A Simple Riskiness Order Leading to the Aumann–Serrano Index of Riskiness*

Sergiu Hart†

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Abstract

We introduce a simple "riskier than" order between gambles, from which the index of riskiness developed by Aumann and Serrano (2008) is directly obtained.

1 Introduction

A *risky asset* (or "gamble") yields uncertain returns according to some probability distribution; these returns may be positive (gains) or negative (losses). There are cases where one gamble is clearly more "risky" than another. However, in general it is not clear how to compare gambles; moreover, whether one is willing to accept a certain gamble depends on the individual risk-posture (such as the degree of "risk-aversion").

Nevertheless, one can try to quantify the intrinsic *riskiness* of a gamble, that is, assign to each gamble a real number that is a measure of its riskiness. Most importantly, the aim is to do it in an *objective* way which is independent of the specific decision-maker. Just as the "return" of the gamble—its expectation—and the "spread" of the gamble—its standard deviation—

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†Institute of Mathematics, Department of Economics, and Center for the Study of Rationality, The Hebrew University of Jerusalem. *E-mail*: `hart@huji.ac.il` *Web site*: `http://www.ma.huji.ac.il/hart`

depend only on the gamble itself—its outcomes and probabilities—so should the riskiness of the gamble.

Two such recent approaches are the "economic" *index of riskiness* developed by Aumann and Serrano (2008),[1] and the "operational" *measure of riskiness* of Foster and Hart (2007).[2] While different, the Aumann–Serrano index and the Foster–Hart measure nevertheless share many useful properties, in addition to being objective; for example, they are monotonic with respect to first-order stochastic dominance: increasing the gains and decreasing the losses lowers the riskiness[3] (see in particular Section 6.1 in Foster and Hart (2007) for a detailed comparison).

The approach of Aumann and Serrano (2008) (henceforth "**A&S**") is based on their *duality axiom*

> "that, roughly speaking, asserts that less risk-averse individuals accept riskier gambles" [quoted from the abstract of A&S].

While on the face of it this axiom seems very reasonable, on closer inspection it turns out to be relatively complex and thus not entirely straightforward. In particular, it involves *two* decision-makers, two gambles, and the index itself.

In this paper we provide an alternative approach that leads to the same index of riskiness. What we do is define a simple *"riskiness" order* on gambles, and then show that this order is represented by the Aumann–Serrano index: a gamble g is riskier than another gamble h (according to our order) if and only if the riskiness index of g is higher than the riskiness index of h. Moreover, this representation is unique, in the sense that an index represents this order if and only if it is a strictly increasing function of the Aumann–Serrano index. We thus follow the standard route of decision theory and consumer theory, which starts with an order on outcomes—a "preference" order—and then represents it by a numerical index—a "utility function."

[1] This index was used in the technical report of Palacios-Huerta, Serrano, and Volij (2004); see the footnote on page 810 of Aumann and Serrano (2008).

[2] For a discussion of some of the earlier work, see Section VIII in Aumann and Serrano (2008) and Section 6 in Foster and Hart (2007).

[3] The standard deviation, often used to measure riskiness, is easily seen to violate this condition.

There are two main advantages to our approach. First, our basic postulate is more transparent: the riskiness order is simpler than the duality axiom. And second, in contrast to A&S, no further assumptions are needed to obtain the index in our setup: neither homogeneity (cf. Theorem A in A&S) nor continuity together with monotonicity (cf. Theorem D in A&S).[4]

Finally, there are certain cases where one gamble g is clearly less risky than another gamble h. One is where some values of g are replaced in h by smaller values (i.e., gains decrease and losses increase). Another is where some values of g are replaced in h by lotteries whose expectations are those values ("mean-preserving spreads"). These two cases correspond to "first-order stochastic dominance" and "second-order stochastic dominance," respectively; see Hadar and Russell (1969), Hanoch and Levy (1969), Rothschild and Stiglitz (1970, 1971). Since the Aumann–Serrano index is monotonic with respect to stochastic dominance of either kind (see Section V.C in A&S), it follows that our riskiness order is an *extension* of both stochastic dominance orders[5]—and, in fact, a *complete* extension.[6]

In Section 2 we present the model and the Aumann–Serrano index. In Section 3 we introduce the riskiness order on gambles and state our main result. A comparison with the A&S approach and their duality axiom is provided in Section 4. The proofs and additional comments are relegated to Section 5.

2 The Model

The model is as in A&S.

A *gamble* g is a real-valued random variable with positive expectation

[4]Section X.N of A&S shows that these additional requirements are indeed necessary in their approach.

[5]I.e., if g is stochastically dominated by h, either first-order or second-order, then g is riskier than h.

[6]See Corollary 2 below. Stochastic dominance (of either kind) provides only a *partial* order between gambles: given two gambles g and h, in general neither one stochastically dominates the other. In contrast, the riskiness order turns out to be *complete*: for any two gambles g and h, either g is riskier than h, or h is riskier than g.

and some negative values (i.e.,[7] $\mathbf{E}[g] > 0$ and $\mathbf{P}[g < 0] > 0$); for simplicity, assume that g takes finitely many values. Let $\mathcal{G}$ denote the collection of all such gambles.

An *agent* (or decision-maker) is characterized by a von Neumann and Morgenstern utility function[8] $u : \mathbb{R} \to \mathbb{R}$ that is assumed to be strictly increasing, strictly concave, and twice continuously differentiable (i.e., of class C^2); let $\mathcal{U}$ denote the collection of all such agents.[9] An agent $u \in \mathcal{U}$ *accepts* a gamble g at wealth level w if $\mathbf{E}[u(w+g)] > u(w)$,and *rejects* g otherwise; i.e, u accepts g at w if and only if the expected utility from accepting is higher than from rejecting.

We restrict ourselves to agents $u \in \mathcal{U}$ whose *decisions are monotonic* relative to wealth ("*monotonic*" agents for short): if u accepts the gamble g at wealth w then u accepts g also at any higher wealth $w' > w$. Letting $\rho_u(w) := -u''(w)/u'(w)$ denote the coefficient of absolute risk aversion of u at w (cf. Arrow 1965, 1971, and Pratt 1964), u is a monotonic agent if and only if ρ_u is a nonincreasing function of the wealth: $\rho_u(w') \leq \rho_u(w)$ for all $w' > w$ (cf. Pratt 1964, Yaari 1969, Dybvig and Lippman 1983). Let $\mathcal{U}_D \subset \mathcal{U}$ denote the collection of all monotonic agents; this class of utilities is known as *DARA*, which stands for "Decreasing Absolute Risk Aversion." A subclass of $\mathcal{U}_D$ that turns out to be particularly relevant here consists of the *CARA* (for "Constant Absolute Risk Aversion") utilities v_α for all $\alpha > 0$, where $v_\alpha(w) := -\exp(-\alpha w)$ for all w (and thus $\rho_{v_\alpha}(w) = \alpha$ for all w); let $\mathcal{U}_C := \{v_\alpha : \alpha > 0\} \subset \mathcal{U}_D$ denote this class of utilities.

For any gamble $g \in \mathcal{G}$, the equation $\mathbf{E}[\exp(-\alpha g)] = 1$ has a unique solution $\alpha > 0$, which we denote $\alpha^*(g)$; the *index of riskiness* R of Aumann–Serrano is then given by

$$R(g) := \frac{1}{\alpha^*(g)} \tag{1}$$

(see A&S, Theorems A and B). Thus among the CARA agents $v_\alpha \in \mathcal{U}_C$, the one with coefficient $\alpha = \alpha^*(g)$ is always indifferent between accepting and

[7] $\mathbf{E}$ and $\mathbf{P}$ denote expectation and probability, respectively.

[8] $\mathbb{R}$ denotes the real line.

[9] We do not distinguish between an "agent" and its "utility function" u; the two terms are used interchangeably.

rejecting g, whereas all those with $\alpha > \alpha^*(g)$ always reject g, and all those with $\alpha < \alpha^*(g)$ always accept g; here "always" stands for "at all wealth levels w."

3 The Riskiness Order

Our approach follows the standard route of decision theory and consumer theory: we start with an "order," and then represent it by an "index" (just as a preference order is represented by a utility function). Specifically, we introduce a "riskiness" order on gambles, and show that it is uniquely represented (up to monotonic transformations) by the Aumann–Serrano index of riskiness.

The order[10] on gambles is defined as follows. We say that an agent $u \in \mathcal{U}$ *totally rejects* a gamble $g \in \mathcal{G}$ if u rejects g at *all* wealth levels[11] w. For two gambles $g, h \in \mathcal{G}$, we say that[12] *g is riskier than h*, written $g \succsim h$, if any monotonic agent $u \in \mathcal{U}_D$ that totally rejects h also totally rejects g. The intuition is that a riskier gamble is rejected more often. However, only when the rejection of h is "total"—a strong premise—do we require that the same hold for g; the definition of $\succsim$ may thus be viewed as a sort of minimal requirement on riskiness.

Our main result is:

Theorem 1 *The riskiness order is represented by the Aumann–Serrano index of riskiness: for any $g, h \in \mathcal{G}$,*

$$g \succsim h \quad \textit{if and only if} \quad R(g) \geq R(h).$$

Immediate consequences are:

[10] An *order* (sometimes called "preorder") $\succsim$ is a binary relation that is reflexive (i.e., $g \succsim g$ for any g) and transitive (i.e., $g \succsim h$ and $h \succsim k$ imply $g \succsim k$, for any g, h, k). It is a *complete* order if any pair g, h can be compared (i.e., either $g \succsim h$ or $h \succsim g$ holds for any g, h).

[11] Unlike the approach of Foster and Hart (2007) which is based on *critical wealth* levels that distinguish between rejection and acceptance, the A&S setup hinges on decisions that are *uniform in wealth*; see Section 6.1 in Foster and Hart (2007).

[12] "g is at least as risky as h" would be more precise, but cumbersome.

Corollary 2 *(i) The riskiness order is a complete order on $\mathcal{G}$.*

(ii) A real-valued function Q represents the riskiness order if and only if Q is ordinally equivalent to R.

(iii) A real-valued function Q that is positively homogeneous of degree one represents the riskiness order if and only if Q is a positive multiple of R.

Thus, while the riskiness order appears by definition to be only a partial order (i.e., not every pair of gambles may be compared), (i) states that it is in fact complete: for any $g, h \in \mathcal{G}$, either $g \succsim h$ or $h \succsim g$. Next, a real-valued function $Q : \mathcal{G} \to \mathbb{R}$ defined on the collection of gambles *represents the riskiness order*[13] $\succsim$ if, for any $g, h \in \mathcal{G}$, we have $g \succsim h$ if and only if $Q(g) \geq Q(h)$. What (ii) says is that each such Q is a monotonic transformation of R; i.e., there exists a strictly increasing function ϕ such that $Q(g) = \phi(R(g))$ for all gambles $g \in \mathcal{G}$. If moreover Q is positively homogeneous of degree one—i.e., $Q(\lambda g) = \lambda Q(g)$ for every $\lambda > 0$ and $g \in \mathcal{G}$—then (iii) says that there exists $c > 0$ such that $Q(g) = cR(g)$ for all $g \in \mathcal{G}$.

Thus we obtain the Aumann–Serrano index of riskiness R directly from the riskiness order, without any further postulates; in contrast, A&S have to appeal in addition either to homogeneity, or to continuity together with monotonicity with respect to first-order stochastic dominance (see Theorems A and D in A&S).

The proof of Theorem 1 is relegated to Section 5.1; the proof of Corollary 2 is completely standard and thus omitted. Finally, the need for monotonic agents is clarified in Section 5.3.

4 The Duality Axiom and the Riskiness Order

The approach of A&S is based on the following duality axiom. They call an agent $i \in \mathcal{U}$ *uniformly no less risk-averse than* agent $j \in \mathcal{U}$, written $i \trianglerighteq j$, if whenever agent i accepts a gamble g at some wealth w, agent j accepts

[13] Just as a utility function represents a preference order.

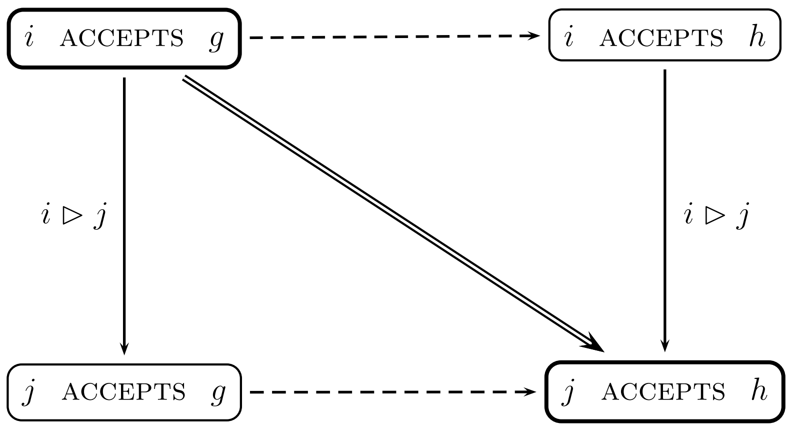

Figure 1: The Duality axiom

that gamble g at any wealth w'; and i is *uniformly more risk-averse than* j, written $i \rhd j$, if $i \trianglerighteq j$ and $j \ntrianglerighteq i$. The duality requirement on a real-valued function Q defined on gambles is that, for any two gambles $g, h \in \mathcal{G}$, any two agents $i, j \in \mathcal{U}$, and any wealth w:

$$\text{Duality.} \quad \left\{ \begin{array}{c} i \rhd j, \\ Q(g) > Q(h), \text{ and} \\ i \text{ accepts } g \text{ at } w \end{array} \right\} \text{ implies } \Big\{ j \text{ accepts } h \text{ at } w \Big\}.$$

While the duality axiom seems appealing, it is relatively complex, and its rationale is not entirely straightforward. It says that if a more risk-averse agent i accepts g, then a less risk-averse agent j should accept a less risky gamble, where "less risky" is taken according to the yet-to-be-determined index Q. Thus one deduces something about the pair (j, h) (namely, that j accepts h) from an assumption on the pair (i, g) (namely, that i accepts g); see Figure 1. This requires replacing one agent with another, and at the same time also one gamble with another (represented by the diagonal arrow in Figure 1). Now the premise that $i \rhd j$ allows one to replace the agents while keeping the gamble fixed (these are the vertical arrows): from (i, g) to (j, g), or, alternatively, from (i, h) to (j, h) (indeed, if i accepts g then j accepts g, and the same holds for h). What is missing is a reason to replace the gambles while keeping the agent fixed (i.e., the horizontal arrows): from (i, g) to (i, h), or from (j, g) to (j, h); this is where the premise $Q(g) > Q(h)$ comes in.

This suggests that one try to deal directly with these "horizontal" implications and dispense with the rest—which is precisely what our riskiness-order approach does.

To see this formally we define another order on gambles: $g \succsim_{AS} h$ if and only if for any two agents $i, j \in \mathcal{U}$ with $i \rhd j$ and any wealth w, if i accepts g at w then j accepts h at w. The duality axiom can then be restated as: for any gambles $g, h \in \mathcal{G}$,

$$\text{Duality (restated).} \quad Q(g) > Q(h) \text{ implies } g \succsim_{AS} h. \tag{2}$$

We have

Proposition 3 *For any two gambles $g, h \in \mathcal{G}$,*

$$g \succsim h \text{ if and only if } g \succsim_{AS} h.$$

Thus, the two orders $\succsim$ and $\succsim_{AS}$ in fact turn out to be identical; after all, our riskiness order $\succsim$ essentially provides the "missing" horizontal implications in Figure 1 (see Section 5.2 for a precise proof). In view of Theorem 1, it follows that

Corollary 4 *A real-valued function Q on gambles represents the $\succsim_{AS}$ order if and only if Q is ordinally equivalent to R.*

However, Corollary 4 does not yet yield the result of A&S, since the duality axiom is weaker than the requirement that Q represents the $\succsim_{AS}$ order (i.e., that $Q(g) \geq Q(h)$ *if and only if* $g \succsim_{AS} h$; compare (2))—which explains why A&S need to appeal to additional axioms: homogeneity (in their Theorem A), or continuity together with monotonicity with respect to first-order stochastic dominance (in their Theorem D; see also Section X.N there for counterexamples without the additional conditions). In Section 5.2 below we will also provide simple alternative proofs for these two results of

A&S, based on Proposition 3 (whose proof is also simple). Finally, Remark (3) in Section 5.3 further clarifies the need for two agents in the duality axiom, and the role of monotonic agents.

In summary, what we have shown is that at the basis of the duality axiom lies our riskiness order, and that using it simplifies and streamlines the Aumann–Serrano approach.

5 Proofs

5.1 The Riskiness Order

Here we prove Theorem 1 of Section 3.

Following Pratt (1964), the *risk premium* $\pi_u(w, g)$ of agent $u \in \mathcal{U}$ for the gamble $g \in \mathcal{G}$ at wealth w is uniquely determined by the equation

$$\mathbf{E}[u(w+g)] = u(w + \mathbf{E}[g] - \pi_u(w,g)). \tag{3}$$

Since u is strictly increasing, we get

Lemma 5 *u rejects g at w if and only if $\pi_u(w,g) \geq \mathbf{E}[g]$.*

The basic result we will use is

Proposition 6 *Let $u_1, u_2 \in \mathcal{U}$ be two utility functions with absolute risk aversion coefficients ρ_1 and ρ_2 and risk premium functions π_1 and π_2, respectively, and let $-\infty \leq a < b \leq \infty$. Then:*

(i) $\rho_1(w) \geq \rho_2(w)$ for every $w \in [a,b]$ if and only if $\pi_1(w,g) \geq \pi_2(w,g)$ for every w and $g \in \mathcal{G}$ with $w + g \in [a,b]$.

(ii) $\rho_1(w) > \rho_2(w)$ for every $w \in [a,b]$ if and only if $\pi_1(w,g) > \pi_2(w,g)$ for every w and $g \in \mathcal{G}$ with $w + g \in [a,b]$.

Proof. Pratt (1964, Theorem 1). □

Recall that, for $\alpha > 0$, the CARA agent $v_\alpha \in \mathcal{U}_C$ satisfies $\rho_{v_\alpha}(w) = \alpha$ for every w. The definition of $\alpha^* \equiv \alpha^*(g)$ implies that $\mathbf{E}[v_{\alpha^*}(w+g)] = v_{\alpha^*}(w)$, or $\pi_{v_{\alpha^*}}(w,g) = \mathbf{E}[g]$ (for any w). Therefore

Lemma 7 *Let $g \in \mathcal{G}$. If $\beta \geq \alpha^*(g)$ then v_β rejects g at all w, and if $\beta < \alpha^*(g)$ then v_β accepts g at all w.*

Proof. Lemma 5 and Proposition 6. □

Proposition 8 *A monotonic agent $u \in \mathcal{U}_D$ rejects a gamble g at all wealths w if and only if $\rho_u(w) \geq \alpha^*(g)$ for all w.*

Proof. Let $\alpha^* \equiv \alpha^*(g)$. If $\rho_u(w) \geq \alpha^*$ for all w, then Proposition 6 with $u_1 = u$ and $u_2 = v_{\alpha^*}$ yields $\pi_1(w, g) \geq \pi_2(w, g)$. Lemmata 5 and 7 imply that $\pi_2(w, g) \geq \mathbf{E}[g]$, and so $\pi_1(w, g) \geq \mathbf{E}[g]$; thus u rejects g at w (by Lemma 5).

Conversely, if $\rho_u(w_0) < \alpha^*$ for some w_0 then $\rho_u(w) \leq \beta$ for all $w \geq w_0$, where $\beta := \rho_u(w_0) < \alpha^*$ (since ρ_u is a decreasing function). Proposition 6 with $u_1 = v_\beta$ and $u_2 = u$ yields $\pi_1(w, g) \geq \pi_2(w, g)$ for all $w \geq w_0$. Lemmata 5 and 7 (recall that $\beta < \alpha^*$) imply that $\pi_1(w, g) < \mathbf{E}[g]$, and so $\pi_2(w, g) < \mathbf{E}[g]$, for all $w \geq w_0$; therefore (by Lemma 5) u accepts g at all w with $w + g \geq w_0$. □

We can now complete the proof of Theorem 1.

Proof of Theorem 1. By Proposition 8, $u \in \mathcal{U}_D$ totally rejects g if and only if $\inf_w \rho_u(w) \geq \alpha^*(g)$. Therefore the definition of $g \succsim h$ is equivalent to "for every $u \in \mathcal{U}_D$, if $\inf_w \rho_u(w) \geq \alpha^*(h)$ then $\inf_w \rho_u(w) \geq \alpha^*(g)$," which in turn is equivalent to "$\alpha^*(h) \geq \alpha^*(g)$" (for one direction take $u = v_{\alpha^*(h)}$). Thus

$$g \succsim h \quad \text{if and only if} \quad \alpha^*(g) \leq \alpha^*(h),$$

and Theorem 1 follows from (1). □

5.2 The Duality Axiom

Here we provide proofs for the statements of Section 4.

Throughout this section one may work either with general agents in $\mathcal{U}$, or only with monotonic ones in $\mathcal{U}_D$; all the results and proofs hold just as well in either case. The reason is that, on the one hand, we do not use the property that ρ is nonincreasing, and on the other hand, all specific agents appearing in the proofs are in fact CARA agents in $\mathcal{U}_C \subset \mathcal{U}_D$.

Proof of Proposition 3. In view of (1), we need to show that

$$g \succsim_{AS} h \quad \text{if and only if} \quad \alpha^*(g) \leq \alpha^*(h).$$

We will use the following characterization of "uniformly more risk-aversion" $\trianglerighteq$ (see (4.1.2) in A&S):

$$i \trianglerighteq j \quad \text{if and only if} \quad \inf_w \rho_i(w) \geq \sup_w \rho_j(w).$$

Assume that $\alpha^*(g) \leq \alpha^*(h)$, and let $i \triangleright j$ be such that i accepts g at w. Let β satisfy $\inf \rho_i \geq \beta \geq \sup \rho_j$; then $i \trianglerighteq v_\beta \trianglerighteq j$. Therefore v_β accepts g at w (since i does so and $i \trianglerighteq v_\beta$), and so by Lemma 7 we have $\beta < \alpha^*(g)$. From $\alpha^*(g) \leq \alpha^*(h)$ it follows that $\beta < \alpha^*(h)$; hence v_β accepts g at w (by Lemma 7 again), and so does j (since $v_\beta \trianglerighteq j$).

Conversely, assume that $\alpha^*(g) > \alpha^*(h)$. Let β satisfy $\alpha^*(g) > \beta > \alpha^*(h)$; then $v_\beta \triangleright v_{\alpha^*(h)}$, but v_β accepts g whereas $v_{\alpha^*(h)}$ rejects h (at any w, by Lemma 7), and so $g \succsim_{AS} h$ does not hold. $\square$

Remark. The proof of Proposition 3 shows that $\succsim_{AS}$ would not be affected if i, j were restricted to being monotonic agents (i.e., $i, j \in \mathcal{U}_D$), or if $i \triangleright j$ were replaced with $i \trianglerighteq j$; the same therefore applies to the duality axiom (recall its "restated" version (2)).

Based on Proposition 3 we can provide simple proofs for the main results—Theorems A and D—of A&S (2008).

Proposition 9 *(i) A continuous and first-order monotonic function $Q : \mathcal{G} \to \mathbb{R}$ satisfies duality if and only if Q is ordinally equivalent to R.*

(ii) A positively homogeneous of degree one function $Q : \mathcal{G} \to \mathbb{R}$ satisfies duality if and only if Q is a positive multiple of R.

Proof. Theorem 1 and Proposition 3 show that the duality axiom (see (2)) is equivalent to

$$Q(g) > Q(h) \quad \text{implies} \quad R(g) \geq R(h), \tag{4}$$

or

$$R(g) > R(h) \quad \text{implies} \quad Q(g) \geq Q(h) \tag{5}$$

(take the negation of (4) and interchange g and h). Thus R satisfies duality; since it is clearly homogeneous, continuous, and first-order monotonic (cf. Section V in A&S), it remains to prove its uniqueness—"ordinal" and "cardinal," respectively—in (i) and (ii). This will readily follow once we show that in each case we have

$$Q(g) > Q(h) \quad \text{if and only if} \quad R(g) > R(h). \tag{6}$$

To prove this in case (i), assume that $Q(g) > Q(h)$; then $Q(g+\varepsilon) > Q(h)$ for small enough $\varepsilon > 0$ (by continuity of Q), implying that $R(g+\varepsilon) \geq R(h)$ (by (4)), and so $R(g) > R(h)$ (by monotonicity of R we have $R(g) > R(g+\varepsilon)$). Conversely, assume that $R(g) > R(h)$; then $R(g+\varepsilon) > R(h)$ for small enough $\varepsilon > 0$ (by continuity of R), implying that $Q(g+\varepsilon) \geq Q(h)$ (by (5)), and so $Q(g) > Q(h)$ (by monotonicity of Q). This completes the proof of (6) in case (i).

The proof in case (ii) is similar. Assume that $Q(g) > Q(h)$; using the homogeneity of Q we get $Q((1-\varepsilon)g) = (1-\varepsilon)Q(g) > Q(h)$ for small enough $\varepsilon > 0$, implying that $(1-\varepsilon)R(g) = R((1-\varepsilon)g) \geq R(h)$ (by (4) and the homogeneity of R), and so $R(g) > R(h)$. Conversely, assume that $R(g) > R(h)$; then $(1-\varepsilon)R(g) = R((1-\varepsilon)g) > R(h)$ for small enough $\varepsilon > 0$, implying that $(1-\varepsilon)Q(g) = Q((1-\varepsilon)g) \geq Q(h)$ (by (5) and the homogeneity of Q), and so $Q(g) > Q(h)$. This completes the proof of (6) in case (ii). □

The role of the additional assumptions in A&S—homogeneity, or continuity and monotonicity—becomes clear now: they are needed to go from (4) to (6).

5.3 Nonmonotonic Agents

In this section we show that one cannot dispense with the requirement that only agents u with monotonic decisions (i.e., $u \in \mathcal{U}_D$) be used in the definition of the riskiness order $\succsim$. Indeed, we provide an example of a *nonmonotonic* agent $u \in \mathcal{U} \setminus \mathcal{U}_D$ and two gambles $g, h \in \mathcal{G}$ such that g is riskier than h (i.e., $g \succsim h$), and u totally rejects h but does not totally reject g.

Example.[14] Since the utility function $u \in \mathcal{U}$ is determined (up to positive affine transformations, which do not affect decisions) by its absolute risk-aversion function $\rho(x) = -u''(x)/u'(x)$ (indeed, $u = \int \exp(-\int \rho))$, it suffices to specify the function ρ. Thus, let $\rho : \mathbb{R} \to \mathbb{R}$ with $\rho(x) > 0$ for every $x \in \mathbb{R}$ satisfy the following two conditions:

(a) there exists $\delta > 0$ such that $\rho(m + y) < 1$ for every integer[15] $m \in \mathbb{Z}$ and every $|y| \leq \delta$; and

(b) $\int_x^{x+1} \rho(y)\,\mathrm{d}y > 1$ for every $x \in \mathbb{R}$.

Informally, (b) says that ρ is on the average larger than 1 on any unit interval, and (a) says that it is smaller than 1 infinitely often, in small intervals around the integers. For an explicit example, take $\rho(x) = 3/2 + \sin(2\pi x - \pi/2)$.

Next, the two gambles g and h are defined as follows. The gamble g takes the values δ and $-\delta$ (where δ is given in (a)) with probabilities $e^\delta/(1 + e^\delta)$ and $1/(1+e^\delta)$, respectively, and the gamble h takes the values 1 and -1 with probabilities $e/(1 + e)$ and $1/(1 + e)$, respectively. It is easy to verify that $\alpha^*(g) = \alpha^*(h) = 1$, and thus $R(g) = R(h) = 1$ and so in particular $g \succsim h$ (see (1) and Theorem 1).

We will now show that u totally rejects h, but u accepts g at all integer wealth levels.

Indeed, for any x

$$\log \frac{u'(x+1)}{u'(x)} = \log u'(x+1) - \log u'(x) = \int_x^{x+1} \frac{u''(y)}{u'(y)}\,\mathrm{d}y = -\int_x^{x+1} \rho(y)\,\mathrm{d}y$$

[14]The example is more complex than needed, since we want to rule out also the possibility discussed in Remark (2) below.

[15]$\mathbb{Z}$ denotes the set of integers (positive, zero, and negative).

which is < -1 by (b). Therefore $u'(x+1) < (1/e)u'(x)$, and so, for any w,

$$\begin{aligned} u(w+1) - u(w) &= \int_w^{w+1} u'(x)\,\mathrm{d}x \\ &< \frac{1}{e}\int_{w-1}^{w} u'(x)\,\mathrm{d}x = \frac{1}{e}\left(u(w) - u(w-1)\right), \end{aligned}$$

which is easily seen to be equivalent to $\mathbf{E}\left[u(w+h)\right] < u(w)$. Therefore, as claimed, h is rejected at all w.

Finally, consider an integer wealth level $m \in \mathbb{Z}$. Since $m+g \in [m-\delta, m+\delta]$ and $\rho(x) < 1 = \alpha^*(g)$ for every $x \in [m-\delta, m+\delta]$ by (a), it follows that u rejects g at m (apply Proposition 6 with $u_1 = v_{\alpha^*(g)}$ and $u_2 = u$, and Lemma 7). □

Remarks. (1) The example is "robust," in the sense that it is preserved under any small enough perturbations (this is due to the inequalities in (a) and (b) being strict); in particular, one can take g and h so that g is strictly riskier than h (i.e., $g \succsim h$ but $h \not\succsim g$).
(2) The example is constructed so that g is accepted not just at a single wealth level (which suffices for g not to be totally rejected), but also at arbitrarily large wealth levels. This implies that, in the definition of the riskiness order $\succsim$, it would not help if one were to replace "total rejection" with "*eventual* total rejection," i.e., rejection from some wealth level on.
(3) The example clarifies the reason that *two* agents are needed in the DUALITY axiom. As we have just seen, for nonmonotonic agents the "horizontal" implications in Figure 1 do not hold. However, having two (possibly nonmonotonic) agents i, j with $i \rhd j$ implies the existence of CARA agents "between" i and j, which are in particular monotonic and thus the "horizontal" implications apply to them (for details see the proof of Theorem A in A&S). Our approach, which considers only monotonic agents, allows us to work with just one agent, and thus dispenses with the need to take two agents i, j with $i \rhd j$.

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