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EQUALLY DISTRIBUTED CORRESPONDENCES

Sergiu HART

Department of Mathematics, Tel Aviv University, Tel Aviv, Israel

Elon KOHLBERG

Department of Economics, Harvard University, Cambridge, Mass., U.S.A.

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The distribution $\mathscr{D}\varphi$ of a correspondence φ is defined, and its connection with the set $\mathscr{D}\mathscr{L}_\varphi$ of distributions of its integrable selections is explored. The main result is that if φ_1 and φ_2 are equally distributed, i.e., if $\mathscr{D}\varphi_1 = \mathscr{D}\varphi_2$, then $\mathscr{D}\mathscr{L}_{\varphi_1}$ and $\mathscr{D}\mathscr{L}_{\varphi_2}$ have the same closure in the weak convergence topology.

1. Introduction

The study of measurable and integrable selections from a correspondence (set-valued function) has been of interest for some time [e.g., see Aumann (1965), Hildenbrand (1974), etc.].

Here we take up the study of *distributions* of such selections. The motivation for doing this comes from Mathematical Economics [see Kannai (1970, section 7); the reader interested in the economic applications is referred to Hart et al. (1974)].

The first question we consider is the following: Does the distribution of a correspondence determine the distributions of its selections? If the underlying space contains atoms, then of course we cannot expect an affirmative answer (e.g., consider the correspondence whose image is always the set $\{1, 2\}$, once on a space consisting of one atom, and once on a space consisting of two atoms). But what if the underlying space is atomless?

Consider the following example [which is essentially a reformulation of an example due to G. Debreu – see Kannai (1970, section 7)].

Let

$$\varphi_1(t) = \{t, -t\} \quad \text{for all} \quad t \in [0, 1],$$

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and

$$\varphi_2(t) = \begin{cases} \{2t, -2t\}, & \text{for} \quad t \in [0, \frac{1}{2}], \\ \{2t-1, 1-2t\}, & \text{for} \quad t \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly, φ_1 and φ_2 must be equally distributed by any reasonable definition of this concept. However, the distributions of their measurable selections are not the same: let

$$f(t) = \begin{cases} 2t, & \text{for} \quad t \in [0, \frac{1}{2}], \\ 1-2t, & \text{for} \quad t \in (\frac{1}{2}, 1], \end{cases}$$

then f is a measurable selection from φ_2, but there is no measurable selection from φ_1 with the same distribution.

The above example, in giving a negative answer to our question, raises serious difficulties as to possible applications of the concept of 'distributions of selections'. However, most of the difficulties can be eliminated provided the following is true: if φ_1 and φ_2 are equally distributed correspondences, then the *closures* of the distributions of their selections are equal. Theorem 1 is the precise formulation of this statement.

In Theorem 3 we show that when the measurable selections are restricted to have a constant integral, the closure of their distributions still depends only on the distribution of the correspondence [this restriction is of importance in Mathematical Economics; indeed, our Theorem 3 is a well-known conjecture of R.J. Aumann – see Kannai (1970, section 7)].

2. Preliminaries

We denote by $\mathbf{R}^l$ the l-dimensional Euclidian space, and by $\mathscr{B}(\mathbf{R}^l)$ the σ-algebra of its Borel subsets.

Let $(A, \mathscr{A}, \nu)$ be a measure space; in the following, all measure spaces will be assumed to be complete [i.e., $\mathscr{A}$ contains all subsets of null sets – see Hildenbrand (1974, D.I)]. Null sets will be systematically ignored.

Let $f:(A, \mathscr{A}, \nu) \to \mathbf{R}^l$ be a measurable function. The *distribution* $\mathscr{D}f$ *of* f is defined as $\nu \circ f^{-1}$ [i.e., the induced measure on $\mathscr{B}(\mathbf{R}^l)$]. The *sequence*[1] $\{f_n\}$ *converges to* f *in distribution* if the sequence $\{\mathscr{D}f_n\}$ converges weakly to $\mathscr{D}f$, where weak convergence of measures is defined as usual by $\mu_n \xrightarrow{W} \mu$ if $\int h \mathrm{d}\mu_n \to \int h \mathrm{d}\mu$ for all real, continuous and bounded functions h. The topology of weak convergence can be metrized, e.g., by the Prohorov metric ρ, defined as

$$\rho(\mu_1, \mu_2) = \inf\{\varepsilon > 0 \,|\, \mu_1(B) \leqq \mu_2(B_\varepsilon)+\varepsilon \text{ and } \mu_2(B) \leqq \mu_1(B_\varepsilon)+\varepsilon \text{ for all Borel subsets } B\},$$

where

$$B_\varepsilon = \{x \,|\, \mathrm{d}(x, B) < \varepsilon\}.$$

[1]Note that the f_n's can be defined on different spaces, but their range must always be in the same space.

Let[2] $\varphi:(A, \mathscr{A}, v) \to \mathbf{R}^l$ be a correspondence, i.e., $\varphi(a)$ is a non-empty subset of $\mathbf{R}^l$ for all $a \in A$. The graph of φ is the set

$$G_\varphi = \{(a, x) | x \in \varphi(a)\}.$$

φ *has measurable graph* [or 'is Borel-measurable', as in Aumann (1965)] if $G_\varphi \in \mathscr{A} \otimes \mathscr{B}(\mathbf{R}^l)$. φ is *integrably bounded* if there exists an integrable function $h:(A, \mathscr{A}, v) \to \mathbf{R}^l$ such that $-h(a) \leqq x \leqq h(a)$ for all $x \in \varphi(a)$ and for all $a \in A$. φ is *closed-valued* if $\varphi(a)$ is a closed set (in $\mathbf{R}^l$) for every $a \in A$.

The set of all integrable selections from φ [i.e., all integrable functions f such that $f(a) \in \varphi(a)$ for all $a \in A$] is denoted $\mathscr{L}_\varphi$, and $\int\varphi = \{\int f | f \in \mathscr{L}_\varphi\}$. For every B, let

$$\varphi^{-1}(B) = \{a \in A | \varphi(a) \cap B \neq \phi\}$$

[the (weak) inverse of φ].

All the definitions up to now are standard [see, e.g., Billingsley (1968) and Hildenbrand (1974)]. At this point we must make precise the notion of 'equally distributed correspondences'. We therefore define the *distribution* $\mathscr{D}\varphi$ *of the correspondence* φ having a measurable graph, by $\mathscr{D}\varphi = v \circ \varphi^{-1}$. This definition is meaningful since by the projection theorem [Hildenbrand (1974, D.I(11))], if φ has a measurable graph then $\varphi^{-1}(B)$ is measurable for all $B \in \mathscr{B}(\mathbf{R}^l)$. Note that $\mathscr{D}\varphi$ is not necessarily additive; if, for all $a \in A$, $\varphi(a)$ consists of just one point, then this definition coincides with the usual one for functions. We say that φ_1 *and* φ_2 *are equally distributed* if $\mathscr{D}\varphi_1 = \mathscr{D}\varphi_2$.

As a Corollary to Theorem 4, we shall prove that, at least for closed-valued correspondences, our definition of equally distributed correspondences coincides with the usual one of functions with the same distribution, when regarding the correspondences as point-valued functions into the set of (closed) subsets of $\mathbf{R}^l$.

The following notations will be used: $\setminus$ for set-theoretic substraction, K^c for the complement of K, and $\int f$ for $\int_A f(a)\mathrm{d}v(a)$. For a set $\mathscr{F}$ of functions with the same range, $\mathscr{D}\mathscr{F}$ will denote the set of their distributions, i.e., $\mathscr{D}\mathscr{F} = \{\mathscr{D}f | f \in \mathscr{F}\}$, and $\overline{\mathscr{D}\mathscr{F}}$ will be its closure with respect to the weak convergence of measures.

3. Statement of the results

In Theorems 1–3, $(A_i, \mathscr{A}_i, v_i)$ (for $i = 1, 2$) will be non-atomic complete probability measure spaces, and $\varphi_i:(A_i, \mathscr{A}_i, v_i) \to \mathbf{R}^l$ correspondences with measurable graphs.

Theorem 1. *Let* φ_1 *and* φ_2 *be integrably bounded. If* φ_1 *and* φ_2 *are equally distributed, then the sets* $\mathscr{D}\mathscr{L}_{\varphi_1}$ *and* $\mathscr{D}\mathscr{L}_{\varphi_2}$ *of distributions of the integrable*

[2]We write $\varphi:A \to \mathbf{R}^l$ also for correspondences, but then we mean $\varphi(a) \subset \mathbf{R}^l$ for all $a \in A$; no confusion will arise, since correspondences are always denoted by the Greek letters φ or ψ.

selections of φ_1 and φ_2, respectively, have the same closure with respect to the weak convergence, i.e.,

$$\overline{\mathscr{D}\mathscr{L}_{\varphi_1}} = \overline{\mathscr{D}\mathscr{L}_{\varphi_2}}.$$

The following theorem is an immediate consequence of Theorem 1.

Theorem 2. *Let φ_1 and φ_2 be integrably bounded and closed-valued. If φ_1 and φ_2 are equally distributed, then $\int\varphi_1 = \int\varphi_2$.*

Theorem 3. *Let φ_1 and φ_2 be integrably bounded and closed-valued, and let $x \in \mathbf{R}^l$. If φ_1 and φ_2 are equally distributed, then the sets of distributions of the integrable selections of φ_1 and φ_2 with integral equal to x, have the same closure with respect to the weak convergence, i.e.,*

$$\overline{\mathscr{D}\{f \in \mathscr{L}_{\varphi_1} | \int f = x\}} = \overline{\mathscr{D}\{g \in \mathscr{L}_{\varphi_2} | \int g = x\}}.$$

Clearly, if $\varphi_i = \psi \circ h_i$, where the functions h_1 and h_2 have the same distribution, then φ_1 and φ_2 are equally distributed (this is, indeed, the case in Mathematical Economics). The following decomposition theorem asserts that this is the general structure of equally distributed correspondences.

Theorem 4. *For $i = 1, 2$, let $(A_i, \mathscr{A}_i, \nu_i)$ be complete probability measure spaces, and let $\varphi_i:(A_i, \mathscr{A}_i, \nu_i) \to \mathbf{R}^l$ be closed-valued correspondences with measurable graphs. Then $\mathscr{D}\varphi_1 = \mathscr{D}\varphi_2$ if and only if there exist a measurable space $(E, \mathscr{E})$, a correspondence $\psi:(E, \mathscr{E}) \to \mathbf{R}^l$ with measurable graph, and measurable functions $h_i:(A_i, \mathscr{A}_i, \nu_i) \to (E, \mathscr{E})$ such that $\varphi_i = \psi \circ h_i$ for $i = 1, 2$, and $\mathscr{D}h_1 = \mathscr{D}h_2$.*

Let $\mathscr{C}^l$ denote the family of all closed and non-empty subsets of $\mathbf{R}^l$, endowed with the topology of 'closed convergence' [e.g., see Hildenbrand (1974, B.II)], and let $\mathscr{B}(\mathscr{C}^l)$ be its Borel σ-field (i.e., the σ-field generated by the open subsets of $\mathscr{C}^l$); a discussion of these concepts will be found before the proof of Theorem 4.

Corollary. For $i = 1, 2$, let $(A_i, \mathscr{A}_i, \nu_i)$ be complete probability measure spaces, and let $\varphi_i:(A_i, \mathscr{A}_i, \nu_i) \to \mathbf{R}^l$ be closed-valued correspondences with measurable graphs. Let $h_i:(A_i, \mathscr{A}_i, \nu_i) \to (\mathscr{C}^l, \mathscr{B}(\mathscr{C}^l))$ be the corresponding point-valued functions, i.e., the functions defined by $h_i(a) = \varphi_i(a) \in \mathscr{C}^l$ for all $a \in A_i$. Then $\mathscr{D}\varphi_1 = \mathscr{D}\varphi_2$ if and only if $\mathscr{D}h_1 = \mathscr{D}h_2$.

4. Proof of the results

In the proof of Theorem 1 we rely on the following lemma:

Lemma. Let $(A, \mathscr{A}, \nu)$ be a non-atomic measure space. Let $\{S_i\}_{i=0}^m \subset \mathscr{A}$ and $\{\alpha_i\}_{i=0}^m$, $\alpha_i \geqq 0$ be such that

$$\nu(\bigcup_{i\in I} S_i) \geqq \sum_{i\in I} \alpha_i, \quad \textit{for all} \quad I \subset \{0, 1 \ldots, m\}, \tag{1}$$

and

$$\nu(\bigcup_{i=0}^{m} S_i) = \sum_{i=0}^{m} \alpha_i. \tag{2}$$

Then there exist disjoint sets $\{T_i\}_{i=0}^m$ such that $T_i \subset S_i$ and $\nu(T_i) = \alpha_i$ for all $i = 0, 1, \ldots, m$.

The proof of this lemma may be carried out in complete analogy with the proof of a well known result in Combinatorics [see Halmos and Vaughan (1950)]. Since it is quite short, we repeat it here.

Proof. We use induction on m. For $m = 0$ it is trivial. Let $m > 0$. We distinguish two cases.

Case (i). For some $I \subsetneqq \{0, 1, \ldots, m\}$, there is equality in (1). Then it is easily verified that both $\{S_i, \alpha_i\}_{i\in I}$ and $\{S_j \backslash \bigcup_{i\in I} S_i, \alpha_j\}_{j\notin I}$ satisfy (1) and (2), and we may apply the induction hypothesis separately to each one of them.

Case (ii). For all $I \subsetneqq \{0, 1, \ldots, m\}$, there is strict inequality in (1). In particular, $\nu(S_0) > \alpha_0$ and $\nu(S_0 \backslash \bigcup_{i\neq 0} S_i) < \alpha_0$. Since ν is non-atomic, we may find an S_0' such that

$$S_0 \backslash \bigcup_{i\neq 0} S_i \subset S_0' \subset S_0,$$

and the replacement of S_0 by S_0' will preserve all inequalities in (1), but at least one of them will be an equality. Clearly, (2) is still valid, and we may proceed as in case (i). Q.E.D.

Proof of Theorem 1

Let $f \in \mathscr{L}_{\varphi_1}$ and let $\varepsilon > 0$. We must find $g \in \mathscr{L}_{\varphi_2}$ such that $\rho(\mathscr{D}f, \mathscr{D}g) < \varepsilon$.

Since f is integrable, there is a compact set K in $\mathbf{R}^l$ such that $\nu_1(f^{-1}(K^c)) < \varepsilon$. Let $K_0 = K^c$ and let $K_1, K_2, \ldots, K_m$ be a partition of K into disjoint Borel sets of diameter less than ε.

Define $\alpha_i = \nu_1(f^{-1}(K_i))$ and $S_i = \varphi_2^{-1}(K_i)$ for $i = 0, 1, \ldots, m$.

Since φ_1 and φ_2 are equally distributed it follows that for every $I \subset \{0, 1, \ldots, m\}$

$$\nu_1(\varphi_1^{-1}(\bigcup_{i\in I} K_i)) = \nu_2(\varphi_2^{-1}(\bigcup_{i\in I} K_i)) = \nu_2(\bigcup_{i\in I} S_i).$$

The K_i's are disjoint, hence

$$\nu_1(\varphi_1^{-1}(\bigcup_{i\in I} K_i)) \geqq \nu_1(f^{-1}(\bigcup_{i\in I} K_i)) = \sum_{i\in I} \nu_1(f^{-1}(K_i)) = \sum_{i\in I} \alpha_i.$$

We now have

$$\nu_2(\bigcup_{i\in I} S_i) \geqq \sum_{i\in I} \alpha_i, \quad \text{for all} \quad I \subset \{0, 1, \ldots, m\}, \quad \text{and}$$

$$\nu_2(\bigcup_{i=0}^{m} S_i) = 1 = \sum_{i=0}^{m} \alpha_i,$$

therefore we may apply the Lemma to get a partition $\{T_i\}_{i=0}^m$ of A_2 such that $T_i \subset S_i = \varphi_2^{-1}(K_i)$ and $\nu_2(T_i) = \alpha_i$.

Define $g|_{T_i}$ to be an integrable selection of the correspondence $\varphi_2 \cap K_i$; since φ_2 is integrably bounded, this selection is possible [Hildenbrand (1974, D.II.4, theorem 2)]. Then $g \in \mathscr{L}_{\varphi_2}$ and it is easy to verify that $\rho(\mathscr{D}f, \mathscr{D}g) < \varepsilon$. Q.E.D.

Proof of Theorem 2

Let $x \in \int\varphi_1$ and let $f \in \mathscr{L}_{\varphi_1}$ be such that $\int f = x$; by Theorem 1 there is a sequence $\{g_n\}$ converging in distribution to f, $g_n \in \mathscr{L}_{\varphi_2}$. Since φ_2 is integrably bounded, $\int g_n \to \int f = x$ [Billingsley (1968, theorem 5.4)]. Therefore x is in the closure of $\int\varphi_2$. But $\int\varphi_2$ is compact [Aumann (1965, theorem 4)], and the proof is completed. Q.E.D.

Proof of Theorem 3

Let $C = \int\varphi_1 = \int\varphi_2$ (by Theorem 2). C is convex and compact [Aumann (1965, theorems 1 and 4)].

We proceed by induction on the dimension of C in $\mathbf{R}^l$. If $\dim(C) = 0$, then the theorem follows at once from Theorem 1.

Next, suppose $\dim(C) = n$.

Let $x \in C$ and let $f \in \mathscr{L}_{\varphi_1}$ be such that $x = \int f$. We must find a $g \in \mathscr{L}_{\varphi_2}$ such that $x = \int g$ and $\rho(\mathscr{D}f, \mathscr{D}g) < \varepsilon$.

Case (i). $x \in$ rel-int C. Let $r > 0$ be the distance of x from the boundary of C. Applying Theorem 1 and the integrable boundedness of φ_2, we can find $g' \in \mathscr{L}_{\varphi_2}$ such that $\rho(\mathscr{D}f, \mathscr{D}g') < \varepsilon/2$ and

$$|x - \int g'| < r \cdot \varepsilon/2.$$

Let y be the intersection of the boundary of C with the half line from $\int g'$ to x, and let $g'' \in \mathscr{L}_{\varphi_2}$ be such that $\int g'' = y$. Clearly, $x = \alpha y + (1-\alpha)\int g'$, where $\alpha < \varepsilon/2$. Applying Lyapunov's (1940) theorem, we obtain a set $S \subset A_2$ satisfying $\nu_2(S) = \alpha$, $\int_S g' = \alpha \int g'$ and $\int_S g'' = \alpha \int g''$. Define g by

$$g(a) = \begin{cases} g'(a), & a \notin S, \\ g''(a), & a \in S, \end{cases}$$

then clearly $\int g = x$ and $\rho(\mathscr{D}f, \mathscr{D}g) < \varepsilon$.

Case (ii). $x \notin$ rel-int C. Let q define a supporting hyperplane such that $q \cdot x = \max q \cdot C$.

Denote

$$\hat{\varphi}_i(a) = \{y \in \varphi_i(a) | q \cdot y = \max q \cdot \varphi_i(a)\}, \quad \text{for} \quad a \in A_i, \quad i = 1, 2.$$

By Hildenbrand (1974, D.II.3, proposition 3), $\hat{\varphi}_i$ have measurable graphs. Also, $\hat{\varphi}_i$ are integrably bounded and closed-valued, and $f \in \mathscr{L}_{\hat{\varphi}_1}$ [by Hildenbrand (1974, D.II.4, proposition 6)].

We now claim that $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are equally distributed. To see this, apply[3] Theorem 4 to get decompositions $\varphi_i = \psi \circ h_i$, then $\hat{\varphi}_i = \hat{\psi} \circ h_i$, where

$$\hat{\psi}(t) = \{y \in \psi(t) | q \cdot y = \max q \cdot \psi(t)\}.$$

But dim $\left(\int \hat{\varphi}_1\right) < n$, hence by the induction hypothesis there is a $g \in \mathscr{L}_{\hat{\varphi}_2} \subset \mathscr{L}_{\varphi_2}$ such that $x = \int g$ and $\rho(\mathscr{D}f, \mathscr{D}g) < \varepsilon$. Q.E.D.

Before proving Theorem 4, we have to introduce the concepts of topology and Borel σ-field on $\mathscr{C}^l$, the family of closed subsets of $\mathbf{R}^l$. The usual way to do this is the following [for discussion, proofs and references see Hildenbrand (1974, B.II)].

Let $X = \mathbf{R}^l \cup \{\infty\}$ be the one-point compactification of $\mathbf{R}^l$, then X is a metric space (it is homeomorphic to the unit sphere of $\mathbf{R}^{l+1}$). Let $\mathscr{K}$ be the family of its closed (i.e., compact) non-empty subsets, endowed with the Hausdorff metric (or, equivalently, with the topology of 'closed convergence'). Let $\mathscr{B}(\mathscr{K})$ be its Borel σ-field, i.e., generated by the open subsets of $\mathscr{K}$. It follows from a result of Dubins and Orenstein [Debreu (1967, (3.1))] that $\mathscr{B}(\mathscr{K})$ is generated by the sets V^w for all open subsets V of X, where

$$V^w = \{K \in \mathscr{K} | K \cap V \neq \phi\}.$$

To each closed subset C of R^l (i.e., $C \in \mathscr{C}^l$), we associate the member $C \cup \{\infty\}$ of $\mathscr{K}$; then $\mathscr{C}^l$ can be regarded as a subset of $\mathscr{K}$, namely

$$\{K \in \mathscr{K} | \infty \in K\}.$$

The topology of 'closed convergence' on $\mathscr{C}^l$ is then the induced topology from $\mathscr{K}$, and $\mathscr{C}^l$ is a compact metric space. Moreover, let $\mathscr{B}(\mathscr{C}^l)$ be its Borel σ-field, then it is easy to see that

(*) $\mathscr{B}(\mathscr{C}^l)$ is generated by the sets U^w for all open subsets U of R^l, where $U^w = \{C \in \mathscr{C}^l | C \cap U \neq \phi\}$.

For any closed-valued correspondence $\varphi: (A, \mathscr{A}, \nu) \to \mathbf{R}^l$, let $h_\varphi: (A, \mathscr{A}, \nu) \to \mathscr{C}^l$ denote the corresponding point-valued function, i.e., the function defined by $h_\varphi(a) = \varphi(a)$ for all $a \in A$.

Proof of Theorem 4

The 'if' part is immediate.

To prove the converse, let $h_i = h_{\varphi_i}$ be defined as above, and let $\psi: (\mathscr{C}^l, \mathscr{B}(\mathscr{C}^l)) \to \mathbf{R}^l$ be defined by $\psi(C) = C$ for all $C \in \mathscr{C}^l$; obviously $\varphi_i = \psi \circ h_i$.

[3]The proof of Theorem 4 does not depend on Theorem 3.

Let U be an open subset of $\mathbf{R}^l$, then $\psi^{-1}(U) = U^w \in \mathscr{B}(\mathscr{C}^l)$. Since $\mathscr{C}^l$ is a complete separable metric space and ψ is a closed-valued correspondence, it follows from [Hildenbrand (1974, D.II.3, proposition 4(b)] that ψ has a measurable graph.

Again, let U be an open subset of $\mathbf{R}^l$, then $h_i^{-1}(U^w) = \varphi_i^{-1}(U)$. Since φ_i has measurable graph, $\varphi_i^{-1}(U) \in \mathscr{A}$ by Hildenbrand (1974, D.II.3, proposition 4(a)), hence h_i is a measurable function.

Moreover, $\mathscr{D}h_i(U^w) = \mathscr{D}\varphi_i(U)$, hence $\mathscr{D}\varphi_1 = \mathscr{D}\varphi_2$ implies $\mathscr{D}h_1 = \mathscr{D}h_2$ by (*). Q.E.D.

The Corollary now follows immediately from Theorem 4 and its proof.

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