Essay 8

On Prize Games

Sergiu Hart[1]

ABSTRACT. We consider the class of hyperplane coalitional games (H-games): the feasible set of each coalition is a half-space, with a slope that may vary from one coalition to another. H-games have turned out in various approaches to the value of general non-transferable utility (NTU) games. In this paper we introduce a simple model — *prize games* — that generates the hyperplane games. Next, we provide an axiomatization for the Maschler & Owen [4] consistent value of H-games.

1. Introduction

A *non-transferable utility (NTU) game in coalitional form* is given by specifying the set of outcomes that are achievable by each subset of players ("coalition"). (For formal definitions, see Section 2.) A special class is that of the *transferable utility (TU) games*, where each coalition's feasible set consists of all payoff vectors that add up to no more than a given amount. Geometrically, the feasible sets are half-spaces with normal vector $(1, 1, \ldots, 1)$. Economic models usually lead to NTU-games. TU-games arise if there is a commodity ("money") which is freely transferable among the players and such that one unit of it has exactly the same (marginal) utility to everyone. Single output production models also lead to TU-games.

Another class of NTU-games is that of the *hyperplane (H) games*, where the feasible set of each coalition is a half-space; however, the normal vectors may change from one coalition to another. Thus every TU-game is an H-game and every H-game is an NTU-game; both inclusions are strict.

Hyperplane games have turned up in the study of *value* concepts (see Maschler & Owen [4, 5], Hart & Mas-Colell [3] for the "consistent value"; and Hart & Mas-Colell [2, Theorem C] for the "egalitarian values"). In all

[1]Department of Mathematics; Department of Economics; and Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, Givat-Ram, 91904 Jerusalem, Israel.

Dedicated to Michael Maschler on his 65th birthday.

cases, the main "distinction" of the H-games has been the fact that they are technically more tractable than the general NTU-games. However, no models where H-games arise naturally have been presented.

We will introduce here (in Section 3) a simple model—*prize games*—that yields the class of H-games. Each coalition has one indivisible "prize", which can be given to only one of its members. The feasible set corresponds to all "lotteries" over who gets the prize, and each player is assumed to possess a von Neumann & Morgenstern utility function (over the lotteries). (Such a construction—bargaining over the probabilities of getting a prize—has been used by Roth & Malouf [6] in bargaining experiments.)

The classical value concept is the *Shapley value* [7] for TU-games. It has a "natural" extension[2] to H-games: the *consistent H-value* of Maschler & Owen [4] . Section 4 includes a definition of this value and some of its properties. We then present (in Section 5) a simple axiomatization for the consistent H-value. It generalizes Young's [8] characterization of the Shapley value in the TU-case.

A few notations: $\Re$ is the real line. For a finite set A, the number of elements of A is denoted $|A|$; 2^A is the set of all subsets of A; $\Re^A$ is the $|A|$-dimensional Euclidean space with coordinates indexed by the elements of A; $\Re^A_+$ and $\Re^A_{++}$ are its non-negative and positive orthants, respectively. The origin $(0,0,\ldots,0)$ of $\Re^A$ is written 0^A or just 0. The unit simplex $\{x \in \Re^A_+ : \sum_{i\in A} x^i = 1\}$ of $\Re^A$ is denoted $\Delta(A)$; it is the set of all probability distributions over the set A. For sets, $A \setminus B$ denotes set subtraction, $A \subset B$ is used for *weak* inclusion, and $\emptyset$ is the empty set.

2. Preliminaries

This section includes the basic definitions of games in coalitional form.

A *non-transferable utility (NTU) game in coalitional form* is a pair (N, V), where N is the *set of players* and V is the *coalitional* or *worth function*, associating to every *coalition* $S \subset N$ the non-empty set $V(S) \subset \Re^S$ of all *feasible payoff vectors* for S. An element $x = (x^i)_{i\in S}$ of $V(S)$ is interpreted as follows: there is a feasible outcome for the coalition S whose utility to each player $i \in S$ equals x^i. We will assume that the origin 0^S of $\Re^S$ belongs to $V(S)$ for all S; this is just a convenient normalization.

Two special classes of NTU-games are as follows.

---

[2]It generalizes both Shapley's formula of "expected marginal contribution in a random order," as well as the "consistency property" of Hart & Mas-Colell [1].

- *Hyperplane (H) games*, where each feasible set $V(S)$ is a half-space with positive normal; *i.e.*, for each coalition $S \subset N$, there are $\lambda_S \in \Re^S_{++}$ and $c_S \in \Re$ such that $V(S) = \{x \in \Re^S : \lambda_S \cdot x \le c_S\}$.
- *Transferable utility (TU) games*, which are H-games with $\lambda_S = (1, 1, \ldots, 1)$ for all S; *i.e.*, there is a real-valued function $v : 2^N \to \Re$ (with $v(\emptyset) = 0$) such that $V(S) = \{x \in \Re^S : \sum_{i \in S} x^i \le v(S)\}$ for all $S \subset N$. For TU-games, we will write (N, V) or (N, v) interchangeably.

We will call the NTU-game (N, V) *monotonic* if

$$V(S) \times \{0^{T \setminus S}\} \subset V(T) \text{ for all } S \subset T \subset N\ . \tag{2.1}$$

That is, a feasible payoff vector for S completed by 0's to the players in $T \setminus S$ yields a feasible payoff vector for T. For TU-games, this becomes $v(S) \le v(T)$ for all $S \subset T \subset N$.

3. Prize Games

In this section we will present the model of "prize games", which generate the hyperplane games.

Consider the following setup, which will be called a *(coalitional) prize model*. There is a finite set of players N. Each coalition $S \subset N$ has a "prize" ζ_S. The prize ζ_S is indivisible, and only one member of S can receive it. Moreover, there are no means of utility transfers between the players. Let '0' stand for "no prize", and let $Z := \{\zeta_S : S \subset N\} \cup \{\text{'0'}\}$ be the set of all outcomes. Each player $i \in N$ has a von Neumann & Morgenstern utility function $u^i : \Delta(Z) \to \Re$; recall that $\Delta(Z)$ is the set of probability distributions ("lotteries") over[3] Z. We normalize u^i so that the utility of getting no prize is 0; *i.e.*, $u^i(\text{'0'}) = 0$. For each $i \in S \subset N$, let $z^i_S := u^i(\zeta_S)$ be player i's utility for the prize of S. We assume that $z^i_S \ge 0$ for all i and S : no prize is worse than "no prize". The prize model is actually specified by N and[4] $z := ((z^i_S)_{i \in S})_{S \subset N} \in \prod_{S \subset N} \Re^S_+$; we will denote it (N, z).

A prize model (N, z) generates a game in coalitional form as follows. The feasible set of each coalition $S \subset N$ consists of all lotteries over which player i in S gets the prize of S; *i.e.*, each $p = (p^i)_{i \in S} \in \Delta(S)$ yields the payoff vector $x = (x^i)_{i \in S}$ with $x^i = p^i z^i_S + (1 - p^i)0 = p^i z^i_S$ for all $i \in S$. Let

[3]Actually, player i's utility needs to be defined only over the prizes of the coalitions containing i; we use the bigger set Z for simplicity of notation.

[4]Such a z is called a *payoff configuration*; it associates a payoff vector to each coalition.

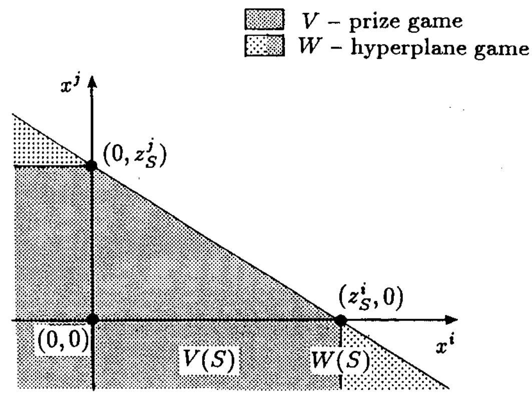

Figure 3.1

$V(S)$ be the comprehensive hull of all these payoff vectors (assume "free disposal"), *i.e.*,

$$V(S) := \{x \in \Re^S : \text{ there is } p \in \Delta(S) \text{ such that } x^i \le p^i z_S^i \text{ for all } i \in S\}.$$

We will call (N, V) the *coalitional prize game* generated by the prize model (N, z).

It is now easy to see that if one considers only non-negative payoffs, then

the class of prize games and the class of hyperplane games coincide

(see Figure 3.1). That is, if (N, V) is a prize game, then there is an H-game (N, W) such that $V(S) \cap \Re_+^S = W(S) \cap \Re_+^S$ for all S (take a hyperplane through the $|S|$ points $\{(0, \ldots, 0, z_S^i, 0, \ldots, 0)\}_{i \in S}$.) Conversely, for every hyperplane game (N, W), there is a prize game (N, V) with $V(S) \cap \Re_+^S = W(S) \cap \Re_+^S$ for all S (define z_S^i by the condition that $(0, \ldots, 0, z_S^i, 0, \ldots, 0)$ lies on the boundary of $W(S)$). Note that if the origin 0^S is an interior point of $V(S)$ for all S or, equivalently, if all the z_S^i are positive rather than just non-negative, then the above correspondence is one-to-one. Moreover, the monotonicity condition (2.1) translates into $z_S^i \le z_T^i$ for all $i \in S \subset T \subset N$: each player prefers the prize of a larger coalition to that of a smaller one. And finally, a TU-game obtains whenever the utilities of all players coincide; *i.e.*, $z_S^i = z_S^j$ for all $i, j \in S \subset N$.

Thus the coalitional prize setup does indeed provide a simple model for the class of hyperplane games.

4. Value

The *value* of a game is usually interpreted as an a priori evaluation of the outcome of the game. It may be viewed as the utility of each player of participating in the game. (For a recent *non-cooperative* model for the value see Hart & Mas-Colell [3]). The classical value concept is the *Shapley* TU-*value* [7].

Let (N, v) be a TU-game. For each player $i \in N$ and each coalition S containing i, let $D^i(S, v) := v(S) - v(S \setminus \{i\})$ be player i's *marginal contribution* to the coalition S. Then the Shapley value of player i in the game (N, v) is

$$\mathrm{Sh}^i(N, v) = (1/n!) \sum_{\pi \in \Pi(N)} D^i(P_\pi^i \cup \{i\}, v), \tag{4.1}$$

where $n := |N|$ is the number of players; $\Pi(N)$ is the set of all $n!$ orders[5] on N; and $P_\pi^i := \{j \in N : \pi^{-1}(j) < \pi^{-1}(i)\}$ is the set of players preceding i in the order $\pi \in \Pi(N)$. Equivalently, endow $\Pi(N)$ with the uniform distribution (*i.e.*, each one of the $n!$ orders on N has equal probability $1/n!$), and let $\mathbf{S}$ be the random coalition $P_\pi^i \cup \{i\}$. (Thus, every $S \subset N$ with $S \ni i$ has probability $(s-1)!(n-s)!/n!$, where $s := |S|$.) Then

$$\mathrm{Sh}^i(N, v) = \mathcal{E}[D^i(\mathbf{S}, v)] \tag{4.2}$$

where $\mathcal{E}$ denotes expectation. We denote the vector $(\mathrm{Sh}^i(N, v))_{i \in N} \in \Re^N$ by $\mathrm{Sh}(N, v)$.

The Shapley value has been extended to the class of hyperplane games by Maschler & Owen [4] as follows: Let (N, V) be a hyperplane game and let $\pi \in \Pi(N)$ be an order on N. The *marginal contributions of the players in the order* π, denoted $d_\pi^i \equiv d_\pi^i(N, V)$ (for all $i \in N$), are defined inductively by

$$d_\pi^i := \max\{x^i : (x^i, y_{(i)}) \in V(P_\pi^i \cup \{i\})\}, \tag{4.3}$$

where $y_{(i)} := (d_\pi^j)_{j \in P_\pi^i}$ is the vector of the marginal contributions of all players preceding i in the order π (already defined by induction). That is, d_π^i is the most that player i can get after each preceding player j got his own[6] d_π^j. For $t = 1, 2, \ldots, n$, let $Q_t \equiv Q_t(\pi) := \{\pi(1), \pi(2), \ldots, \pi(t)\}$ be

[5] An *order* on N is a one-to-one function $\pi : \{1, 2, \ldots, n\} \rightarrow N$; *i.e.*, $(\pi(1), \pi(2), \ldots, \pi(n))$ is a permutation of N.

[6] This suggests that "(marginal) surplus generated by i," or "value added by i" (as in, for instance, "value added tax") are perhaps better names for d^i. We keep the name "marginal contribution" since it is the one that has been classically used for TU-games.

the set of the first t players in the order π. The definition above is plainly equivalent to

$$(d_\pi^k)_{k\in Q_t} \in \partial V(Q_t), \qquad \text{for all } t = 1, 2, \ldots, n, \tag{4.4}$$

where $\partial V(S)$ stands for the boundary of the set $V(S)$ (this is the set of Pareto efficient vectors in $V(S)$). For example, if $N = \{1, 2, \ldots, n\}$ and π is the natural order $(1, 2, \ldots, n)$, then $d_\pi^1 = \max\{x^1 : x^1 \in V(1)\}$; $(d_\pi^1, d_\pi^2) \in \partial V(12)$ determines d_π^2; $(d_\pi^1, d_\pi^2, d_\pi^3) \in \partial V(123)$ determines d_π^3; and so on (see Figure 4.1, for the orders $(1,2,3)$ and $(2,1,3)$).

The *consistent H-value* φ of Maschler & Owen [4] may be defined by:

$$\varphi^i(N, V) := (1/n!) \sum_{\pi\in\Pi(N)} d_\pi^i(N, V) \tag{4.5}$$

for all $i \in N$: the value to each player is his average (over all orders) marginal contribution. For TU-games $d_\pi^i(N, v) = v(P_\pi^i \cup \{i\}) - v(P_\pi^i) = D^i(P_\pi^i \cup \{i\}, v)$, thus φ is the Shapley value (recall (4.1)). The fact that the boundary of $V(N)$ is a hyperplane guarantees that $\varphi(N, V)$ is an efficient payoff vector, as an average of the efficient vectors $d_\pi = (d_\pi^i)_{i\in N}$. (Again, we write $\varphi(N, V)$ for the vector $(\varphi^i(N, V))_{i\in N} \in \Re^N$.)

Note that in a TU-game, the marginal contribution of i depends only on the set of preceding players P_π^i; in an H-game, it may well depend also on the order of the players inside P_π^i (again, see Figure 4.1). Is there a way to define the marginal contribution of a player i to a coalition S (containing i) in an H-game so that, as in a TU-game, it depends only on the set S? The problem is that one needs to compare two *sets*, $V(S)$ and $V(S \setminus \{i\})$. In general there is no number x^i such that $V(S) = \{x^i\} \times V(S \setminus \{i\})$ (unless—as in a TU-game—the two sets are "parallel", meaning that $(\lambda_S^j)_{j\in S\setminus\{i\}}$ and $\lambda_{S\setminus\{i\}}$ are proportional).

We are thus led to the idea to "summarize" the possibilities of $S \setminus \{i\}$ by one payoff vector, rather than having a whole set. The natural candidate is, of course, the value of the subgame[7] $(S \setminus \{i\}, V)$. We then define the marginal contribution of player i as the most he can get in $V(S)$ when the players in $S \setminus \{i\}$ get the value of $(S \setminus \{i\}, V)$.

Formally, let (N, V) be a hyperplane game, and assume by induction that the values $\varphi(T, V) \in \Re^T$ of its subgames (T, V), for all $T \subsetneq N$, are given. Let $i \in N$ be a player and let S be a coalition containing i. We define

---

[7]Given a game (N, V) and a coalition $T \subset N$, we write (T, V) for the *subgame* whose player set is T and whose coalitional function is the restriction of V to the subsets of T (*i.e.*, to 2^T).

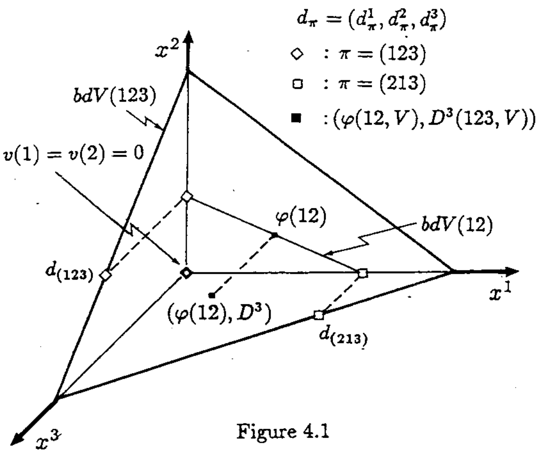

Figure 4.1

the *marginal contribution of player i to the coalition S* as[8]

$$D^i(S, V) := \max\{x^i : (x^i, \varphi(S \setminus \{i\}, V)) \in V(S)\}. \tag{4.6}$$

Equivalently, $D^i(S, V)$ is uniquely determined by the condition

$$(D^i(S, V), \varphi(S \setminus \{i\}, V)) \in \partial V(S). \tag{4.7}$$

Note that in order to define the marginal contributions one needs to know the values of the subgames (T, V) for all subsets T. For TU-games we get of course $D^i(S, v) = v(S) - v(S \setminus \{i\})$, since $\varphi = \text{Sh}$ yields efficient payoff vectors (moreover, in this case of TU-games we don't need to know $\varphi(T, v)$, only that it is efficient).

Proposition 4.1. *The consistent H-value φ satisfies*

$$\varphi(N, V) = (1/|N|) \sum_{i \in N} (D^i(N, V), \varphi(N \setminus \{i\}, V)); \tag{4.8}$$

and

$$\varphi^i(N, V) = \mathcal{E}[D^i(\mathbf{S}, V)] \tag{4.9}$$

[8]Note that d_π^i was called "marginal contribution of i *in the order* π," whereas $D^i(S)$ is the "marginal contribution of i *to the coalition* S." In H-games, these are distinct notions (again, see Figure 4.1).

for every hyperplane game (N, V) and $i \in N$. Moreover, if (N, V) is a monotonic hyperplane game, then $\varphi(N, V)$ is non-negative and it depends only on the non-negative parts of the feasible sets (i.e., *if $V(S) \cap \Re_+^S = W(S) \cap \Re_+^S$ for all S then $\varphi(N, V) = \varphi(N, W)$*).

Proof: [9] By induction: In (4.5), consider all orders where player i comes last. The (conditional) average marginal contribution of each $j \neq i$ is $\varphi^j(N \setminus \{i\}, V)$ by the induction hypothesis; efficiency then implies that the i-th coordinate is the "remainder," namely, $D^i(N, V)$ (see (4.7)). This proves (4.8), from which (4.9) follows, again by induction. The "moreover" statements for monotonic games are immediate by any of the formulas (4.5), (4.8) or (4.9). ∎

The interpretation of formula (4.9) (which is the same as (4.2) for the Shapley value) is clear: The value to each player is his expected marginal contribution. As for formula (4.8), it may be viewed as follows: One takes the value payoff vectors of the n subgames with $n-1$ players, "completes" each one to an efficient payoff vector for N (see (4.7)), and then averages these vectors – to obtain the value of the n-player game. Thus the value to player i is the average of his marginal contribution $D^i(N, V)$ to the grand coalition N and of his values when each one of the other players "drops out" (*i.e.*, $\varphi^i(N \setminus \{j\}, V)$ for all $j \neq i$). Note that, in particular, we obtain that the Shapley TU-value satisfies (4.8).

It is interesting to interpret the various formulas for the consistent H-value in the context of the prize games model. The value associates to a prize game (N, z) a probability distribution $\rho(N, z) \in \Delta(N)$, with $\rho^i(N, z)$ being the probability that i gets the "grand" prize ζ_N. The connection between ρ and φ is of course $\varphi^i(N, V) = \rho^i(N, z) z_N^i$ for all $i \in N$ (where (N, V) is the corresponding coalitional prize game). We will refer to ρ as the *value probability.*

Consider for instance formula (4.8), and assume that the prize game is monotonic. As noted in Section 3, this implies in particular that $z_N^j \geq z_{N\setminus\{i\}}^j$ for all $j \neq i$. Let $\theta_i^j := z_{N\setminus\{i\}}^j / z_N^j$; *i.e.*, player j is indifferent between getting the prize $\zeta_{N\setminus\{i\}}$ of $N \setminus \{i\}$ for sure, and getting the grand prize ζ_N of N with probability θ_i^j (and nothing with probability $1 - \theta_i^j$).

Write $\rho_i^j := \rho^j(N \setminus \{i\}, z)$ for the value probability of j in the $N \setminus \{i\}$ game; then j is indifferent between getting $\zeta_{N\setminus\{i\}}$ with probability ρ_i^j, and getting the grand prize ζ_N with probability $\rho_i^j \theta_i^j$. The remaining probability, namely $\delta^i := 1 - \sum_{j \in N\setminus\{i\}} \rho_i^j \theta_i^j$, is i's *marginal probability* (indeed,

[9]Note that formula (4.8) is essentially formula (5.1) in Maschler & Owen [4]. In (4.9), S is the random coalition of i together with all players preceding i in a random order, and $\mathcal{E}$ denotes expectation (see (4.2) above).

$\delta^i z_N^i = D^i$; see (4.7)). Then $\rho^i = (1/n)(\delta^i + \sum_{j\in N\setminus\{i\}} \rho_j^i \theta_j^i)$; *i.e.*, the value probability of i in (N, z) is the average of his marginal probability δ^i and of his value probabilities $\rho_j^i\theta_j^i$ in the subgames $(N \setminus \{j\}, z)$, evaluated in terms of the grand prize. Similar interpretations may be given to the other formulas, namely (4.5) and (4.9).

5. Axiomatization

There are to date four approaches to the consistent H-value: First, generalizing the random order formulas for the Shapley value (see Maschler & Owen [4], and also the previous Section). Second, using an appropriate "consistency" or "reduced game" property (Maschler & Owen [4]). Third, a dynamic procedure (Maschler & Owen [4]). And fourth, via a noncooperative foundation (Hart & Mas-Colell [3]). We will present here another approach; it is axiomatic, and based on the "marginality" idea introduced by Young [8] for TU-games.

A *solution function* ψ on a class Γ of games is a mapping that associates to every game (N, V) in Γ a payoff vector[10] $\psi(N, V)$ that is feasible for the grand coalition N; *i.e.*, $\psi(N, V) \in V(N)$. We only consider H-games; the domain Γ will thus always be included in the class of all H-games.

The solution function ψ is *efficient* if it always yields (Pareto) efficient payoff vectors; *i.e.*, $\psi(N, V) \in \partial V(N)$ for every game $(N, V) \in \Gamma$. Two players $i, j \in N$ are called *substitutes* in a game (N, V) if for all $S \subset N \setminus \{i, j\}$, all $x_S \in \Re^S$ and all $\xi \in \Re$, we have $(x_S, \xi) \in V(S \cup \{i\})$ if and only if $(x_S, \xi) \in V(S \cup \{j\})$. The solution function ψ is *symmetric*[11] if $\psi^i(N, V) = \psi^j(N, V)$ whenever i and j are substitutes in (N, V).

Efficiency and symmetry are standard requirements. We now introduce another postulate. The solution function ψ satisfies *marginality* if $\psi^i(N, V) = \psi^i(N, W)$ whenever the marginal contributions of player $i \in N$ are identical in the two games (N, V) and (N, W); *i.e.*, if $D^i(S, V) = D^i(S, W)$ for all $S \subset N$ with $S \ni i$, then $\psi^i(N, V) = \psi^i(N, W)$. Marginal contributions to a coalition are defined as in the previous Section (see (4.7)) using the solution ψ for the subcoalitions, namely

$$D^i(S, V) := \max\{x^i : (x^i, \psi(S \setminus \{i\}, V)) \in V(S)\} ; \tag{5.1}$$

equivalently, $D^i(S, V)$ is uniquely determined by the condition

$$(D^i(S, V), \psi(S \setminus \{i\}, V)) \in \partial V(S). \tag{5.2}$$

---

[10]We are thus considering only *one-point* solutions.
[11]This is also called "equal treatment property."

Marginality says that if a player has the same marginal contributions in two games, then his value must be the same in the two games.

We can now state the main characterization result.

Theorem 5.1. *Let ψ be a solution function on the class of hyperplane games. Then ψ satisfies efficiency, symmetry and marginality if and only if ψ is the consistent H-value.*

Marginality implies that the value is a function of the marginal contributions only. However, it could be any function, not necessary linear. It is remarkable that efficiency and symmetry suffice to determine this function uniquely (as the average (4.9)).

Proof: The three axioms applied to TU-games uniquely characterize the Shapley value there; this is the result of Young [8, Theorem 2]. For general hyperplane games, we will prove by induction that ψ must coincide with the consistent H-value φ. Let (N, V) be a hyperplane game, and assume that $\psi(S, V) = \varphi(S, V)$ for all $S \subsetneq N$. Fix $i \in N$. Define a TU-game (N, w) as follows: $w(S) := 0$ if $i \notin S$ and $w(S) := D^i(S, V)$ if $i \in S$. Then $D^i(S, w) = w(S) - w(S \setminus \{i\}) = D^i(S, V)$ for all S containing i, which by marginality implies that $\psi^i(N, w) = \psi^i(N, V)$. But (N, w) is a TU-game, therefore (by (4.2) and then (4.9)) $\psi^i(N, V) = \psi^i(N, w) = \mathrm{Sh}^i(N, w) = \mathcal{E}[D^i(\mathbf{S}, w)] = \mathcal{E}[D^i(\mathbf{S}, V)] = \varphi^i(N, V)$. The converse, *i.e.*, that φ satisfies the three axioms, is immediate (again, see (4.9)). ∎

The same result holds for *monotonic* games.

Theorem 5.2. *Let ψ be a solution function on the class of monotonic hyperplane games. Then ψ satisfies efficiency, symmetry and marginality if and only if ψ is the consistent H-value.*

Proof: First, note that the characterization of Young [8] for the Shapley TU-value holds when restricted to the space of monotonic TU-games (the proof described there for superadditive games actually applies to any cone of TU-games that includes all the unanimity games). Next, the only change needed in the proof of Theorem 5.1 is to make w monotonic. This is done as follows: for each S not containing i define $w(S) := \sum_{T \subsetneq S} D^i(T \cup \{i\}, V)$ and $w(S \cup \{i\}) := \sum_{T \subset S} D^i(T \cup \{i\}, V)$. Since (N, V) is monotonic we have $D^i(T \cup \{i\}, V) \geq 0$, from which the monotonicity of (N, w) follows immediately. ∎

Remarks 5.3.

(i) Efficiency and symmetry are actually needed only for TU-games.

(ii) The results here may be extended to general (non-hyperplane) NTU-games; with an appropriate definition of marginal contributions, the three axioms of efficiency, symmetry and marginality characterize the consistent NTU-value of Maschler & Owen [5].

Acknowledgments: Research partially supported by the National Science Foundation. I wish to acknowledge useful discussions on these topics with Robert J. Aumann, Michael Maschler, Andreu Mas-Colell and Eyal Winter.

6. References

[1] S. Hart and A. Mas-Colell, "Potential, value and consistency," *Econometrica* **57** (1989) 589–614.

[2] S. Hart and A. Mas-Colell, "Egalitarian solutions of large games: I. A continuum of players," DP #1, The Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, mimeo, 1992.

[3] S. Hart and A. Mas-Colell, "A model of n-person noncooperative bargaining," DP #10, The Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, mimeo, 1992.

[4] M. Maschler and G. Owen, "The consistent Shapley value for hyperplane games," *International Journal of Game Theory* **18** (1989) 389–407.

[5] M. Maschler and G. Owen, "The consistent Shapley value for games without side payments," in: *Rational Interaction*, R. Selten, Ed., Springer-Verlag, 1992, pp. 5–12.

[6] A. E. Roth and M. K. Malouf, "Game-theoretic models and the role of information in bargaining", *Psychological Review* **86** (1979) 574–594.

[7] L. S. Shapley, "A Value for n-Person Games," in: *Contributions to the Theory of Games II, Annals of Mathematics Studies* **28**, H. W. Kuhn & A. W. Tucker, Eds., Princeton University Press, 1953, pp. 307–317.

[8] H. P. Young, "Monotonic solutions of cooperative games," *International Journal of Game Theory* **14** (1985) 65–72.