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ENDOGENOUS FORMATION OF COALITIONS[1]

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In order to develop a theory of coalition formation and maintenance, we first establish a valuation criterion for each individual player in a given coalition structure. Various stability concepts based on it are then developed and studied.

1. INTRODUCTION

THE FORMATION OF COALITIONS is a fundamental problem in game theory. By a "coalition" is meant a group of "players" (e.g., economic or political agents) which decide to act together, as one unit, relative to the rest of the players. This includes the instances of syndicates, unions, cartels, blocks, political parties, parliamentary coalitions, etc. We would like to emphasize that forming a coalition does *not* eliminate the individual players as decision makers. In all interactions with the other players, coalition members act as one unit (it may be useful to think of a "representative agent" taking their place);[2] however, this arrangement will continue only as long as each player finds it desirable to act this way. Further bargaining occurs among the members of each coalition on how to divide what they obtained together. Thus, the existence of coalitions implies that the interactions among the players will be conducted on two levels: first, *among* the coalitions, and second, *within* each coalition.

Most of the existing models in economics and game theory assume that the coalition structure is given exogenously; instead, we try here to obtain it as an *endogenous outcome* of our model. Namely, we want to be able to predict which coalitions will in fact form in each given situation.[3]

Our theory combines two kinds of game theoretic concepts: value and stability. The basic idea is, first, to evaluate the players' prospects in the various coalition structures, and then, based on these "values," to find which ones are stable. We will call this value "coalition structure value," or "*CS-value*" for short. The reason we are considering "coalition structures" (i.e., partitions of the set of players into disjoint coalitions) rather than just "coalitions" is that, in general, players may find it to their advantage to join forces in some situations, and to act separately in others—all depending on the way the other participants are organized. This implies that both the value and the stability concepts should depend on the entire coalition structure. The CS-value we analyze in this paper was first developed by Owen [7].

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[2]The representative is really a "fiction;" he is not a player. He does not have any objective function to optimize!

[3]We mention here the von Neumann and Morgenstern [14] solution, which is sometimes interpreted as identifying, although implicitly only, coalition formation.

Most of the (game theoretic) concepts incorporating coalition structures are based on the assumption that the outcome of each coalition should be efficient for that coalition. Our approach considers, instead, outcomes that are "overall" efficient, no matter how the players are organized. Thus, we assume as a postulate that society as a whole operates efficiently; the problem we address here is how are the benefits distributed among the participants. With this view in mind, coalitions do not form in order to obtain their "worth" and then "leave" the game. But rather, they "stay" in the game and bargain as a unit with all the other players. This means that coalitions try to obtain as much as possible by not letting the others exploit their (individual) weaknesses when they are separated. As an everyday example of such a situation, "I will have to check this with my wife/husband" may (but not necessarily) lead to a better bargaining position, due to the fact that the other party has to convince *both* the player and the spouse.

When evaluating coalition structures, the bargaining among the coalitions and within each one of them should both be taken into account. A natural question arises whether these two are consistent; namely, whether the bargaining procedure followed by the *individual* members within each coalition is identical to the one used by the *coalitions* among themselves. A remarkable property of the CS-value is this consistency.

Having obtained a "value" for each player in each coalition structure—which is a *unique* evaluation of that player's prospects there, and is measured in his own utility scale—we can now address the question of which coalitions are stable. We want to find those coalition structures for which no players have better alternatives (again, according to the CS-value). We use the notion of "strong equilibrium:" no group of players, whether from the same coalition or from different ones, can get together and form new coalition(s) in such a way that they are all better off.

At this point, it is useful to analyze some examples. The first one is due to Roth [9]; it is a three-person game without side payments:

$$V(\{i\}) = \{x \in \mathbb{R}^3 \mid x \leqq 0\}, \qquad \text{for} \quad i = 1,2,3,$$

$$V(\{1,2\}) = \{x \in \mathbb{R}^3 \mid x \leqq (\tfrac{1}{2}, \tfrac{1}{2}, 0)\},$$

$$V(\{1,3\}) = \{x \in \mathbb{R}^3 \mid x \leqq (\tfrac{1}{4}, 0, \tfrac{3}{4})\},$$

$$V(\{2,3\}) = \{x \in \mathbb{R}^3 \mid x \leqq (0, \tfrac{1}{4}, \tfrac{3}{4})\},$$

$$V(\{1,2,3\}) = \{x \in \mathbb{R}^3 \mid x \leqq y \text{ for some } y \text{ in the convex hull of } (\tfrac{1}{2}, \tfrac{1}{2}, 0), (\tfrac{1}{4}, 0, \tfrac{3}{4}) \text{ and } (0, \tfrac{1}{4}, \tfrac{3}{4})\}.$$

In this case, the unique nontransferable-utility (NTU) value (cf. both Harsanyi [3], and Shapley [12]) is $(1/3, 1/3, 1/3)$. However, players 1 and 2, by forming a coalition, can get for sure $1/2$ each, in which case 3 will get 0. The reason player 3 gets a positive amount—of $1/3$—is that, whenever he joins either player 1 or player 2, he makes a positive contribution to him: in $V(\{i\})$, i gets 0, whereas in $V(\{i,3\})$, i gets $1/4$ (for $i=1$ or $i=2$). By forming a union, players 1 and 2 "make" 3 a null player—i.e., his contribution becomes zero. Therefore, a stable coalition structure consisting of $\{1,2\}$ and $\{3\}$ will in fact form in this situation; the corresponding CS-value is then $(1/2, 1/2, 0)$.

Our second example is a slight modification of Shafer's [**10**, Example 2]. There are three traders and two commodities, which may be thought of as "left-gloves" and "right-gloves" (or "shoes", etc., as long as they are not interchangeable). The first two traders have utility only for *pairs* of gloves, while the third's depends only on the number of gloves and not on their "handedness" (he uses them only for their leather). Let $\epsilon > 0$ be small; then Table I sums up the data.

To find the Shapley NTU-value allocation, note that every trader has some positive contribution (there is at least one positive λ), hence all λ's must be positive, and the allocation must be in the interior of the Pareto efficient surface. Efficiency now implies that trader 1 must have equal amounts of left- and right-gloves (otherwise, the excess could be transferred to trader 3); the same is true for 2, hence also for 3 (total endowment is $(1,1)$). From this it follows that all exchanges on the Pareto surface are in "pairs" of gloves, which makes all utilities identical, hence all λ's equal. The NTU-value allocation is shown in Table II.

Trader 3, which started with only (ϵ, ϵ), receives approximately $1/6$ of the goods, no matter how small $\epsilon > 0$ is! The reason is, as in the previous example, that whenever 3 joins one of the other traders, say 1, they have together $(1, \epsilon)$; it then follows that at best, trader 1 can get (ϵ, ϵ), leaving player 3 with $(1-\epsilon, 0)$. Therefore, when 3 is "second" in a random order (which has probability $1/3$), he will get almost half of the goods in the market; this leads to his value of about $1/6$.

TABLE I

Trader i	Initial endowment a^i	Utility function $u^i(x_L, x_R)$
1	$(1-\epsilon, 0)$	$\min\{x_L, x_R\}$
2	$(0, 1-\epsilon)$	$\min\{x_L, x_R\}$
3	(ϵ, ϵ)	$\frac{1}{2}(x_L + x_R)$

TABLE II

Trader	NTU-Value: Utility	NTU-Value: Commodities
1	$\frac{5}{12} - \frac{5}{12}\epsilon$	$(\frac{5}{12} - \frac{5}{12}\epsilon, \frac{5}{12} - \frac{5}{12}\epsilon)$
2	$\frac{5}{12} - \frac{5}{12}\epsilon$	$(\frac{5}{12} - \frac{5}{12}\epsilon, \frac{5}{12} - \frac{5}{12}\epsilon)$
3	$\frac{1}{6} + \frac{5}{6}\epsilon$	$(\frac{1}{6} + \frac{5}{6}\epsilon, \frac{1}{6} + \frac{5}{6}\epsilon)$

What can 1 and 2 do? By forming a "union" (thus inducing the coalition structure $\{\{1,2\},\{3\}\}$), they eliminate the possibility that 3 will be able to "catch" one of them "alone." In this case, 3 will become a dummy (contributing precisely ϵ), and the value allocation will be $((1/2)(1-\epsilon))$ to each of 1 and 2, and ϵ to 3.

In both these examples it appears that 1 and 2 should form a coalition, obtain their worth $V(\{1,2\})$, and then "leave" the game. Is it reasonable? To see this, let us change the previous example as follows: $a^1 = (1-2\epsilon, 0)$, $a^2 = (0, 1-\epsilon)$, $a^3 = (2\epsilon, \epsilon)$. In this case, the NTU-value to players 1 and 2 will still be close to $5/12$, and to player 3, close to $1/6$ (again, $\epsilon > 0$ is small). If 1 and 2 form a coalition, together they have $(1-2\epsilon, 1-\epsilon)$, which gives them a total utility of $1-2\epsilon$, if they just get their worth. However, after deciding to form a coalition, they do not "leave" the game but rather bargain with 3, *as a unit*, for some exchange of right-gloves for left-gloves, which will improve *everyone's* outcome. For example, they may get to $(1-(7/4)\epsilon, 1-(7/4)\epsilon)$ for $\{1,2\}$ and $((7/4)\epsilon, (7/4)\epsilon)$ for 3 (an increase of $(1/4)\epsilon$ for both $\{1,2\}$ and 3). This illustrates our basic principle stated previously: coalitions form in order to obtain the best shares for their members, when considering *efficient* outcomes.

Another question arising in this example is: how will 1 and 2 divide their total outcome? It seems reasonable that 2 should get a little more than 1, but how much? We regard this as a bargaining problem between 1 and 2 (i.e., within the coalition $\{1,2\}$); they can agree on any division of $(1-(7/4)\epsilon, 1-(7/4)\epsilon)$, and if they don't, the coalition will break and each will get his value in the game played by the three traders as individuals. However, since it is clear that both 1 and 2 are in a much better position together than separate, they will indeed reach an agreement, which is a "fair" division of their total outcome, based on their "relative power." This is precisely the content of our Consistency Theorem.

All the examples above are of games without side payments (or, economies without transferable utilities). However, it seems to us that the formation of coalitions does not depend on this assumption; indeed, in the last two examples, the goods themselves served as a convenient medium of utility exchange (compare this with Shafer's **[10]** value). Therefore, we will deal in this paper with the simpler case of games with side payments and we hope that the theory presented here can be extended later to the more general case.

The rest of the paper is divided into two parts. In Section 2 the CS-value is constructed axiomatically, and then its fundamental consistency properties are exhibited. Section 3 is devoted to the stability problem; we define our concepts and discuss various extensions.

2. VALUE

This section is devoted to the study of the CS-value. As we stated in the introduction, this value is the same as Owen's [7] "Value of Games With A Priori Unions."

Owen's objective in the above mentioned paper was to extend the notion of Shapley value to games in which the players are already organized into "unions" (or, a coalition structure). He obtained the value by "properly" decomposing the game and then using the usual Shapley value; an axiomatic characterization was then also given.

Our purpose here is different: we aim to investigate the way such coalition structures are formed. Thus, we construct the value axiomatically, where each axiom is justified and motivated by our view that the CS-value should provide an evaluation of the players' prospects in the various situations. As a result, we are able to provide both a better understanding of the axioms, as well as a replacement of Owen's Axiom A3 (which, in our view, is the most difficult to accept) by a much weaker and more natural one (our Axiom 4). We also present additional consistency properties of the CS-value.

Let U be an infinite set, the *universe of players* (we follow Shapley's **[11]** approach). A *game* v is a real function defined on all subsets of U, satisfying $v(\emptyset) = 0$. A set $N \subset U$ is a *carrier of* v if, for all $S \subset U$, $v(S) = v(S \cap N)$; in this paper we consider only *games with finite carriers*. We call $v(S)$ the *worth of* S.

A *coalition structure* $\mathscr{B}$ is a finite partition $\mathscr{B} = \{B_1, B_2, \ldots, B_m\}$ of U (i.e., $\bigcup_{k=1}^{m} B_k = U$ and $B_k \cap B_l = \emptyset$ for $k \neq l$). For a subset of players N (usually taken to be a carrier of some game), we will denote by $\mathscr{B}_N$ the restriction of $\mathscr{B}$ to N; namely, $\mathscr{B}_N = \{B_k \cap N \mid k = 1, 2, \ldots, m\}$, which is a partition of N (empty sets $B_k \cap N$ will be discarded).

A coalition structure value (CS-value) is an operator ϕ which assigns to every game v with finite carrier, every coalition structure $\mathscr{B}$, and every player $i \in U$, a real number $\phi^i(v, \mathscr{B})$. Equivalently, one may think of $\phi(v, \mathscr{B})$ as a (finitely) additive measure on U, defined by

$$\phi(v, \mathscr{B})(S) = \sum_{i \in S} \phi^i(v, \mathscr{B}),$$

for $S \subset U$.

We will consider the following axioms on ϕ (assumed to hold for all games v and v' and all coalition structures $\mathscr{B}$ and $\mathscr{B}'$).

AXIOM 1 (Carrier): *Let N be a carrier of v; then*

(i) $$\phi(v, \mathscr{B})(N) \equiv \sum_{i \in N} \phi^i(v, \mathscr{B}) = v(N);$$

(ii) *if* $\mathscr{B}_N = \mathscr{B}'_N$, *then* $\phi(v, \mathscr{B}) = \phi(v, \mathscr{B}')$.

Let π be a permutation of the players; i.e., a one-to-one mapping of U onto itself. For $S \subset U$, we write πS for the image of S under π; given a game v, we define a new game πv by

$$(\pi v)(S) = v(\pi S)$$

for all $S \subset U$.

Axiom 2 (Symmetry): *Let π be a permutation of the players. Then*

$$\phi(\pi v, \pi \mathscr{B}) = \pi \phi(v, \mathscr{B}).$$

The sum $v + v'$ of two games v and v' is the game given by

$$(v + v')(S) = v(S) + v'(S),$$

for all $S \subset U$.

Axiom 3 (Additivity):

$$\phi(v + v', \mathscr{B}) = \phi(v, \mathscr{B}) + \phi(v', \mathscr{B}).$$

Given a game v and a coalition structure $\mathscr{B} = \{B_1, B_2, \ldots, B_m\}$, we say that *the game among coalitions is inessential* if

$$v\Big(\bigcup_{k \in K} B_k\Big) = \sum_{k \in K} v(B_k)$$

for all subsets K of $\{1, 2, \ldots, m\}$; i.e., v restricted to the field generated by $\mathscr{B}$ is additive.

Axiom 4 (Inessential Game): *Let v and $\mathscr{B} = \{B_1, B_2, \ldots, B_m\}$ be such that the game among coalitions is inessential. Then*

$$\phi(v, \mathscr{B})(B_k) = v(B_k)$$

for all $k = 1, 2, \ldots, m$.

We discuss now the four axioms. The "carrier" axiom actually contains three parts. If i is a null player in a game v (i.e., $U \backslash \{i\}$ is a carrier of v; or equivalently, $v(S \cup \{i\}) = v(S)$ for all $S \subset U$), then his value is 0 in all coalition structures. Moreover, if such i "moves" from one B_k to another, it does not affect anyone's value. And last, for all coalition structures the value is efficient.

The efficiency of the CS-value is an essential feature. It differs from the other approaches (e.g., Aumann and Drèze [2], Myerson [5], Shenoy [13], and also the various bargaining sets) where each coalition $B_k \in \mathscr{B}$ gets only its worth (i.e., $v(B_k)$). Our view is that the reason coalitions form is not in order to get their worth, but to be in a better position when bargaining with the others on how to divide the maximal amount available (i.e., the worth of the grand coalition, which for superadditive games is no less than $\sum_{k=1}^{m} v(B_k)$). We assume that the amount $v(N)$ (where N is any carrier of v) will be distributed among the players and thus that all collusions and group formations are done with this in mind. A coalition B forms when all its members commit themselves to bargain with the others as one unit.[4]

This further clarifies Axiom 4: When the game between the coalitions is inessential, each coalition gets only its worth and there is no surplus to be bargained over.

The other two axioms are straightforward. We note that the symmetry axiom involves permuting both the game and the coalition structure (i.e., "changing the names" of the players). As for the additivity of the value with respect to the game, it is assumed for a fixed coalition structure. It may be replaced by the following axiom: The CS-value of a probabilistic mixture of two games (i.e., with probability p the game v is played, and with probability $p' = 1 - p$ the game v' is played) is the same mixture of the CS-values of the two games. This is consistent with the interpretation of value as the expected utility of playing the game.

Although Theorem 2.1 below indicates that our axioms characterize Owen's value, the significant difference between our axioms and Owen's is to be found in our Axiom 4. Owen's corresponding Axiom A3 assumed that for *all* games v and all coalition structures $\mathscr{B}$, the total value of each coalition B_k in $\mathscr{B}$ depends only on the restriction of v to (the field generated by) $\mathscr{B}$. In contrast, we assume this to hold for *inessential games only*, in which case it is easier to justify. The problem of bargaining among coalitions arises when there is a surplus which is available when such coalitions combine together. Since there is no such surplus in inessential games (among coalitions), there is nothing to divide, thus each one gets its worth. What is surprising is that this natural condition, in conjunction with Axioms 1–3, implies Owen's Axiom A3. See also the remark following the proof of Theorem 2.1 and Proposition 2.2.

THEOREM 2.1: *There is a unique CS-value ϕ satisfying Axioms* 1–4; *it is Owen's* [7] *value.*

Let N be a finite set of players, and $\mathscr{B} = \{B_1, B_2, \ldots, B_m\}$ a coalition structure. A complete (linear) *order on N is consistent* with $\mathscr{B}$ if, for all $k = 1, 2, \ldots, m$ and all $i, j \in B_k$, all elements of N between i and j also belong to B_k. A *random order on N consistent with $\mathscr{B}$* (or, *given $\mathscr{B}$*) is a random variable whose values are the orders on N that are consistent with $\mathscr{B}$, all equally probable (i.e., each with probability $(m!\,b_1!\,b_2! \ldots b_n!)^{-1}$, where b_k is the number of elements in $B_k \cap N$). The interpretation is as follows: the players arrive randomly, but such that all members of the same coalition do so successively. This is the same as randomly ordering first the coalitions and then the members within each coalition.

[4]The question of when this commitment is credible will be answered in the next section; for a stable coalition, it is self-enforcing.

PROPOSITION 2.2: *The unique CS-value* ϕ *satisfying Axioms* 1–4 *is given by*

$$\phi^i(v,\mathscr{B}) = E\left[v\left(\mathscr{P}^i \cup \{i\}\right) - v\left(\mathscr{P}^i\right)\right], \tag{2.3}$$

where the expectation E *is over all random orders on a carrier* N *of* v *that are consistent with* $\mathscr{B}$, *and* $\mathscr{P}^i$ *denotes the* (*random*) *set of predecessors of* i.

Theorem 2.1 and Proposition 2.2 will be proved together. Although the main ideas are standard, our use of the weaker Axiom 4 necessitates some additional work.

PROOF OF THEOREM 2.1 AND PROPOSITION 2.2: It can easily be checked that the operator given by formula (2.3) indeed satisfies Axioms 1–3; as for Axiom 4, let i_k be the first member of B_k in a random order; the fact that all the other players of B_k follow i_k implies that

$$\sum_{j \in B_k}\left[v\left(\mathscr{P}^j \cup \{j\}\right) - v\left(\mathscr{P}^j\right)\right] = v\left(\mathscr{P}^{i_k} \cup B_k\right) - v\left(\mathscr{P}^{i_k}\right),$$

which, for an inessential game among coalitions, equals $v(B_k)$ (since $\mathscr{P}^{i_k}$ is a union of B's). Thus all four axioms are indeed satisfied.

In order to complete the proof, we have to show uniqueness, namely that Axioms 1–4 determine ϕ. Axiom 3 (additivity) implies that it suffices to check basic games only, i.e., games v of the form

$$v(S) = \begin{cases} c & \text{if } S \supset R, \\ 0 & \text{otherwise,} \end{cases}$$

where c is a fixed constant and $R \subset U$ is a fixed finite and nonempty set.

Denote $R_k = R \cap B_k$, and assume, without loss of generality, that R_k is not empty for $k = 1, 2, \dots, l$ (where $1 \leq l \leq m$). Let[5] $\rho = \max_k |R_k|$, and let R'_k, for $k = 1, 2, \dots, l$, be disjoint sets of players, such that $|R'_k| = \rho$ and $R'_k \supset R_k$ (here we use the fact that U is an infinite set). Let $R' = \bigcup_{k=1}^{m} R'_k$, $\mathscr{B}' = \{R'_1, R'_2, \dots, R'_l, U/R'\}$, and

$$v'(S) = \begin{cases} c, & \text{if } S \supset R', \\ 0, & \text{otherwise.} \end{cases}$$

Consider $\phi(v', \mathscr{B}')$. All players outside R' get 0, and all players in R' are identical (since R'_k are of the same size ρ, for all $1 \leq k \leq l$). Axioms 1 and 2 imply that, for all $i \in R'$,

$$\phi^i(v', \mathscr{B}') = \frac{c}{|R'|} = \frac{c}{\rho \cdot l},$$

[5]The number of elements of a finite set A is denoted by $|A|$.

hence, for all $1 \leq k \leq l$,

$$\phi(v',\mathscr{B}')(R_k') = \frac{c}{l} .$$

Next, consider $\phi(v - v',\mathscr{B}')$. It is an inessential game among coalitions, since

$$(v - v')\Big(\bigcup_{k \in K} R_k'\Big)$$

is either $c - c = 0$ or $0 - 0 = 0$, according to whether $K \supset \{1,2, \ldots, l\}$ or not. Therefore, by Axiom 4,

$$\phi(v - v',\mathscr{B}')(R_k') = (v - v')(R_k') = 0,$$

for all $1 \leq k \leq l$. Axiom 3 now implies

$$\begin{aligned}\phi(v,\mathscr{B}')(R_k') &= \phi(v',\mathscr{B}')(R_k') + \phi(v - v',\mathscr{B}')(R_k') \\ &= \frac{c}{l} + 0 = \frac{c}{l} .\end{aligned}$$

But R is a carrier of v and $\mathscr{B}_R = \mathscr{B}_R'$; therefore (Axiom 1)

$$\phi(v,\mathscr{B})(R_k) = \phi(v,\mathscr{B}')(R_k) = \phi(v,\mathscr{B}')(R_k') = \frac{c}{l} ,$$

for all $1 \leq k \leq l$. Finally, symmetry among the members of the same R_k gives

$$\phi^i(v,\mathscr{B}) = \begin{cases} \dfrac{c}{l \cdot |R_k|} , & \text{if} \quad i \in R \cap B_k \equiv R_k , \\ 0, & \text{otherwise,} \end{cases}$$

which implies the uniqueness of ϕ. *Q.E.D.*

Remark: It is clear from the use of Axiom 4 in the proof that it may be replaced by a slightly weaker axiom, namely Axiom 4′.

Axiom 4′ (Null Game): *If*

$$v\Big(\bigcup_{k \in K} B_k\Big) = 0$$

for all $K \subset \{1,2, \ldots, m\}$, *then* $\phi(v,\mathscr{B})(B_k) = 0$ *for all* $k = 1,2, \ldots, m$.

(Note that we assume only that the total CS-value of each coalition B_k in $\mathscr{B}$ is zero, and not that each player gets zero.)

Two other alternative axioms that may also be used in place of Axiom 4 are as follows:

AXIOM 5 (Dummy Coalition): *If B_l is such that*

$$v\Big(\bigcup_{k\in K} B_k \cup B_l\Big) = v\Big(\bigcup_{k\in K} B_k\Big) + v(B_l)$$

for all $K \subset \{1,2,\ldots,m\}$, then $\phi(v,\mathscr{B})(B_l) = v(B_l)$.

AXIOM 5′ (Null Coalition): *If B_l is such that*

$$v\Big(\bigcup_{k\in K} B_k \cup B_l\Big) = v\Big(\bigcup_{k\in K} B_k\Big)$$

for all $K \subset \{1,2,\ldots,m\}$, then $\phi(v,\mathscr{B})(B_l) = 0$.

It is clear that Axiom 5 (5′) implies Axiom 4 (4′); however, together with 1–3, they all become equivalent.

For a game (v,N) with finite carrier N, let $\mathrm{Sh}\, v$ denote its Shapley value; i.e., $\mathrm{Sh}^i v$ is the value of player i, and $(\mathrm{Sh}\, v)(S) = \sum_{i\in S}\mathrm{Sh}^i v$ for $S \subset U$.

COROLLARY 2.4: *For all $B_k \in \mathscr{B}$,*

$$\phi(v,\mathscr{B})(B_k) = (\mathrm{Sh}\, v_{\mathscr{B}})(B_k),$$

where $(v_{\mathscr{B}},\mathscr{B})$ is the game v restricted to the field generated by $\mathscr{B}$ (i.e., each $\mathscr{B}_k \in \mathscr{B}$ is a "player").

PROOF: The proof is immediate from formula (2.3) *Q.E.D.*

Thus, the total CS-value of each coalition in $\mathscr{B}$ is precisely the Shapley value of the game played by the (representatives of the) coalitions in $\mathscr{B}$. Note that if one wants the CS-value to satisfy the "null player axiom" (i.e., adding null players does not change the value), one is necessarily led to regard each coalition in the coalition structure as one "representative," independent of the number of original players it is composed of!

Axiom 1(ii) implies that only the partition of a carrier N of v needs to be specified. Thus, we will abuse our notation by writing $\phi(v,\mathscr{B}_N)$ for $\phi(v,\mathscr{B})$. Another notation will be useful: For any set $S = \{i_1, i_2, \ldots, i_s\}$, if

$$\mathscr{B}_S = \{\{i_1\},\{i_2\},\ldots,\{i_s\}\},$$

i.e., the partition of S is into singletons (one player sets), then we will write $\mathscr{B}_S = \langle S\rangle$; in contrast, $\mathscr{B}_S = \{S\}$ means that all members of S are "together" in one coalition.

COROLLARY 2.5: *Let N be a carrier of v. Then*

$$\phi(v,\{N\}) = \phi(v,\langle N\rangle) = \mathrm{Sh}\, v.$$

PROOF: Again, it follows from (2.3). *Q.E.D.*

How is the CS-value related to the two bargaining processes among the coalitions (i.e., the elements of $\mathscr{B}$), and within each coalition? By Corollary 2.4, the former is obtained by replacing each $B_k \in \mathscr{B}$ with a single player, and then taking the Shapley value of the resulting game, "the game among coalitions."

Consider now one coalition $B_k \in \mathscr{B}$. How should its members divide the total amount $\phi(v,\mathscr{B})(B_k)$ which they receive? Their "relative power" must be taken into account, and one way to measure it is by comparing their "prospects" should B_k break apart. For example, suppose $N = \{1,2,3\}$, and $\mathscr{B}_N = \{\{1,2\}, \{3\}\}$. The coalition $\{1,2\}$ gets $\phi(v,\mathscr{B})(\{1,2\})$; if they do not agree on the division of this amount and "split," they will get $\phi(v,\langle N\rangle)(\{i\})$, for $i = 1$ and $i = 2$. This may serve as a "disagreement point" in the two-person bargaining problem where the set of possible "agreements" corresponds to all the various ways of dividing $\phi(v,\mathscr{B})(\{1,2\})$. According to the Nash [6] solution, each $i = 1,2$ will receive $\phi(v,\langle N\rangle)(\{i\}) + (1/2)[\phi(v,\mathscr{B})(\{1,2\}) - \phi(v,\langle N\rangle)(\{1,2\})]$. It turns out that, for $i = 1,2$, this quantity is precisely $\phi(v,\mathscr{B})(\{i\})$!

What happens if B_k contains more than two players? One may choose the n-person generalization of the Nash solution (see Harsanyi [3]), but this is not completely satisfactory, for two reasons. First, all possible subcoalitions of B_k should be taken into account, and not only the singletons. And second, it would be preferable to use the same solution concept for the bargaining *within* B_k as the one used for the bargaining *among* the coalitions. It is precisely this consistency that we are seeking.

To formalize this discussion, let v, $\mathscr{B} = \{B_1, B_2, \ldots, B_m\}$ and $B_k \in \mathscr{B}$ be given. Define a new game w_k on B_k by

$$(2.6) \qquad w_k(S) = \phi(v,\mathscr{B}\,|\,S)(S),$$

for all $S \subset B_k$, where $\mathscr{B}\,|\,S$ is the coalition structure obtained from $\mathscr{B}$ by replacing B_k with S and $B_k \backslash S$; i.e.,

$$(2.7) \qquad \mathscr{B}\,|\,S = \{B_1, \ldots, B_{k-1}, S, B_k \backslash S, B_{k+1}, \ldots, B_m\}.$$

Note that $w_k(\emptyset) = 0$, hence w_k is indeed a game; moreover, it has a finite carrier whenever v does.

THEOREM 2.8 (Consistency): *Given a game v and a coalition structure $\mathscr{B} = \{B_1, B_2, \ldots, B_m\}$, the following equality holds:*

$$\phi^i(v,\mathscr{B}) = \phi(w_k, \langle B_k\rangle)(\{i\}),$$

where $i \in B_k \in \mathscr{B}$ and w_k is given by (2.6) *and* (2.7).

This result shows that the CS-value enjoys the following consistency property: the bargaining procedure within coalitions may be derived from the one among coalitions. Indeed, since $S \in \mathscr{B}\,|\,S$ and $\{i\} \in \langle B_k\rangle$, both $w_k(S)$ and $\phi(w_k, \langle B_k\rangle)(\{i\})$ are CS-values to *coalitions* in the corresponding coalition structures; they do not depend in any way on the division of payoffs within coalitions.

To further explain this result, suppose that a rule prevails in society which, given a game and a coalition structure, assigns a payoff (value) to each coalition (in the coalition structure). Then the payoff to each player is determined by the value of a new game, played by all the members of his coalition (and by them only). The worth of each subcoalition is given by its payoff (value) if it were to break away. However, this is only a "gedanken experiment," since the threat of breaking away is never carried out; it is only a way of evaluating the "relative power" of the various players. Therefore, the coalition structure of the rest of the players (i.e., those outside the coalition), is assumed to be unchanged. Our result is that the value obtained this way coincides with the CS-value.

In (2.7) we defined $\mathscr{B} \mid S$, for $S \subset B_k$, by replacing B_k with S and $B_k \backslash S$. By doing so we assumed that the members of the complement of S "stay together." An alternative possibility is that, when S "leaves," the rest of B_k will break apart into individuals (singletons). In this case, we shall define $\mathscr{B} \mid S$ by

(2.9) $$\mathscr{B} \mid S = \{B_1, \ldots, B_{k-1}, S, \{j_1\}, \ldots, \{j_t\}, B_{k+1}, \ldots, B_m\},$$

where $B_k \backslash S = \{j_1, j_2, \ldots, j_t\}$. This induces by (2.6) a new game w_k on B_k. However, we have the following result:

THEOREM 2.10 (Consistency): *Theorem* 2.8 *is true with* $\mathscr{B} \mid S$ *given by* (2.9) *instead of* (2.7).

This means that the CS-value enjoys the consistency property independently of the way w_k is defined; although (2.7) and (2.9) define two different games w_k, Theorem 2.10 states that their values coincide. This additional feature is important particularly in view of the well-known ambiguity of the "complement's behavior:" when a coalition acts together, how does its complement react—as a coalition, or as individuals? In our case, it does not matter!

Owen's decomposition shows still another consistency of the CS-value, with $\mathscr{B} \mid S$ given by

$$\{B_1, \ldots, B_{k-1}, S, B_{k+1}, \ldots, B_m\}.$$

Namely, S replaces B_k, and the rest of the players in $B_k \backslash S$ "disappear" (i.e., become null).

PROOF OF THEOREMS 2.8 AND 2.10: We have to show that $\psi(v, \mathscr{B})$, given by

$$\psi^i(v, \mathscr{B}) = \phi(w_k, \langle B_k \rangle)(\{i\}),$$

for all $B_k \in \mathscr{B}$ and $i \in B_k$, satisfies Axioms 1–4.

Symmetry is immediate, since all definitions are independent of the "names" of the players. The additivity of ϕ with respect to v (for fixed $\mathscr{B}$), hence also of w_k, implies the additivity of ψ. As for Axiom 4, note that

$$\psi(v, \mathscr{B})(B_k) = \phi(w_k, \langle B_k \rangle)(B_k) = w_k(B_k),$$

since B_k is a carrier of w_k. But

$$w_k(B_k) = \phi(v, \mathscr{B} \mid B_k)(B_k) = \phi(v, \mathscr{B})(B_k)$$

(since $\mathscr{B} \mid B_k = \mathscr{B}$), hence

$$\psi(v, \mathscr{B})(B_k) = \phi(v, \mathscr{B})(B_k);$$

ϕ satisfies Axiom 4, therefore ψ does it too. Moreover,

$$\psi(v, \mathscr{B})(U) = \sum_{k=1}^{m} \psi(v, \mathscr{B})(B_k) = \sum_{k=1}^{m} \phi(v, \mathscr{B})(B_k) = \phi(v, \mathscr{B})(U),$$

hence ψ is efficient. Also, if i is a null player, then "moving" him from one B_k to another does not affect ϕ (by Axiom 1(ii)), hence it does not affect the w_k's and ψ. To complete the proof, we have to show that in this case $\psi^i(v, \mathscr{B}) = 0$. Indeed, let $i \in B_k$; then, for all $S \subset B_k$ with $S \not\ni i$,

$$\begin{aligned} w_k(S \cup \{i\}) &= \phi(v, \mathscr{B} \mid (S \cup \{i\}))(S \cup \{i\}) \\ &= \phi(v, \mathscr{B} \mid (S \cup \{i\}))(S) \\ &= \phi(v, \mathscr{B} \mid S)(S) = w_k(S) \end{aligned}$$

(we used Axiom 1(i) and (ii) for the carrier $U \setminus \{i\}$). Hence i is a null player in w_k, therefore a null coalition (see Axiom 5′) in $(w_k, \langle B_k \rangle)$, and $\psi^i(v, \mathscr{B}) = 0$ follows. *Q.E.D.*

REMARKS: (i) In view of Corollary 2.5, one may replace $\langle B_k \rangle$ by $\{B_k\}$ in Theorems 2.8 and 2.10.

(ii) For two players, the Shapley value and the Nash solution coincide; this proves our claim in the example discussed before Theorem 2.8.

3. STABILITY

As we stated in the introduction, we deal with the question of stability of coalitions in the broader sense of stability of coalition *structures*. Our objective is to identify those structures which the players do not wish to upset—since they have no better alternatives. However, in what sense should the expression "better alternative" be understood? This will be explored in this section.

We start with the fact that for every game and every coalition structure we have a well-defined "value." This CS-value should not be interpreted as a definitive prediction on what the outcome will be; rather, it is an expected outcome based on all the various possibilities, and including all the bargainings (within coalitions and among them). This is a standard way (see Roth **[8]**) for viewing the value. E.g., the Shapley value of the three-person majority game is $1/3$ to each player. This does not imply that in an actual play of the game, each player will get $1/3$. Rather, two players will form a "winning majority" and get

$1/2$ each, leaving 0 for the third player. Since on the average each player will be in the majority in two out of every three times the game is played, his expected payoff is $2/3 \cdot 1/2 + 1/3 \cdot 0 = 1/3$. Thus we think of $\phi^i(v, \mathscr{B})$ as the (expected) utility of playing the game v in "position" i (i.e., as player i) when the players are organized in coalitions according to $\mathscr{B}$.

Since $\phi^i(v, \mathscr{B})$ is always measured in player i's utility scale (for games with side payments it is the same scale for everybody; however, this argument becomes important in games without side payments), he is able to compare his "prospects" in the various coalition structures and to decide whether or not he wants to change the coalition he is in.

One advantage of using the CS-value is that it "summarizes" in *one number* all the possibilities of a player in every situation (in our case, in every coalition structure). In contrast, other game theoretic concepts compare *sets* of outcomes.

We do not present here a dynamic process according to which coalition structures form, but rather, we only specify which ones are stable. It is clear that any such dynamic process, if it converges at all, must lead to a stable coalition structure. Thus, the characterization of those which are stable may be regarded as a first step in the analysis. Moreover, dynamic theories usually rely on additional (arbitrary) assumptions (in our case, for example, the order in which players "talk" to one another) which significantly affect the outcome. Our stable coalition structures may be regarded as "universal" outcomes, independent of the specifications of the process.

We come now to the precise definition of stability. It is clear that one should require that no player can, unilaterally, improve his outcome. However, this is a very weak condition. The most a single player can do is to refuse to join a coalition. Thus, we are led to consider stability with respect to deviations by any group of players (from the same coalition, or from different ones). In game theoretic terms, this corresponds to the notion of *strong equilibrium* (cf. Aumann [**1**]) rather than Nash equilibrium.

How can a set of players change a coalition structure? If we think of coalitions as being based on unanimous consent (i.e., no player may be forced to join a coalition), any group of players can break apart the coalitions to which they belong (and only those!). Moreover, they can organize themselves in any way, not necessarily as a single coalition. E.g., if the coalition structure is[6] [12 | 34 | 567], then player 1 may generate [1 | 2 | 34 | 567]; players 1 and 3 may induce [1 | 2 | 3 | 4 | 567] or [13 | 2 | 4 | 567].

One problem is: What happens to those coalitions from which one or more players depart? Do they "fall apart," or do they still "stick together"? In the previous example [12 | 34 | 567], if 5 leaves his coalition, do we get [12 | 34 | 5 | 6 | 7] or [12 | 34 | 5 | 67]? There is no universally correct answer to this problem. On one hand, the view of a coalition as resulting from a unanimous agreement among all its members to be together, suggests that if some players leave, the agreement

[6]From now on, we will simplify our notation for coalition structures; [12 | 34 | 567] means $\{\{1,2\}, \{3,4\}, \{5,6,7\}\}$. Also, we consider only finitely many players (a finite carrier of the game).

breaks down and the rest of the members become singletons (at least in the first stage). On the other hand, especially in large games,[7] the fact that a (small) number of players leave a coalition should not influence the others' agreement to act together. These two views precisely correspond to the two kinds of behavior of the complementary coalition in (2.7) and (2.9).

We now present two models of stability, each one being based on the strong equilibria of an appropriate game corresponding to the above description. Let (v, N) be a fixed game, where $N = \{1, 2, \ldots, n\}$ is a carrier of v.

MODEL γ: The game $\Gamma \equiv \Gamma_{v,N}$ is given by:[8]

(3.1) The set of *players* is N.

(3.2) For each $i \in N$, the set of *strategies* of i is $\Sigma^i = \{S \subset N \mid i \in S\}$.

(3.3) For each n-tuple of strategies $\sigma = (S^1, S^2, \ldots, S^n) \in \Sigma^1 \times \Sigma^2 \times \cdots \times \Sigma^n$ and each $i \in N$, the *payoff* to i is $\phi^i(v, \mathscr{B}_\sigma^{(\gamma)})$, where

$$T_\sigma^i = \begin{cases} S^i, & \text{if } \quad S^j = S^i \quad \text{for all } \quad j \in S^i, \\ \{i\}, & \text{otherwise} \end{cases},$$

and $\mathscr{B}_\sigma^{(\gamma)} = \{T_\sigma^i \mid i \in N\}$.

MODEL δ: The game $\Delta \equiv \Delta_{v,N}$ is given by (3.1), (3.2), and (3.4):

(3.4) For each n-tuple of strategies $\sigma = (S^1, S^2, \ldots, S^n) \in \Sigma^1 \times \Sigma^2 \times \cdots \times \Sigma^n$ and each $i \in N$, the *payoff* to i is $\phi^i(v, \mathscr{B}_\sigma^{(\delta)})$, where

$$\mathscr{B}_\sigma^{(\delta)} = \{T \subset N \mid i, j \in T \text{ if and only if } S^i = S^j\}.$$

The first game Γ is interpreted as follows: each player chooses the coalition to which he wants to belong. A coalition forms if and only if all its members have in fact chosen it; the rest of the players become singletons. In the other game Δ, the choice of a strategy by a player means the largest set of players he is willing to be associated with in the same coalition. Each set of all the players who chose the same S then forms a coalition (which may, in general, differ from S). This means that a coalition corresponds to an equivalence class, with respect to equality of strategies. For example, let $N = \{1,2,3,4\}$, and $S^1 = \{1,2,3\}$, $S^2 = \{1,2,3\}$, $S^3 = \{1,2,3\}$, $S^4 = \{1,2,3,4\}$; then $\mathscr{B}^{(\gamma)} = \mathscr{B}^{(\delta)} = [123 \mid 4]$. If we replace S^1 by $\hat{S}^1 = \{1\}$, then $\hat{\mathscr{B}}^{(\gamma)} = [1 \mid 2 \mid 3 \mid 4]$, whereas $\hat{\mathscr{B}}^{(\delta)} = [1 \mid 23 \mid 4]$. In general, if one or more players leave a coalition to which they belong, the other members of that

[7] This suggestion is due to L. S. Shapley.

[8] We thank L. S. Shapley for pointing out that this game appears in von Neumann and Morgenstern [**14**, Section 26].

coalition become singletons in the game Γ, while in the game Δ they all form one new coalition.

For a coalition structure $\mathscr{B}$ and a player $i \in N$, let $S^i_{\mathscr{B}}$ be that element of $\mathscr{B}$ to which i belongs: $i \in S^i_{\mathscr{B}} \in \mathscr{B}$ (this defines $S^i_{\mathscr{B}}$ uniquely); put $\sigma_{\mathscr{B}} = (S^i_{\mathscr{B}})_{i \in N}$. If the players choose $\sigma_{\mathscr{B}}$, then in both Γ and Δ the coalition structure that results is clearly $\mathscr{B}$.

DEFINITION 3.5: The coalition structure $\mathscr{B}$ is *γ-stable* (*δ-stable*) in the game (v, N) if $\sigma_{\mathscr{B}}$ is a strong equilibrium in $\Gamma_{v,N}$ ($\Delta_{v,N}$, respectively); i.e., if there exists no nonempty $T \subset N$ and no $\hat{\sigma}^i \in \Sigma^i$ for all $i \in T$, such that $\phi^i(v, \hat{\mathscr{B}}) > \phi^i(v, \mathscr{B})$ for all $i \in T$, where $\hat{\mathscr{B}}$ corresponds to $((\hat{\sigma}^i)_{i \in T}, (\sigma^j_{\mathscr{B}})_{j \in N \setminus T})$ by (3.3) ((3.4), respectively).

The test of a meaningful theory clearly lies in its applications. Indeed, one should analyze various games and classes of games, and characterize in each case the corresponding stable coalition structures. This is taken up in a forthcoming paper (Hart and Kurz [**4**]). It includes the study of all three-player games, four-player symmetric games, weighted majority games, and others.

A general question arises: Do stable coalition structures always exist?

PROPOSITION 3.6: *There exist games for which every coalition structure is neither γ-stable nor δ-stable.*

Examples of such games are given in Hart and Kurz [**4**].

The fact that there is no universal existence theorem should not be surprising. Allowing group deviations, by their nature, leads to the possibility of circularity. In fact, all game theoretic concepts which are based on some form of group stability behave similarly.[9] This, however, does not detract from the importance of studying the stability properties of a game. In those cases in which a stable coalition structure exists the theory provides a useful prediction. We are reminded, in this connection, that the fact that the core may be empty in some cases does not remove the value of this concept when used to understand the solution of a game.

There are a few additional concepts which can be examined. First, one may broaden the definition of stability. For example, one may use the core concept instead of that of strong equilibrium. More precisely, a coalition structure will be said to be stable if no group of players can *certainly* become better off. Depending upon the definition of "certainly," we obtain two new concepts. In one case, it is "to guarantee;" in the other, "not to be prevented from." This corresponds precisely to the α-Core and the β-Core (cf. Aumann [**1**]).

From the (normal form) game $\Gamma_{v,N}$ (or, equivalently, $\Delta_{v,N}$) one obtains two games without side payments $(V^{(\alpha)}, N)$ and $(V^{(\beta)}, N)$, as follows: For each

[9]The various bargaining sets are not empty; however, the coalition structure is *given* there *exogenously*.

$S \subset N$, let $\Sigma^S = \prod_{i \in S} \Sigma^i$, then

$$V^{(\alpha)}(S) = \Big\{ (x^i)_{i \in S} \in \mathbb{R}^S \mid \text{there is } (\sigma^i)_{i \in S} \in \Sigma^S \text{ such that for all } (\sigma^j)_{j \in N \setminus S} \in \Sigma^{N \setminus S}, \ \phi^i(v, \mathscr{B}_\sigma) \geq x^i \text{ for all } i \in S \Big\},$$

$$V^{(\beta)}(S) = \Big\{ (x^i)_{i \in S} \in \mathbb{R}^S \mid \text{for all } (\sigma^i)_{j \in N \setminus S} \in \Sigma^{N \setminus S} \text{ there is } (\sigma^i)_{i \in S} \in \Sigma^S \text{ such that } \phi^i(v, \mathscr{B}_\sigma) \geq x^i \text{ for all } i \in S \Big\}.$$

We recall that a vector $x = (x^i)_{i \in N}$ in $\mathbb{R}^N$ belongs to the *core* of a game without side payments (V, N) if there exists no nonempty coalition $T \subset N$ and no $y \in V(T)$ such that $y^i > x^i$ for all $i \in T$.

DEFINITION 3.7: Given a game (v, N), the coalition structure $\mathscr{B}$ is *α-stable* (*β-stable*) if $\phi(v, \mathscr{B})$ belongs to the core of $(V^{(\alpha)}, N)$ $((V^{(\beta)}, N)$, respectively).

It is clear that $V^{(\beta)}(S) \supset V^{(\alpha)}(S)$ for all S, hence β-stability implies α-stability. In order to compare with γ- and δ-stability, the following description is useful: $\mathscr{B}$ is *not* α-stable if there exists a nonempty $T \subset N$ and $(\hat{\sigma}^i)_{i \in T} \in \Sigma^T$ such that, for all $(\hat{\sigma}^j)_{j \in N \setminus T} \in \Sigma^{N \setminus T}$, $\phi^i(v, \hat{\mathscr{B}}) > \phi^i(v, \mathscr{B})$ for all $i \in T$ (where $\hat{\mathscr{B}}$ corresponds to $\hat{\sigma}$). $\mathscr{B}$ is *not* β-stable if there exists a nonempty $T \subset N$ such that, for every strategy $(\hat{\sigma}^j)_{j \in N \setminus T} \in \Sigma^{N \setminus T}$ there exists $(\hat{\sigma}^i)_{i \in T} \in \Sigma^T$ with $\phi^i(v, \hat{\mathscr{B}}) > \phi^i(v, \mathscr{B})$ for all $i \in T$ (again, $\hat{\mathscr{B}}$ corresponds to $\hat{\sigma}$). In the first case, the members of T can guarantee an improvement, independent of what the others will do; in the latter, they cannot be prevented from getting better off, for anything the rest do (but, dependent on what they do). Since in the definition of γ- and δ-stability, it is assumed that the complement of T does not change its strategies, it follows that β-stability is implied both by γ-stability and by δ-stability. For a given game v, let $\mathscr{S}^{(\alpha)}(v)$, $\mathscr{S}^{(\beta)}(v)$, $\mathscr{S}^{(\gamma)}(v)$, and $\mathscr{S}^{(\delta)}(v)$ denote the set of all α-, β-, γ-, and δ- (respectively) stable coalition structures. We thus have for every game v:

$$\mathscr{S}^{(\alpha)}(v) \supset \mathscr{S}^{(\beta)}(v) \supset \left[\mathscr{S}^{(\gamma)}(v) \cup \mathscr{S}^{(\delta)}(v) \right].$$

These two new notions of stability are useful whenever the participants are in some sense "careful"—they take into account the possible reactions of the others, when deciding upon a change in the coalition structure. Incidentally, there are games for which no coalition structure is even α-stable (cf. Hart and Kurz [**4**, Proposition 5.14]).

A second direction of study is to consider *hierarchical coalition structures*, with coalitions whose members are themselves coalitions of the original players, and so on. There is no difficulty in defining the CS-value in this case, using similar axioms (see Owen [7, pp. 85–86]).

Thirdly, one may look for *stable sets* of coalition structures, with respect to the domination relation (i.e., $\mathscr{B}_1$ dominates $\mathscr{B}_2$ if there exists a nonempty coalition $T \subset N$ that can induce $\mathscr{B}_1$ from $\mathscr{B}_2$, and such that $\phi^i(v, \mathscr{B}_1) > \phi^i(v, \mathscr{B}_2)$ for all $i \in T$). Since the number of coalition structures is finite, such sets always exist (note that $\mathscr{B}$ is a stable coalition structure if and only if $\{\mathscr{B}\}$ is a stable set).

The reader interested in the application[10] of these concepts is referred to Hart and Kurz [4].

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[10]We would like to point out that there is a *finite* procedure for finding all stable coalition structures in any game (this is not the case for other concepts, e.g., the von Neumann–Morgenstern solution).

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