|\Sigma| = d$, and H a finite index subgroup. Then H has a generating set Σ' satisfying*

(a): $|\Sigma'| \leq d \cdot [G : H]$
(b): $K(H, \Sigma') \geq K(G, \Sigma)$.

Proof. Denote by n the index $[G : H]$ and let $T = \{t_1, \dots, t_n\}$ be a right transversal for H in G, i.e. right coset representatives and denote by $g \to \bar{g}$ the map from $G \to T$ such that

$$g\bar{g}^{-1} \in H.$$

As is well known H is generated by Σ' the non trivial elements of the form $(t_i s)(\overline{t_i s})^{-1}$, $t_i \in T$, $s \in \Sigma$ (cf. [MKS, p.89]). For any unitary representation ρ of H on a Hilbert space V with no non zero fixed vector let $(\tilde{V}, \tilde{\rho}) = \mathrm{Ind}_H^G(\rho)$ denoted the induced representation. Recall that $\tilde{V}$ is the space of functions $f : G \to V$ that satisfy

$$f(hg) = \rho(h) f(g)$$

with G acting on the right. Such a function is determined as soon as $f(t_i)$, $t_i \in T$ is given and hence $\tilde{V}$ may be identified with V^n as the direct sum of n-copies of V. One checks directly or via Frobenius reciprocity that $\tilde{\rho}$ doesn't contain the trivial representation.

Now given $v_0 \in V$, define $f \in \tilde{V}$ by setting $f(t_i) = v_0/\sqrt{n}$ for $i = 1, \dots, n$. Then $||f|| = 1$ and therefore by the definition of $K(G, \Sigma)$ there is some $s \in \Sigma$ such that

$$\begin{aligned} K(G,\Sigma) \leq ||\tilde{\rho}(s) f - f||^2 &= \sum_{i=1}^n ||f(t_i s) - f(t_i)||^2 \\ &= \sum_{i=1}^n ||f(t_i s (\overline{t_i s})^{-1} \cdot \overline{t_i s}) - f(t_i)||^2 \\ &= \sum_{i=1}^n ||\rho(t_i s (\overline{t_i s})^{-1}) \cdot f(\overline{t_i s}) - f(t_i)||^2 \\ &= \sum_{i=1}^n ||\rho(t_i s (\overline{t_i s})^{-1} \tfrac{v_0}{\sqrt{n}} - \tfrac{v_0}{\sqrt{n}}||^2 \\ &= \frac{1}{n} \sum_{i=1}^n ||\rho(t_i s (\overline{t_i s})^{-1}) v_0 - v_0||^2 \end{aligned}$$

and thus for one of the elements in Σ', say u we get

$$K(G, \Sigma) \leq ||\rho(u) v_0 - v_0||^2.$$

Since v_0 was arbitrary we obtain as required $K(H, \Sigma') \geq K(G, \Sigma)$. □

Proposition 3.10. *Let A be an abelian group generated by k elements, Σ, with $K(A, \Sigma) = \varepsilon$. Then $|A| \leq C^k$ where $C = [\frac{2\pi}{\sqrt{\varepsilon}}] + 1$.*

Proof. Divide the unit circle into C equal arcs starting at 1. This gives a subdivision on the k-torus, Π^k, into C^k boxes. Any unitary character of A determines a point of Π^k. If $|A| > C^k$ then there are two distinct characters that land in the same box and thus their quotient would be a non trivial character χ satisfying

$$|\chi(s) \cdot 1 - 1|^2 < \left(\frac{2\pi}{C}\right)^2$$

for all $s \in \Sigma$. Since χ defines a one dimensional unitary representation we conclude that

$$\varepsilon < \left(\frac{2\pi}{C}\right)^2$$

which contradicts our choice of C. Thus $|A| \leq C^k$ are required. □

Finally we can complete the proof of 3.5 as an immediate consequence of propositions 3.8 and 3.9 choosing for the constant in the theorem $([\frac{2\pi}{\sqrt{\varepsilon}}]+1)^d$.

4. $\{SL_n(p)\}$-FIXED n VERSUS FIXED p

This section is devoted to examples of finitely generated amenable groups Γ. Using theorem 3.1 this leads us to finite quotient groups which are not expanders with respect to the generators that come from Γ.

Example 4.1. Let $K = \prod_{n=5}^{\infty} S_n$, where S_n is the permutation group of $\{1, 2, \ldots, n\}$. Let $\tau = (\tau_n)$, $\sigma = (\sigma_n)$ be the elements of K defined by:

$$\tau_n = (12), \text{ all } n; \quad \sigma_n = (1\ 2\ 3 \cdots n), \text{ all } n;$$

i.e. σ_n is the cyclic permutation on the n numbers $1\ 2\ 3\ \cdots n$. Let $\Gamma = \langle \sigma, \tau \rangle \subset K$ denoted the countable group generated by τ and σ.

We claim that

(i): Γ is an amenable group

(ii): if $\Gamma_+ = \Gamma \cap (\prod_{n=5}^{\infty} A_n)$, where $A_n < S_n$ is the alternating group then Γ_+ is of index 4 in Γ and is dense in $\prod_{n=5}^{\infty} A_n$.

Proof. (i) Clearly Γ is finitely generated. Let Δ_k denote the subgroup of Γ generated by $\{\tau, \sigma\tau\sigma^{-1}, \sigma^2\tau\sigma^{-2}, \ldots \sigma^{k-1}\tau\sigma^{-(k-1)}\}$. The projection of Δ_k to S_n for $n \leq k$ is surjective. On the other hand, the projection of Δ_k to $\prod_{n=k+1}^{\infty} S_n$ is a finite group, it is in fact S_k diagonally embedded in that product. Thus Δ_k is a finite group. Hence $\Delta = \bigcup_{1}^{\infty} \Delta_k$ is a locally finite group and thus is amenable.

Now $\sigma^{-1}\Delta\sigma \supset \Delta$ and thus$\tilde{\Delta} = \bigcup_{n=1}^{\infty} \sigma^{-n}\Delta\sigma^n$ is amenable as an increasing union of amenable groups and it is the normal closure of τ in Γ. Hence $\Gamma/\tilde{\Delta}$ is a cyclic group generated by the image of σ. It follows that Γ, as amenable extension of an amenable group is itself amenable.

(ii) For even n, both τ_n and σ_n are odd permutations, while for odd n, τ_n is odd and σ_n is even. Thus for a word ω in τ and σ, the sign of $(\omega)_n$ is constant along even n's and odd n's separately and the two homomorphisms

$$\begin{aligned} \mathrm{sg}_0(\omega) = &\text{ sign of the projection of } \omega \text{ into } S_n \text{ for } n \text{ even} \\ \mathrm{sg}_1(\omega) = &\text{ sign of projection of } \omega \text{ into } S_n \text{ for } n \text{ odd} \end{aligned}$$

are well defined and together they define an epimorphism from Γ onto $\{\pm 1\} \times \{\pm 1\}$. Let Γ_+ denote the kernel. It is of index 4 and Γ_+ is projected onto A_n for all n. In order to see that the image of Γ_+ is dense in $\prod_5^\infty A_n$ it suffices to show that its image in any finite product $H = \prod_{n=5}^{h} A_n$ is onto. In fact the image of Γ_+ is a subgroup L of H which is mapped onto A_n for all $5 \leq n \leq h$. Each of the simple groups A_n is a Jordan-Hölder factor of H, as well as of L. Since these groups are all different the order of L equals the order of H whence $L = H$ as required. □

P.M. Neumann called our attention that the group Γ above was already been studied for different reasons by B.H. Neumann in 1937! (see [N]).

Now theorem 3.1 implies that the family of Cayley graphs $X_n = X(S_n, \{\tau_n, \sigma_n\})$ is not a family of expanders. This is well known and easy to prove directly cf. ([L1] example 4.3.3.(c)) where three different proofs are given. The easiest one uses the fact that diameter $(X(S_n, \{\tau_n, \sigma_n\}))$ is like n^2 while the diameter of expanders grow like the logarithm of the order of the group which would be $n \log n$ in this case. We mention in passing here that the following important problem is still open:

Problem 4.2. Are there bounded sets of generators, Σ_n for S_n such that $X(S_n, \Sigma_n)$ is an expander family?

This problem is not yet solved even allowing for non constructive methods. Note that by a recent result of Alon and Roichman [AR] every finite group is an expander with respect to $O(\log |G|)$ generators- but the issue here is the bound set of generators.

Here is another example with similar flavor.

Example 4.3. Let p be a fixed prime and denote by K^p the product $\prod_{n=2}^{\infty} SL_n(p)$. Consider the embedding of $SL_2(p)$ in $SL_n(p)$ in the upper left corner. Now let Γ be the subgroup of K^p generated by $\tau = (\tau_n)$, $u = (u_n)$, $\sigma = (\sigma_n)$ where:

$$\tau_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(p) \subseteq SL_n(p)$$

$$u_n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(p) \subseteq SL_n(p)$$

$$\sigma_n = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 1 \\ (-1)^{n-1} & \cdots & & 0 \end{pmatrix} \in SL_n(p).$$

We claim that Γ is a finitely generated amenable group which is dense in K^p. The proof is very similar to the one we gave for example 4.1, and we omit the details.

In [BKL] it was known that the Cayley graphs $Y_n = X(SL_n(p); \{\tau_n, u_n, \sigma_m\})$ have **logarithmic** diameter. In spite of this, using example 4.3 and theorem 3.1 we have:

Corollary 4.4. *The family of Cayley graphs $X(SL_n(p); \{\tau_n, u_n, \sigma_n\})$ is not an expander family.*

A more elementary proof of this pact was suggested to us by Yael Luz: $SL_n(p)$ acts transitively on the non zero vectors of F_p^n. If the Y_n would be expanders then their quotient graphs Z_n would be expanders, where the vertices of Z_n are $F_p^n \setminus \{0\}$ and $\alpha \in Z_n$ is adjacent to $\tau_n^{\pm 1}(\alpha)$, $u_n^{\pm 1}(\alpha)$, $\sigma_n^{\pm 1}(\alpha)$. Let $A_n \subseteq Z_n$ be the subset $\{\epsilon_3, \dots, \epsilon_{[n/2]}\}$ where (ϵ_i) is the standard basis of F_p^n then $\tau_n^{\pm}(A_n) = u_n^{\pm}(A_n) = A_n$ while $|\sigma_n^{\pm}(A_n) \Delta A_n| \leq \frac{5}{n}|A_n$, hence the Z_n's are not expanders.

Example 4.3 is especially interesting in light of the following:

Proposition 4.5. *For fixed n, the compact group $K_n = \prod_{p-prime} SL_n(p)$ does not contain a finitely generated dense amenable group. For $n \geq 3$, K_n contains a dense group with property (T).*

Proof. Let Γ be a finitely generated subgroup of K_n. For each $g \in \Gamma \setminus \{e\}$ there is an n-dimensional representation $\rho : \Gamma \to SL_n(p)$ for some p with $\rho(g) \neq 1$. By a lemma due to Wilson [W] (or implicitly to Malcev; see also [LMS] for explicit formulation) there exists a finite family of fields $\{K_i\}_{i \in J}$ and representations $\rho_i : \Gamma \to SL_n(K_i)$ such that $\bigcap_{i \in J} \ker \rho_i = \{e\}$. Hence Γ is imbedded in $\bigoplus_{i \in J} SL_n(K_i)$. Now if Γ is in addition amenable then by the dichotomy of Tits [T] it follows that $\rho_i(\Gamma)$ is virtually solvable for each $i \in J$ whence Γ itself is virtually solvable, i.e. has a finite index solvable subgroup. The group Γ cannot be dense in K_n since $PSL_n(p)$ are simple groups of unbounded order. This proves the first assertion of the proposition.

For the second assertion consider the diagonal embedding of $SL_n(\mathbb{Z})$ in $\prod_p SL_n(p)$. This is clearly injective and the image is dense since for every $m \in \mathbb{Z}$ the canonical projection $SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/m\mathbb{Z})$ is onto. This is a special case of **strong approximation** which can be established here by elementary methods. □

Remarks 4.6.

(1): The group $\bigoplus_p SL_n(p)$ is an amenable group dense in K_n but is not finitely generated.

(2): K_2 does not have a dense subgroup with property T. If Γ were such a group then Γ would be separated by finitely many two dimensional representations as in the proof of proposition 4.5. But by a result of Zimmer every subgroup of $SL_2(K)$ with property T is finite (cf. [L1, thm. 3.4.7]). On the other hand K_2 does have a dense subgroup with property (τ), i.e. all **finite** representations are bounded away from the trivial one (equivalently, Γ has property (τ) if all of its finite quotients with respect to a fixed generating set are an expander family, see [L1,§4]). Such a group is $SL_2(2) \times SL_2(\mathbb{Z}[\frac{1}{2}])$, see [L1, example 4.3.3E].

(3): We do not know whether or not K^p contains a dense subgroup with property T. This appears to be quite unlikely.

The known examples of discrete groups Γ with property T are all obtained from lattices in semisimple Lie groups G which have property T. In all cases where G has property T it has been shown that all lattices Γ are arithmetic, [M3] for groups with rank ≥ 2 and [GS] for $Sp(n,1)$ and F_4. These cases will now considered separately.

(i): When Γ is a lattice in a s.s Lie group of rank ≥ 2 Serre [Se] conjectured (and his conjecture has been established in most cases) that there is an affirmative answer to the congruence subgroup problem (CSP). This in turn implies that they cannot be mapped densely into K^p.

(ii): Lattices in rank 1 Lie groups like $Sp(n,1)$ and F_4 have many infinite quotient groups [Gr] which give many more examples of exotic groups with property T. Nonetheless, the issue as to what kind of finite quotients those groups have is not so clear. Serre conjectured that such lattices should have a negative answer to the CSP. However, in light of the recent proof of super rigidity for these groups it is not unreasonable to modify his conjecture so that these lattices also satisfy the CSP. (even though they have many infinite normal groups). If this is the case then indeed also this kind of groups with property T cannot be densely embedded in $\prod_p SL_n(p)$.

5. Open problems and remarks

As we have already said in the introduction, the fundamental problem that motivated the work in this paper is whether the expander property for a family of finite groups is a property of the groups themselves or does it also depend upon the choice of generators. The contrasting results of §2 and §3 about expander and non expander families illuminate same aspects of the following open problem:

Problem 5.1. Let $\{G_i\}$ be a family of finite groups with two sets of generators $\{\Sigma_i\}$, $\{\Sigma_i'\}$ and the size of all them being bounded by a fixed constant k. Suppose that $\{G_i, \Sigma_i\}$ is an expander family, i.e. the Cayley graphs $X(G_i, \Sigma_i)$ are expanders, is also $\{G_i, \Sigma_i'\}$ an expander family.

We don't know of any family of groups that is an expander with respect to one set of generators but not with respect to another. This is rather surprising in view of the fact that in the expander families of §2 the choice of generators was crucial. In the other direction we also don't know of **even one** family of groups that is an expander family with respect to all bounded sets of generators. A natural candidate for such a family would be

$$\{SL_2(p) : p \text{ a prime}\}.$$

Computations carried out in Laffery-Rockmore [LR] suggest indeed that this might even be an expander family even if one were to choose for each p the "worst" possible generating set with only two elements. One consequence of the expander property is that the diameters of the graphs in question are logarithmic in the size of the graph. Here is weaker open question:

Problem 5.2. Is

$$\text{diameter}(X(SL_2(p), \Sigma)) \leq C \cdot \log p$$

for some fixed constant C and **all** sets of generators Σ? Is it true at least for random sets of k-generators where k is fixed and $p \to \infty$.

On the other side of the coin, we did present in §3 at least some examples of families of groups which are not expanders with respect to any sets of generators of bounded size.

A problem related to 5.1 but concerning infinite groups is as follows. If Γ is an infinite group with property T, and Σ is a finite set of generators then it is known that $K(\Gamma, \Sigma) > 0$ where $K(\Gamma, \Sigma)$ was defined in 3.4. Very little is known about exact values of $K(\Gamma, \Sigma)$ (see [Bu]) and even the following qualitative question seems to be open:

Problem 5.3. Let Γ be a group with property T. Is

$$\inf\{K(\Gamma, \Sigma) : \Sigma \text{ a set of generators for } \Gamma\}$$

strictly positive?

Since property T yields expanders while amenable groups give non expanders the following strategy, for answering problem 5.1 in the negative, suggests itself: find an infinite family of finite groups which are quotients of a finitely generated amenable group A and also a quotients of a group B with property T. For example, if K were a profinite group and A and B were finitely generated dense in K, with A amenable and B with property T then we would achieve this goal. In fact we don't believe that this is possible. Indeed proposition 4.4 above and 5.5 below lend support to the following:

Conjecture 5.4. Let K be a compact group. If A and B are both finitely generated subgroups dense in K with A amenable and B having property T then K is finite.

Proposition 5.5. *Conjecture 5.4 is valid if K is a compact Lie group, or more generally K is a linear group over some field.*

Proof. If K is a compact Lie group it is linear over $\mathbb{R}$ so that in both cases we have $K \hookrightarrow GL_n(F)$ for some field F. From the Tits alternative [T] $A \subset G$ either contains a free group on two generators or is a finite extension of a solvable group. Since A is amenable the first possibility is ruled out and thus A is virtually solvable and since it is dense in K, K is also virtually solvable which implies that B is virtually solvable. A subgroup of finite index of a group with property T has property T and if a solvable group has property T it must be finite whence B itself is finite and thus so is its closure K. □

Another approach is via invariant measures. If K, a compact group, has a dense subgroup with property T then K has an affirmative answer to the Banach-Ruziewicz problem. That means that Haar measure μ on K is the unique finitely additive measure on K, defined on μ-measurable sets, that is invariant under all group translations. (see [L1] and the references there). On the other hand, for a countable amenable group $A \subset K$ we can find many finitely additive measures on K that are invariant under A (this follows easily from the existence of approximately invariant subsets in K, cf. [Sc]). Unfortunately invariance under A doesn't imply invariance under K for finitely additive measures so this doesn't settle conjecture 5.4. In light of examples 4.1 and 4.3 the following test cases are of interest:

Problem 5.6. Let K be one of the groups:

$$\prod_{n=2}^{\infty} SL_n(p), \quad \prod_{n=1}^{\infty} S_n$$

and let μ denote Haar measure on K. Does K have a finitely additive probability measure defined on the μ-measurable sets which is different from μ.

An affirmative reply to 5.6 would imply that K doesn't contain any dense subgroups with property T. This fact alone would also follow from conjecture 5.4

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