Canonical Subgroups over Hilbert Modular Varieties

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The Problem. Associate in a natural way to a g-dimensional abelian variety A/R, $[R : \mathbb{Z}_p] < \infty$, with real multiplication an invariant isotropic subgroup H such that $pH = \{0\}$, and $\sharp H = p^g$.

Motivation. Properties of the U-operator on overconvergent modular forms; used to prove classicality results for modular forms, study of p-adic families of modular forms, special values of L-functions, and modularity of Galois representations. Cf. Besser's talk.

If A has ordinary reduction: classical ✓
(There is a unique way to lift the kernel of Frobenius $\mathrm{Fr} : \overline{A} \to \overline{A}^{(p)}$, where $\overline{A} = A \pmod{pR}$.)

Therefore, the problem is:

- **Extend this to non-ordinary abelian varieties;**
- **Do it in families.**

Reformulation

For appropriate moduli schemes X, Y over $\mathbb{Z}_p$, $\mathfrak{X}_{\mathrm{rig}}, \mathfrak{Y}_{\mathrm{rig}}$ the associated (Raynaud) generic fibers, we have a diagram,

$$\begin{array}{ccc} (A, H) & \rightsquigarrow & \mathfrak{Y}_{\mathrm{rig}} \\ & & \pi \downarrow \; \uparrow s \\ A & \rightsquigarrow & \mathfrak{X}_{\mathrm{rig}}, \end{array}$$

and the section s exists over the ordinary locus.

The Problem. Extend s "as much as possible".

The main theorem

Theorem (G. - Kassaei)

Let $\{\tilde{h}_\beta\}_{\beta\in\mathbb{B}}$ be (Zariski local) lifts of the partial Hasse invariants. Let $\mathcal{U} \subset \mathfrak{X}_{\mathrm{rig}}$ be

$$\mathcal{U} = \{P : \nu(\tilde{h}_\beta(P)) + p\nu(\tilde{h}_{\sigma^{-1}\circ\beta}(P)) < p,\ \forall\beta \in \mathbb{B}\}.$$

There exists a section $s^\dagger : \mathcal{U} \to \mathfrak{Y}_{\mathrm{rig}}$, extending the section s on the ordinary locus.

What comes into the proof?

1. Stratifications of $\overline{X}, \overline{Y}$ (the special fibers).
2. Study of $\pi : \overline{Y} \to \overline{X}$ on completed local rings.
3. "Dissection" of $\mathfrak{Y}_{\mathrm{rig}}$, the generic fiber of Y, using g different valuations.
   ($g = [L : \mathbb{Q}]$, where L is the totally real field acting.)

Remark. *The structure suggests strategy should be applicable to many Shimura varieties of PEL type.*

Notation

- L - totally real field, $[L:\mathbb{Q}] = g$.
- p unramified in L.

$$\mathbb{B} = \mathrm{Hom}(L, \mathbb{Q}_p^{ur}) = \coprod_{\mathfrak{p}|p} \mathbb{B}_{\mathfrak{p}} \circlearrowleft \sigma.$$

(σ = Frobenius, lift of $x \mapsto x^p$.)

- For $S \subseteq \mathbb{B}$, let $S^c = \mathbb{B} \setminus S$ and

$$\ell(S) = \{\sigma^{-1} \circ \beta : \beta \in S\}, \quad r(S) = \{\sigma \circ \beta : \beta \in S\}.$$

- κ = minimal field $\supseteq \mathcal{O}_L/\mathfrak{p}$, $\forall \mathfrak{p}|p$.
- $\boxed{\mathcal{O}_L \otimes_{\mathbb{Z}} W(\kappa) \cong \oplus_{\beta \in \mathbb{B}} W(\kappa)_\beta}$ induces a decomposition of any $\mathcal{O}_L \otimes W(\kappa)$-module.

Moduli Spaces

$X/W(\kappa)$ parameterizes $\underline{A} = (A, \iota, \alpha, \lambda_A)/S$, where S is a $W(\kappa)$-scheme and:

$A \rightarrow S$ abelian scheme, of rel. dim'n g, $\quad \iota : \mathcal{O}_L \hookrightarrow \mathrm{End}_S(A)$,

$\alpha =$ rigid $\Gamma_{00}(N)$-level structure.

$\lambda_A : (\mathfrak{a}, \mathfrak{a}^+) \xrightarrow{\cong} (\mathcal{P}_A, \mathcal{P}_A^+)$ a polarization: $A \otimes \mathfrak{a} \cong A^t$,

$\mathcal{P}_A = \mathrm{Hom}_{\mathcal{O}_L}(A, A^t)^{sym}$ with the positive cone of polarizations.

$Y/W(\kappa)$ parameterizes $(\underline{A}, H)$ such that:

H is killed by p, degree p^g, $\mathcal{O}_L$-invariant, isotropic

Equivalently,

$$(f : \underline{A} \rightarrow \underline{B}),$$

such that $\deg(f) = p^g, \mathrm{Ker}(f) \subseteq A[p]$, $f^*\mathcal{P}_B = p\mathcal{P}_A$.

Atkin-Lehner: $w(f : \underline{A} \rightarrow \underline{B}) = (f^t : \underline{B} \rightarrow \underline{A})$, $\quad f \circ f^t = p$

Invariants for $\overline{X}$

For $\underline{A}/k$, $k \supseteq \kappa$ perfect. Let

$$\alpha_A = \mathrm{Ker}(\mathrm{Fr}_A) \cap \mathrm{Ker}(\mathrm{Ver}_A).$$

Decomposition of the Dieudonné modules:

$$\mathbb{D}(A[p]) = \oplus_{\beta\in\mathbb{B}}\, k^2 \quad \supseteq \quad \mathbb{D}(\alpha_A) = \oplus_{\beta\in\mathbb{B}} \underbrace{\mathbb{D}(\alpha_A)_\beta}_{0 \text{ or } 1 \text{ dim'l}}$$

The type of $\underline{A}$ is

$$\tau(\underline{A}) = \{\beta \in \mathbb{B} : \mathbb{D}(\alpha_A)_\beta \neq \{0\}\}.$$

Define strata of $\overline{X}$,

$$\boxed{\begin{array}{ll} W_\tau \leftrightsquigarrow \{\underline{A} : \tau(\underline{A}) = \tau\} & \text{(locally closed)}, \\ Z_\tau \leftrightsquigarrow \{\underline{A} : \tau(\underline{A}) \supseteq \tau\} & \text{(closed)}. \end{array}}$$

Theorems (G. - Oort)

1. $\overline{W_\tau} = Z_\tau = \coprod_{\tau' \supseteq \tau} W_{\tau'}$. So $\{W_\tau : \tau \subseteq \mathbb{B}\}$ is a stratification of the moduli space $\overline{X}$ by 2^g strata.
2. W_τ is non-singular, quasi-affine of dimension $g - \sharp\,\tau$.
3. $\exists h_\beta$, a Hilbert modular form of weight $p \cdot \sigma^{-1} \circ \beta - \beta$, such that $(h_\beta) = Z_\beta$.
   (In classical terms: weight $(0, \dots, 0, p, -1, 0 \dots, 0)$.)
4. $\widehat{\mathcal{O}}_{\overline{X},P} \cong k[[t_\beta : \beta \in \mathbb{B}]]$ and if $h_\beta(P) = 0$ then we may identify h_β with t_β.
5. The kernel of the q-expansion map on the graded ring of Hilbert modular forms modulo p is the ideal $\langle h_\beta - 1 : \beta \in \mathbb{B} \rangle$.

Invariants for $\overline{Y}$

Given $\underline{A} \underset{f^t}{\overset{f}{\rightleftarrows}} \underline{B}$, we have

$$\bigoplus_\beta \mathrm{Lie}(\underline{A})_\beta \underset{\oplus_\beta \mathrm{Lie}(f^t)_\beta}{\overset{\bigoplus_\beta \mathrm{Lie}(f)_\beta}{\rightleftarrows}} \oplus_\beta \mathrm{Lie}(\underline{B})_\beta.$$

Define

- $\varphi = \varphi(f) = \{\beta \in \mathbb{B} : \mathrm{Lie}(f)_{\sigma^{-1}\circ\beta} = 0\}$,
- $\eta = \eta(f) = \{\beta \in \mathbb{B} : \mathrm{Lie}(f^t)_\beta = 0\}$,
- $I = \ell(\varphi) \cap \eta$ (the "critical indices").

Properties:

1. $(\varphi \triangle \eta)^c \supseteq \tau(\underline{A}) \supseteq \varphi \cap \eta$.
2. $\eta \supseteq \ell(\varphi)^c$.

A pair (φ, η) (for $\varphi, \eta \subseteq \mathbb{B}$) is admissible if

$$\eta \supseteq \ell(\varphi)^c.$$

Exist 3^g such pairs.

Define strata in $\overline{Y}$:

$$W_{\varphi,\eta} \leftrightsquigarrow \{(f : \underline{A} \to \underline{B}) : \varphi(f) = \varphi, \eta(f) = \eta\} \quad \text{(loc. closed)},$$
$$Z_{\varphi,\eta} \leftrightsquigarrow \{(f : \underline{A} \to \underline{B}) : \varphi(f) \supseteq \varphi, \eta(f) \supseteq \eta\} \quad \text{(closed)}.$$

Theorems

1. $\overline{W_{\varphi,\eta}} = Z_{\varphi,\eta} = \coprod_{(\varphi',\eta')\geq(\varphi,\eta)} W_{\varphi',\eta'}$ and so $\{W_{\varphi,\eta}\}$ is a stratification of $\overline{Y}$ with 3^g strata.
2. $W_{\varphi,\eta}$ and $Z_{\varphi,\eta}$ are non-singular, equi-dimensional of dimension $2g - (\sharp\,\varphi + \sharp\,\eta)$.
3. There are 2^g maximal strata, given by $Z_{\varphi,\ell(\varphi)^c}, \quad \varphi \subseteq \mathbb{B}$. There are 2^r horizontal components, where $r = \sharp\{\mathfrak{p}|p\}$.
   Two of which are $\overline{Y}_F = Z_{\mathbb{B},\emptyset} \leftrightsquigarrow (\underline{A}, \mathrm{Ker}(\mathrm{Fr}_A))$
   $$\overline{Y}_V = Z_{\emptyset,\mathbb{B}} \leftrightsquigarrow (\underline{A}, \mathrm{Ker}(\mathrm{Ver}_A)).$$
4. $w(Z_{\varphi,\eta}) = Z_{r(\eta),\ell(\varphi)}$.
5. $\pi(Z_{\varphi,\eta}) = Z_{\varphi\cap\eta}$.
6. If $C \subseteq Z_{\varphi,\eta}$ is an irreducible component then
   $$C \cap \overline{Y}_F \cap \overline{Y}_V \neq \emptyset.$$

7. Let $\overline{Q} \in \overline{Y}$ be a closed k-point, then

$$\widehat{\mathcal{O}}_{Y,\overline{Q}} \cong \frac{W(k)[[\ \{x_\beta : \beta \in I\}, \{y_\beta : \beta \in I\}, \{z_\beta : \beta \in I^c\}\ ]]}{\langle \{x_\beta y_\beta - p : \beta \in I\} \rangle}.$$

Moreover, the variables can be chosen so that: If

$$\varphi \supseteq \varphi' \supseteq \varphi - r(I), \quad \eta \supseteq \eta' \supseteq \eta - I,$$

and (φ', η') is admissible, write

$$\varphi' = \varphi - J, \quad \eta' = \eta - K,$$

then $Z_{\varphi',\eta'}$ is described in $\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}}$ by the ideal

$$\langle \{x_\beta : \beta \in I - K\}, \{y_\beta : \beta \in I - \ell(J)\} \rangle.$$

Moreover, if $\overline{Q} \in Z_{\varphi',\eta'}$ then (φ', η') are as above.

Key Lemma

Let $\beta \in \varphi \cap \eta = \tau$, $\pi(\overline{Q}) = \overline{P}$,

$$\pi^* : \widehat{\mathcal{O}}_{\overline{X},\overline{P}} \longrightarrow \widehat{\mathcal{O}}_{\overline{Y},\overline{Q}}.$$

Then:

$$\pi^*(t_\beta) = \begin{cases} ux_\beta + vy^p_{\sigma^{-1}\circ\beta} & \sigma\circ\beta \in \varphi, \quad \sigma^{-1}\circ\beta \in \eta, \\ ux_\beta & \sigma\circ\beta \in \varphi, \quad \sigma^{-1}\circ\beta \notin \eta, \\ vy^p_{\sigma^{-1}\circ\beta} & \sigma\circ\beta \notin \varphi, \quad \sigma^{-1}\circ\beta \in \eta, \\ 0 & \sigma\circ\beta \notin \varphi, \quad \sigma^{-1}\circ\beta \notin \eta, \end{cases}$$

where u, v are units.

Ideas coming into the proof

- Study the situation on components of $\mathrm{Spf}(\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}})$; they correspond to strata $Z_{\varphi',\eta'}$ passing through $\overline{Q}$.
- Gain data on $\pi^*(t_\beta)$; roughly, $\pi^*(t_\beta) = ux_\beta^M + vy_{\sigma^{-1}\circ\beta}^N$.
- Globalize so as to be able to study these expressions on components of $Z_{\varphi',\eta'}$ but at other points then $\overline{Q}$.
- Reduce to computation at a "special superspecial point" (using that any component C of any strata $Z_{\varphi,\eta}$ intersects $\overline{Y}_F \cap \overline{Y}_V$).

$\mathfrak{X}_{\mathrm{rig}}, \mathfrak{Y}_{\mathrm{rig}}$ are the rigid spaces associated to $X^{\wedge \overline{X}}, Y^{\wedge \overline{Y}}$. Given $P \in \mathfrak{X}_{\mathrm{rig}}$, get $\overline{P} = \mathrm{sp}(P) \in \overline{X}$. The variables $t_\beta \in \widehat{\mathcal{O}}_{X,\overline{P}}$ are functions on $\mathrm{sp}^{-1}(\overline{P})$ = residue "disc" about P. Let

$$\nu(P) = (\nu_\beta(P)), \quad \nu_\beta(P) = \begin{cases} \nu(t_\beta(P)) & \beta \in \tau(\overline{P}), \\ 0 & \text{else.} \end{cases}$$

($\nu(x) = \min(\mathrm{val}(x), 1)$.) For $Q \in \mathfrak{Y}_{\mathrm{rig}}$, $\overline{Q} = \mathrm{sp}(Q) \in \overline{Y}$, let

$$\nu(Q) = (\nu_\beta(Q)), \quad \nu_\beta(Q) = \begin{cases} 1 & \beta \in \eta(\overline{Q}) - I(\overline{Q}), \\ \nu(x_\beta(P)) & \beta \in I(\overline{Q}), \\ 0 & \beta \notin I(\overline{Q}). \end{cases}$$

$\nu(P), \nu(Q)$ belong to the valuation cube $\Theta = [0,1]^{\mathbb{B}}$.
$\nu(Q) + \nu(wQ) = \underline{1}$ (easy!).

The Cube Theorem

Parameterize the "open faces" of Θ by $\underline{a} = (a_\beta)$, $a_\beta \in \{0, 1, *\}$. For such $\underline{a}$ define

$$\varphi(\underline{a}) = \{\beta \in \mathbb{B} : a_\beta \neq 0\}, \quad \eta(\underline{a}) = \{\beta \in \mathbb{B} : a_{\sigma^{-1}\circ\beta} \neq 1\}.$$

There is a 1 : 1 order-reversing correspondence

$$\{\text{open faces of } \Theta\} \leftrightsquigarrow \{\text{strata } W_{\varphi,\eta}\}.$$
$$\mathscr{F}_{\underline{a}} \longmapsto W_{\varphi(\underline{a}),\eta(\underline{a})}$$

- $\nu(Q) \in \mathscr{F}_{\underline{a}} \iff \overline{Q} \in W_{\varphi(\underline{a}),\eta(\underline{a})}$.
- $\nu(Q) \in \mathrm{Star}(\mathscr{F}_{\underline{a}}) \iff \overline{Q} \in Z_{\varphi(\underline{a}),\eta(\underline{a})}$.

The open faces of Θ produce a "dissection" of $\mathfrak{Y}_{\mathrm{rig}}$:

$$\mathscr{F}_{\underline{a}} \longleftrightarrow \{Q : \nu(Q) \in \mathscr{F}_{\underline{a}}\}.$$

The canonical subgroup theorem

$$\mathcal{U} := \{P \in \mathfrak{X}_{\mathrm{rig}} : \nu_\beta(P) + p\nu_{\sigma^{-1}\circ\beta}(P) < p, \forall \beta \in \mathbb{B}\}$$

$$\mathcal{V} := \{Q \in \mathfrak{Y}_{\mathrm{rig}} : \underbrace{\nu_\beta(Q) + p\nu_{\sigma^{-1}\circ\beta}(Q)}_{\lambda_\beta(Q)} < p, \forall \beta \in \mathbb{B}\}$$

Theorem

$\pi(\mathcal{V}) = \mathcal{U}$ and there is a section

$$s^\dagger : \mathcal{U} \to \mathcal{V},$$

extending the canonical section on the ordinary locus.

Ideas in the proof

- Define for $\mathfrak{p}|p$,
  $\mathcal{V}_{\mathfrak{p}} := \{Q : \lambda_\beta(Q) < p,\ \forall \beta \in \mathbb{B}_{\mathfrak{p}}\}$
  $\mathcal{W}_{\mathfrak{p}} := \{Q : \lambda_\beta(Q) > p,\ \forall \beta \in \mathbb{B}_{\mathfrak{p}}\}$

  We first show that:
  - $\mathcal{U}$ is admissible.
  - $\pi^{-1}(\mathcal{U}) = \mathcal{V} \coprod \mathcal{W}$, admissible disjoint union, where
    $$\mathcal{W} = \bigcup_{\emptyset \neq S \subseteq \{\mathfrak{p}|p\}} \left[\bigcap_{\mathfrak{p} \in S} \mathcal{W}_{\mathfrak{p}} \cap \bigcap_{\mathfrak{p} \notin S} \mathcal{V}_{\mathfrak{p}} \right].$$

  This uses the notion of tubular neighborhoods (cf. Grosse-Kloenne's talk) and our strata on $\overline{Y}$.
- $\pi|_{\pi^{-1}(\mathcal{U})}$ is finite-flat.
- The connected components of $\mathcal{V}$ are in bijection with those of $\mathcal{U}$.
- We calculate that $\pi|_{\mathcal{V}}$ has degree 1 by restricting to $\mathrm{sp}^{-1}(W_{\mathbb{B},\emptyset}) \subseteq \mathrm{sp}^{-1}(\overline{Y}_F)$.

Further properties

- We can determine what happens under

$$P \mapsto s^{\dagger}(P) \mapsto w \circ s^{\dagger}(P) \mapsto \pi \circ w \circ s^{\dagger}(P),$$

  and so can iterate the construction to construct higher level canonical subgroups ($\subseteq A[p^n]$).
- Can prove functorial behavior relative to morphisms between Hilbert modular varieties. In particular,
  (i) decent to $\mathbb{Q}_p$ of the canonical subgroup, and
  (ii) prove $\mathcal{U}$ is maximal (in a suitable sense) for the construction of the canonical subgroup.

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