ECONOMETRICA

VOLUME 43 July, 1975 NUMBER 4

VALUES OF MARKETS WITH A CONTINUUM OF TRADERS[1]

BY ROBERT J. AUMANN

It is shown that in markets with a continuum of traders, the allocations associated with the Shapley value are the same as the competitive allocations.

1. INTRODUCTION

THE "EQUIVALENCE PRINCIPLE" for the core[2] states that the core of a perfectly competitive market coincides with the set of its competitive allocations. It is the object of this paper to establish a similar equivalence principle for the Shapley value [**5**, **21**, and **23**], a game theoretic concept quite different from the core. Since the Shapley value can be interpreted in terms of "marginal worth", it is perhaps *the* game theoretic concept most closely related to traditional economic ideas.

By a "perfectly competitive" market we mean one in which there are many traders, with each trader holding only a negligible proportion of the total resources of the economy. This situation may be modelled by a continuum of traders—specifically, by a non-atomic continuum, as in Aumann [**1**].

We will be dealing with ordinary Walrasian pure exchange economies; no assumption of transferable utility will be made. The value was originally defined [**21**] for games with transferable utility only, but was later [**23**] adapted to apply also to games without transferable utility. It is this adaptation, itself adapted to non-atomic markets, that we shall use in this paper. Values are ordinarily expressed in units of utility; since we are dealing with markets that are defined a priori in terms of preferences rather than in terms of utilities, we shall wish to refer directly to allocations rather than to utilities. Allocations corresponding to values in such a set-up will be called "value allocations"; a precise definition will be given in Section 5 below.

To establish the equivalence principle for the value, we will have to make certain differentiability assumptions on the traders' preferences, assumptions that are not necessary for the core equivalence principle. Preferences satisfying these assumptions will be called "uniformly smooth." This notion is closely related to Debreu's "smooth preferences" [**10**]. We might say that the preferences in a market are uniformly smooth if and only if they are smooth in Debreu's sense and the properties embodying this smoothness hold uniformly over all the traders in the market. This, too, will be spelled out precisely below (Sections 4 and 5).

[1] Text of the Walras-Bowley Lecture delivered at the winter meeting of the Econometric Society held in Toronto in December, 1972. This work was supported in part by National Science Foundation Grant GS-40104 at the Institute for Mathematical Studies in the Social Sciences, Stanford University, and in part by CORE, the Center for Operations Research and Econometrics at the Catholic University of Louvain.

[2] For a comprehensive treatment of the core equivalence principle, see Hildenbrand [**14**].

Recall that an allocation $\boldsymbol{x}$ is called *competitive* if there is a price vector p such that $(p, \boldsymbol{x})$ is a competitive equilibrium. We are now ready for the statement of our main theorem:

THEOREM 1 (Value Equivalence Theorem): *In a non-atomic market with uniformly smooth preferences, the set of value allocations coincides with the set of competitive allocations.*

It should be noted that this result depends strongly on the non-atomicity of the market, i.e., on its perfectly competitive nature. The theorem is not true *in either direction* for finite markets.[3] This is to be contrasted with the situation for the core: competitive allocations are in the core for any market, even a finite one; there the continuum is needed only to establish the converse, that all core allocations are competitive.

The Value Equivalence Theorem can be interpreted in terms of social welfare functions, and the concept of an individual's contribution to society. Given utility functions for the traders, we may, for any allocation $\boldsymbol{x}$ and coalition S, consider the sum of the utilities of the members of S for the bundles assigned to them under $\boldsymbol{x}$. Let us think of this sum as the "welfare" of the coalition S under the allocation $\boldsymbol{x}$; in other words, assume the existence of an additively separable social welfare function consistent with individual preferences. The "marginal contribution" of an individual trader is then the amount of welfare added to society by his presence —i.e., the difference between the maximum levels of welfare that society as a whole can achieve, with that trader and without him. It can be shown (see the Appendix) that, under the conditions of Theorem 1, *the marginal contribution of a trader is almost surely the same as his "true" contribution*: If the traders are ordered at random, then the marginal contribution of a trader almost surely equals the amount added by him to the maximum welfare achievable by the coalition of traders preceding him in the random order. This justifies our reference to the marginal contribution of a trader simply as his "contribution." Our theorem then makes two assertions: First, that an allocation is competitive if its utility to each trader is precisely his contribution; and conversely, that for each allocation $\boldsymbol{x}$, individual utility functions can be chosen (consistent with the preferences), so that the contribution of each trader is precisely the utility of the bundle assigned to him by $\boldsymbol{x}$.

In brief, *an allocation is competitive if and only if for some choice of the individual utility functions, it yields to each trader precisely his contribution to the welfare of society.*

Among the precursors of this result are results of Shapley [**22**], Champsaur [**6**], and Aumann and Shapley [**5**, Theorem J and Proposition 31.7]. The literature will be more thoroughly discussed in Section 12.

The next two sections are devoted to a review of the Shapley value, first for finite games and then for games with a continuum of players. Section 4 is devoted to

[3] Counter-examples may be found toward the end of Section 11.

the relevant definitions of differentiability and smoothness. In Section 5 the market model is presented, "value allocation" is defined, and the main results are stated. In Section 6 we again discuss the intuitive content of the main theorem, in a more leisurely manner than above. Section 7 is devoted to an illustrative example, and Section 8 to a suggestive but unrigorous demonstration of the main results, based on the calculus of infinitesimals. In Section 9 we discuss the intuitive meaning of the differentiability and smoothness conditions, and Section 10 is devoted to a similar discussion of the uniformity conditions. In Section 11 we discuss the extraordinary relation, implicit in our results, between the basically cardinal concept of Shapley value and the ordinal concept of competitive equilibrium. Section 12 is devoted to a discussion of the literature, and Sections 13 through 15 to proofs. An appendix clarifies the precise relationship between the marginal and "true" contributions of a trader.

Readers who wish to skim through the paper should perhaps read only Sections 5 and 8, and the intuitive discussion immediately following the statement of the Value Equivalence Theorem in Section 1.

2. THE VALUE IN FINITE GAMES

Let N be a finite set; call the members of N *players* and the subsets of N *coalitions*. A *game on* N is a function v that associates with each coalition S a real number $v(S)$ (the *worth* of S), such that $v(\emptyset) = 0$. A *payoff vector on* N is a measure on the subsets of N. Intuitively, a payoff vector is simply a function that assigns a real number (the payoff) to each player; such functions are in an obvious one-to-one correspondence with the measures on N, and for reasons that are both technical and conceptual,[4] it is convenient to treat a payoff vector as a measure.

A *null player* in a game v is a player i such that $v(S \cup \{i\}) = v(S)$ for all coalitions S. Players i and j are called *substitutes* if $v(S \cup \{i\}) = v(S \cup \{j\})$ whenever S contains neither i nor j. For a fixed player set N, a *value* is a function φ that associates with each game v a payoff vector φv satisfying the following conditions:

CONDITION 2.1 (Additivity): $\varphi(v + w) = \varphi v + \varphi w$.

CONDITION 2.2 (Symmetry): $(\varphi v)(\{i\}) = (\varphi v)(\{j\})$ *whenever i and j are substitutes.*

CONDITION 2.3 (Efficiency): $(\varphi v)(N) = v(N)$.

CONDITION 2.4 (Null player condition): $(\varphi v)(\{i\}) = 0$ *whenever i is a null player.*

PROPOSITION 2.1: *For each finite player set N there is one and only one value φ; it is given by the formula $(\varphi v)(\{i\}) = E(v(S_i \cup \{i\}) - v(S_i))$, where S_i is the set of all players preceding i in a random order on N, and E is the expectation operator when all $|N|!$ such orders are assigned equal probability.*

[4] We wish to think of the payoff to a coalition, not only to an individual.

For a proof of Proposition 2.1 that is even simpler than the original proof [**21**], see [**5**, Appendix A].

3. THE VALUE IN CONTINUOUS GAMES

In this section we take for the underlying player space a measurable space $(T, \mathscr{C})$, i.e., a set T together with a σ-field $\mathscr{C}$ of subsets of T. The members of $\mathscr{C}$ are *coalitions*, but one should not think of the points t of T as individual players; rather, one should think of an individual player as an infinitesimal subset dt of T (cf. [**5**, Section 29]).

A *game on* $(T, \mathscr{C})$ is a function v from $\mathscr{C}$ to the real numbers such that $v(\emptyset) = 0$. There are various ways of extending the definition of value from the finite games studied in Section 2 to the more general situation we have here; they are studied in detail in [**5**]. Here we will adopt a definition due to Kannai [**15**] (see also [**5**, Section 17]).

Let v be a game and S a coalition; we wish to define $(\varphi v)(S)$. Intuitively, one proceeds by dividing both S and $T\backslash S$ into a large but finite number of "small" sets W_i, say n in all (see Figure 1). If one considers each of these small sets as an individual player, then one gets a finite game by restricting v to unions of the W_i. In this finite game the coalition S—or more precisely the coalition of those W_i whose union is S—has a value, which we denote $\varphi_n(S)$. Now if we let $n \to \infty$ and the W_i shrink, $\varphi_n(S)$ may or may not tend to a limit. If it does, and if this limit is independent of the various choices that must be made, then we denote the limit $(\varphi v)(S)$, and call it the "asymptotic value of S."

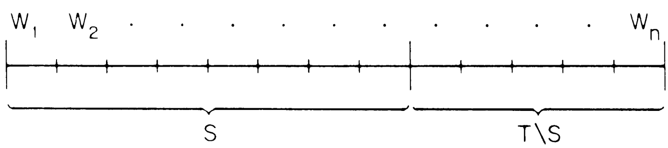

FIGURE 1

Formally, let $\Pi_1, \Pi_2, \ldots$ be a sequence of partitions of T into measurable sets (i.e., members of $\mathscr{C}$), such that each Π_m refines the previous partition Π_{m-1}. Assume moreover that the sequence is *separating*, i.e., that if s and t are distinct points in T, then for m sufficiently large, s and t are in different members of Π_m; this is a precise expression of the idea that the sets in the partitions shrink. For example, if Π_m is the partition of the unit interval $[0, 1]$ into the subintervals $[0, 1/2^m], (1/2^m, 2/2^m], \ldots, (1 - 1/2^m, 1]$, then the sequence $\{\Pi_m\}$ is separating. For each m, let v_m be the finite game on Π_m defined by $v_m(\Xi) = v(\bigcup_{W \in \Xi} W)$, and let φv_m be its value. Let $\Pi_1 = \{S, T\backslash S\}$, and for each m, let $S_m = \{W \in \Pi_m : W \subset S\}$; S_m is the coalition in the finite game v_m corresponding to S in the original game v. Now let $m \to \infty$; if $(\varphi v_m)(S_m)$ has a limit, and if this limit is independent of the choice of the sequence $(\Pi_1, \Pi_2, \ldots)$ (when chosen in accordance with the above conditions), then the limit is denoted $(\varphi v)(S)$. If, moreover, this is the case of all

$S \in \mathscr{C}$, then φv is called the *asymptotic value* of v. Note that the asymptotic value, if it exists, is a finitely additive measure on $(T, \mathscr{C})$. Note also that when T is finite, the asymptotic value coincides with the finite value.

The asymptotic value obeys conditions analogous to Conditions (2.1) through (2.4); see [**5**, Sections 2 and 17, in particular Theorem F]. Specifically, we note the *efficiency condition*:

$$(\varphi v)(T) = v(T). \tag{3.1}$$

4. DIFFERENTIABILITY, SMOOTHNESS, AND UNIFORM SMOOTHNESS

We will be working in a Euclidean space E^n. Superscripts will be used to denote coordinates. For x and y in E^n we write $x > y$ when $x^i > y^i$ for all i, and $x \geqslant y$ when $x^i \geqslant y^i$ for all i. The symbol $x \cdot y$ will denote the scalar product $\Sigma_i x^i y^i$. We will also use standard matrix notation, so that $x \cdot y = xy^*$, where both x and y are considered row vectors and * denotes "transpose." The symbol 0 denotes the origin of E^n as well as the number 0; no confusion will result. The non-negative orthant $\{x \in E^n : x \geqslant 0\}$ in E^n is denoted Ω.

Unless otherwise specified, the word "function" will mean "real valued function."

A function u on Ω is *continuously differentiable* (on Ω) if it can be extended to an open neighborhood (in E^n) of Ω on which it is continuously differentiable. An equivalent definition [**29**] is that u is continuous on Ω and continuously differentiable in the interior of Ω, and each of the partial derivatives of u can be continuously extended from the interior of Ω to Ω. In that case, the row vector of the partial derivatives of u is denoted u'; on the boundary of Ω, u' is defined to be the unique continuous extension of u' in the interior. A function on Ω with values in a finite dimensional Euclidean space is *continuously differentiable* (C^1 for short) if each of its components is continuously differentiable. A real-valued function u on Ω is *twice continuously differentiable* (C^2 for short) if both u and u' are continuously differentiable; the matrix of second derivatives is denoted u''.

A function u on Ω is called *quasi-concave* if for all $\alpha \in [0, 1]$ and all x and y in Ω, $u(\alpha x + (1 - \alpha)y) \geqslant \min(u(x), u(y))$.

If u is a C^2 function on Ω, define

$$u^H = \begin{vmatrix} u'' & u'^* \\ u' & 0 \end{vmatrix}, \tag{4.1}$$

where * denotes "transpose." This is the *bordered Hessian* of u, i.e., the $(n + 1) \times (n + 1)$ determinant in which the $n \times n$ matrix u'' is bordered by the row vector u', by the column vector u'^*, and by the number 0. When $u' > 0$, define

$$u^c = \frac{-u^H}{\|u'\|_2^{n+1}}, \tag{4.2}$$

where $\| \ \|_2$ is the Euclidean norm. It may be verified [**10**] that u^c depends only on the level (or indifference) surfaces of u, and not on u itself. The *Gaussian curvature* of the level surface of u at x is then defined as $u^c(x)$.

When $n = 2$, then $u^c(x) = 1/r$, where r is the radius of the circle that "best fits" the level curve of u at x. Thus if $u^c(x) = 0$, then $r = \infty$, so that the "circle" that best fits the level curve is a straight line. Further discussion of the Gaussian curvature can be found in Section 9 and also in [**10**].

Let $(T, \mathscr{C})$ be a measurable space. For each $t \in T$, let w_t be a function from Ω to some Euclidean space E^l (possibly $l = 1$). The w_t are *uniformly bounded* if there is a point b in E^l such that $w_t(x) < b$ for all t in T and x in Ω. They are *uniformly bounded in compact sets* if for each compact subset C of Ω, there is a point b in E^l such that $w_t(x) < b$ for all t in T and x in C. They are *uniformly positive* (*or bounded away from zero*) *in compact sets* if for each compact subset C of Ω, there is a point $b > 0$ in E^l such that $w_t(x) > b$ for all t in T and x in C. They are *simultaneously measurable* if $w_t(x)$ is measurable as a function of (t, x), i.e., measurable in the product field $\mathscr{C} \times \mathscr{B}$, where $\mathscr{B}$ is the σ-field of Borel sets in Ω.

A family $\{u_t\}$ of functions on Ω is called *bounded differentiable* if the u_t are simultaneously measurable, uniformly bounded, and continuously differentiable on Ω, and the derivatives u_t' are, in compact sets, uniformly bounded and uniformly positive. Finally, $\{u_t\}$ will be called *uniformly smooth* if it is bounded differentiable and the following conditions are satisfied:

CONDITION 4.1: *The u_t are quasiconcave and twice continuously differentiable on Ω.*

CONDITION 4.2: *The Gaussian curvatures u_t^c are uniformly positive in compact sets.*

CONDITION 4.3: *The second derivatives u_t'' are uniformly bounded in compact sets.*

5. DESCRIPTION OF MODEL AND STATEMENT OF RESULTS

The market model is basically the same as in [**1**]. It consists of: (i) a measurable space $(T, \mathscr{C})$, called the *trader space*, together with a σ-additive, non-negative measure μ on $\mathscr{C}$, with $\mu(T) < \infty$; (ii) the non-negative orthant Ω—called the consumption set—of a Euclidean space E^n, whose dimension n represents the number of different goods being traded in the market; (iii) a measurable function $\boldsymbol{a}$ from T to Ω called the *initial allocation*; and (iv) for each t in T, a relation $\succ_t$ on Ω, called the *preference relation* of t. The system consisting of these four items is called a *market*. The market is called *non-atomic* if μ is non-atomic.

We will assume that the measurable space $(T, \mathscr{C})$ is *standard*, i.e., that it is finite, denumerable, or isomorphic[5] to the unit interval $[0, 1]$ with the Borel sets. This assumption is less restrictive than it sounds; any Borel subset of any Euclidean space (or indeed, of any complete separable metric space) has this property.

[5] Two measurable spaces are *isomorphic* if there is a one-to-one transformation from one onto the other that preserves measurability in both directions.

If f is a function on T, we often write $\int_S f$ instead of $\int_S f(t)\mu(dt)$, and $\int f$ instead of $\int_T f$. We will make the following assumptions:

ASSUMPTION 5.1: $\int \boldsymbol{a} > 0$.

ASSUMPTION 5.2: *$\boldsymbol{a}$ is uniformly bounded.*[6]

Assumption 5.1 is fairly standard; it means that each commodity is actually present in the market, though not all traders need initially hold a positive amount of it. Assumption 5.2 is less familiar; actually we make it only as a kind of normalization, and at the expense of certain complications below, could dispense with it. This matter will be discussed further in Section 10.

In the non-atomic case, it is best to view an individual trader as an "infinitesimal subset" dt of T, rather than as a single point in T. Individual bundles are also infinitesimal; thus the initial bundles of the individual trader dt is $\boldsymbol{a}(t)\mu(dt)$, whereas the point $\boldsymbol{a}(t)$ itself is to be thought of as dt's initial bundle *density* (initial bundle of dt divided by his measure). Similarly, $x \succ_t y$ means that dt prefers $x\mu(dt)$ to $y\mu(dt)$, i.e., he prefers the bundle density x to the bundle density y. This point of view is expounded at some length in [**5**, Section 29]; see also [**26**, Footnote 12].

A *utility* for a relation $\succ$ on Ω is a real-valued function u on Ω such that $u(x) > u(y)$ if and only if $x \succ y$. The preferences $\succ_t$ are called *uniformly smooth* if there exists a uniformly smooth family of utilities for them. An intuitive discussion of uniform smoothness will be given in Section 10.

Let $U = \{u_t\}$ be a bounded differentiable[7] family of utilities for the preferences. Define a game v_U on $(T, \mathscr{C})$ by

$$(5.1) \qquad v_U(S) = \max \left\{ \int_S u_t(\boldsymbol{x}(t))\mu(dt) : \int_S \boldsymbol{x} = \int_S \boldsymbol{a} \text{ and } \boldsymbol{x}(t) \in \Omega \text{ for all } t \text{ in } S \right.$$

Intuitively, $v_U(S)$ is the maximum total utility that S can guarantee to itself by redistributing its total initial bundle $\int_S \boldsymbol{a}$ among its members. We remark that the maximum in the definition of v_U is attained [**4**], and the game v_U has an asymptotic value φv_U [**5**, Proposition 31.7].

An *allocation* is a measurable function $\boldsymbol{x}$ from T to Ω such that $\int \boldsymbol{x} = \int \boldsymbol{a}$. It is called a *value allocation* if there exists a bounded differentiable family U of utilities for the preferences such that for all t,

$$(5.2) \qquad (\varphi v_U)(dt) = u_t(\boldsymbol{x}(t))\mu(dt),$$

i.e., such that for all S,

$$(5.3) \qquad (\varphi v_U)(S) = \int_S u_t(\boldsymbol{x}(t))\mu(dt).$$

In words, (5.2) says that $\boldsymbol{x}$ yields each trader precisely the utility that he is assigned by the value. An allocation $\boldsymbol{x}$ is called *competitive* if there exists a p in Ω with

[6] That is, there is a point b in Ω such that $\boldsymbol{a}(t) < b$ for all t.
[7] See Section 4.

$p \neq 0$ (called the vector of *prices*) such that for almost all t in T, $\boldsymbol{x}(t)$ is maximal with respect to $\succ_t$ in the *budget set* $\boldsymbol{B}_p(t) = \{x \in \Omega : p \cdot x \leqslant p \cdot \boldsymbol{a}(t)\}$.

This completes the presentation of the basic definitions. The first result is:

PROPOSITION 5.1: *In a non-atomic market, every value allocation is competitive.*

Conversely, we have:

PROPOSITION 5.2: *In a non-atomic market, assume that either* (i) *the preferences are uniformly smooth, or* (ii) *there exists a bounded differentiable family of concave utilities for the preferences. Then every competitive allocation is a value allocation.*

Proposition 5.2 requires either smoothness or concavity; for Proposition 5.1 neither requirement is needed. But Proposition 5.1 does require differentiability, which is implicit in the definition of "value allocation".

The Value Equivalence Theorem—stated in the introduction—is, of course, an immediate consequence of Propositions 5.1 and 5.2.

We mention also that if in the definition of value allocation we replace "asymptotic value" by "mixing value" [**5**, Section 14, p. 115], then the Value Equivalence Theorem and Propositions 5.1 and 5.2 remain true without any change. The proofs are as in Sections 13 through 15; one merely has to replace the word "asymptotic" by the word "mixing".

6. INTERPRETATION

The simplest interpretation is essentially the one sketched in the introduction (after the statement of the Value Equivalence Theorem). It involves an additively separable social welfare function, defined by a family $U = \{u_t\}$ of utilities. It is best to think of trader dt's utility for the bundle $x\mu(dt)$ as $u_t(x)\mu(dt)$. Then $\int_T u_t(\boldsymbol{x}(t))\mu(dt)$ is society's level of welfare under the allocation $\boldsymbol{x}$; and $v_U(S)$ is the maximum level of welfare that the coalition S can achieve by its own efforts. Thus $v_U(S)$ measures the power or bargaining strength of the coalition S; and *given the bargaining strengths*, φv_U may be considered the "equitable" payoff distribution. Hence a value allocation (see (5.2)) is one that yields each trader precisely his "equity" in the market; i.e., an allocation $\boldsymbol{x}$ whose utility $u_t(\boldsymbol{x}(t))\mu(dt)$ to each trader dt is precisely $(\varphi v_U)(dt)$.

Here we used the word "equity" in describing the value, whereas in the introduction we referred to the traders' "contributions". The assertion that the value constitutes an equitable payoff distribution is based on its axiomatic definition (see Conditions 2.1 through 2.4). That it also may be thought of in terms of the players' contributions is a consequence of Proposition 2.1 (see also Step 5 of Section 7, and the Appendix). These two interpretations are simply two sides of the same coin—two ways of looking at the same concept.

It should be noted that for a given family U of utilities, there will in general[8] be at most one such x, and usually none; to obtain the set of all value allocations, one must allow U to vary. An illustrative example will be analyzed in Section 7.

There is a certain conceptual analogy between the process of determining a value allocation and that of determining a competitive allocation. To find a competitive allocation, one assigns prices to the commodities. Given the initial allocation, these prices generate individual demands. If the demands thus generated are feasible—i.e., if they can be obtained by reallocating the total initial supply—then we have a competitive allocation.

To find a value allocation, one assigns a utility function to each trader. Given the initial allocation, these utility functions generate a payoff distribution—the value. If this payoff distribution is feasible—i.e., if it can be obtained by reallocating the total initial supply—then we have a value allocation. The utility functions here play a role similar to that of the prices in the case of competitive allocations. Prices enable us to evaluate bundles of goods; utility functions enable us to evaluate coalitions of players. Workers in general equilibrium theory have in the past been loath to compare utilities; but there is some reason to believe that such comparisons play a significant role in economic activity. Workers in areas such as growth, income distribution, and taxation have recognized this and have been quite willing to assume social welfare functions and the utility comparisons they imply. The fact that there are certain conceptual difficulties involved in such comparisons does not permit us, ostrich-like, to ignore them. The analysis outlined here suggests that society may be generating utility comparisons by means of a mechanism similar to the familiar supply and demand mechanism that generates prices; more bluntly, that "social welfare" is a function of power rather than an ethical construct.

The above interpretation of the Value Equivalence Theorem—i.e., in terms of social welfare functions—is open to a certain objection; though not crippling, it bears discussion. The games v discussed in Sections 2 and 3 are usually interpreted as "side payment" or "transferable utility" (TU) games. Essentially, this means that the payoffs are assumed to be in a single, desirable commodity; that $v(S)$ is the *total* payoff that S can get for its members; and that S can get for its members *any* payoff totalling $v(S)$, as is for example the case when the players can freely transfer the single good among each other. The markets defined in Section 6, on the other hand, cannot be considered TU games: the payoff is in terms of the personal utilities u_t, which are not "commodities", and are certainly not transferable. How, then, can TU theory be relevant to these markets?

Naturally, one can answer that though the TU view of a game v is the usual one, it is not the only possible one, and that in the current application the social welfare view is preferred. But it is also possible to justify the notion of value allocation directly in terms of the TU view. One can think of markets in terms of production, as follows: there are n raw materials, not in themselves consumable, and one desirable consumer commodity, which for simplicity we shall call *food*. Each

[8] For example, whenever the u_t are strictly concave.

trader dt starts out with a bundle $\boldsymbol{a}(t)\mu(dt)$ of raw materials, and with no food. The traders may exchange raw materials. If, after the exchange, dt holds a bundle $x\mu(dt)$ of raw materials, he can produce from it an amount $u_t(x)\mu(dt)$ of food. Transfers of food are forbidden.

A little reflection will convince the reader that this is conceptually entirely equivalent to the usual view of a market.

Since food transfers are forbidden, this cannot be viewed as a TU game. But suppose now that food transfers are legalized.[9] Then each coalition S can produce an amount $v_U(S)$ of food, which it can divide among its members in any way it pleases. Therefore the situation is well represented by the TU game v_U.

In general, the value φv_U of v_U is not achievable without actual transfers of food (cf. Section 7). Specifically, we will have

$$(6.1) \qquad (\varphi v_U)(dt) = u_t(\boldsymbol{x}(t))\mu(dt) + \zeta(dt),$$

where $\boldsymbol{x}$ is an allocation of resources that achieves $v_U(T)$, i.e., such that $v_U(T) = \int_T u_t(\boldsymbol{x}(t))\mu(dt)$, and $\zeta(dt)$ is the net intake of food by dt resulting from food transfers (rather than from his own production). These transfers clearly sum to 0, i.e., $\zeta(T) = \int_T \zeta(dt) = 0$; but for individual traders they will usually not vanish, i.e., the measure ζ will not vanish identically.

Suppose now that ζ does happen to vanish identically; then (6.1) reduces to (5.2), i.e., $\boldsymbol{x}$ is a value allocation. This means that the value φv_U is achievable directly by means of the resource allocation $\boldsymbol{x}$, without any transfers of food. In this case $\boldsymbol{x}$ is called a *value allocation.*

If $\boldsymbol{x}$ is a value allocation, one can think of the traders going to market under the impression that food transfers will be permitted. This leads them to trade so as to arrive at the value of the associated game v_U; i.e., they will reach the allocation $\boldsymbol{x}$, but will undertake no subsequent food transfers. After going home, they hear that food transfers have been forbidden. Fortunately, nobody wanted to transfer food anyway. Thus, to arrive at a value allocation, one must only think about food transfers—imagine them—they don't actually have to be possible. The "transferable utility" does not really have to be transferable, because in the end nobody wants to transfer it.

Basically, this argument uses what is sometimes called the principle of independence of irrelevant alternatives (cf. [**18**]). We agree that when food may be transferred, then among all possible outcomes, an outcome that is in some sense best is to give to player dt the allocation $\boldsymbol{x}(t)\mu(dt)$. But this allocation does not require the transfer of food, and so a fortiori it should be best when such transfers are forbidden.

We close this section with a few words about the notion of "equity" and its connection with the Shapley value. We stated above that the value of a game v is calculated by taking into account the bargaining strengths of players and coalitions, as reflected in the worths $v(S)$. Thus the more a player can get for himself, acting alone or in concert with other players, the more the value will

[9] Economies of this kind (i.e., with legalized food transfers) are called *TU markets.*

assign to him. Some readers may feel that this smacks more of power politics—of "might makes right"—than of what is commonly considered equity or fairness. Wouldn't it be more appropriate to adopt a point of view similar to that of Rawls [**20**] or of some of the recent literature on income redistribution,[10] which ignore the power structure of society and treat people in a basically egalitarian way?

We think not. Criteria of equity that do not take power considerations into account have a pleasing air of symmetry and abstract perfection; they appeal to our philosophical, ethical senses. But what would make society adopt such criteria? And even if adopted, what chance do they have of surviving the attacks of pressure groups, large and small? For better or for worse, economic realities are dictated not by ethical considerations, but by the exercise of economic power. We are not suggesting that the more egalitarian considerations are not worthy of study; but we do think that the kind of criterion exemplified by the Shapley value—which some readers might prefer to call a "reasonable compromise" rather than an "equitable solution"—is more relevant to economic reality.

7. AN EXAMPLE

This section is devoted to a trivial but nonetheless instructive example. Consider an economy in which there is only one commodity, which is desired by all traders (i.e., for all $t, x \succ_t y$ if and only if $x > y$). Let the initial bundle density be 1 for half the traders (say the set S_1), and 3 for the other half (say $S_3 = T \backslash S_1$). Of course one would not expect any trading to take place in such a situation, but let us follow through our formal analysis and look for value allocations. Let u be any strictly concave function on the nonnegative reals that is bounded, differentiable, and strictly increasing (e.g., $1 - 1/(x+1)$). Set $u_t = u$ for all t; for simplicity, let $\mu(T) = 1$. Then $v_U(S_1) = (1/2)u(1)$, $v_U(S_3) = (1/2)u(3)$, and $v_U(T) = u(2)$. Since u is strictly concave, it follows that $v_U(T) > v_U(S_1) + v_U(S_3)$; in other words, trading is socially beneficial. To obtain the full benefit from trading, society must choose an allocation that actually yields a total utility of $v_U(T) = u(2)$. The only such allocation is $\boldsymbol{x}_0(t) \equiv 2$. But this cannot be a value allocation, as it is highly inequitable: it does not differentiate at all between the players in S_1 and S_3, and in fact assigns to traders in S_3 bundles smaller than their initial bundles! Hence there is no value allocation at all associated with this choice of the u_t—no allocation that is both socially optimal, and divides the benefits from trading equitably among the traders.

If one calculates[11] φv_U, one finds that it assigns densities $u(2) - u'(2)$ and $u(2) + u'(2)$ to the players in S_1 and S_3, respectively[12] (see Figure 2). But to achieve such a payoff distribution one would have to start by assigning a density of 2 units of the single commodity to all traders, and then one would have to "transfer utility"

[10] See, for example, Sheshinski [**25**] and the papers quoted there.

[11] Step 5 of Section 8 or Lemma 13.2.

[12] When $u(x) = 1 - 1/(x+1)$, then $u(2) = u'(2) = 5/9 = u(1) + 1/18$, and $u(2) + u'(2) = 7/9 = u(3) + 1/36$. Thus the value dictates a gain in utility (over that of the initial bundle) for all traders, but a smaller gain for S_3 than for S_1. "Equitable" is not necessarily "equal".

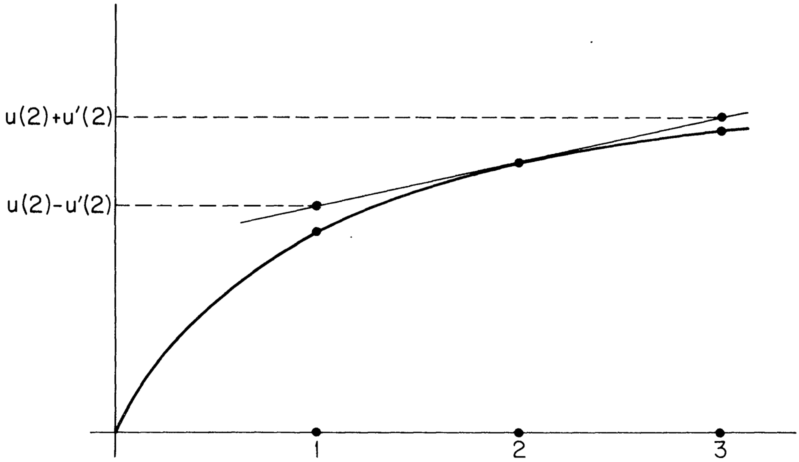

FIGURE 2

in the amount of $u'(2)$ from S_3 to S_1—an impossible (indeed meaningless) proceeding.

Since this choice of U does not lead to a value allocation, the question arises, which choice does? A trivial answer is provided by taking $u_t = w$ for all t, where $w(x) = x$ in an interval including $[1, 3]$. In that case, there is no social benefit from trading, and so the initial allocation is also a (in fact, *the*) value allocation.

But there is also a more instructive answer. Consider the u_t in our first example, i.e., $u_t(x) = 1 - 1/(x + 1)$. If we multiply these u_t by some $\alpha > 1$ for $t \in S_3$, and leave them unchanged for $t \in S_1$, then we are assigning a higher social benefit to consumption by S_3 than by S_1—consumption by S_3 is considered "more valuable".[13] As a result, maximum social welfare will be attained by assigning a higher commodity density to S_3 than to S_1. In particular, if α is sufficiently high, maximum social welfare will be attained by transferring goods from S_1 to S_3. We have already seen that if $\alpha = 1$, then maximum social welfare is attained by transferring from S_3 to S_1. For an appropriate intermediate α, therefore, maximum social welfare is attained precisely at the initial allocation, i.e., without any transfers. Of course this does not necessarily imply that the value is then also achieved without any transfers, but it may be verified that that is in fact the case.

When there is more than one commodity, value allocations will in general involve transfers of commodities; but the utilities must be chosen so that no "transfer of utility" is required.

8. AN UNRIGOROUS DEMONSTRATION OF THE RESULTS

The proof of our results relies heavily on the theory of non-atomic games developed in Aumann-Shapley [5]; when spelled out, it is long and involved. On the other hand, it is comparatively easy to "demonstrate" the theorem in

[13] Because the members of S_3 are more powerful, in that their initial bundles are greater.

terms of the calculus of infinitesimals and one or two other rough conceptual tools; that is what we shall do in this section. Such a demonstration, though unrigorous, gives insight into the situation and shows why one would expect the theorem to hold.

In Steps 1 through 5 of the demonstration, $U = \{u_t\}$ is a fixed bounded differentiable family of utilities, and $\boldsymbol{x}$ is an allocation at which $v_U(T)$ is achieved, i.e., such that

$$(8.1) \qquad v_U(T) = \int_T u_t(\boldsymbol{x}(t))\mu(dt).$$

The universal quantifier ("for all t") is to be understood in each one of these steps.

STEP 1: $(\boldsymbol{x}(t), u_t(\boldsymbol{x}(t)))$ *is on the boundary of the convex hull of the graph of* u_t.

When $n = 1$, this asserts that the situation cannot be as pictured in Figure 3. Indeed, if it were, we could split dt into two players, dt_1 and dt_2, and give them densities x_1 and x_2, respectively. The total utility that dt would get in this way would be $y\mu(dt)$ rather than $u_t(\boldsymbol{x}(t))\mu(dt)$. Thus dt, and so also T as a whole, would

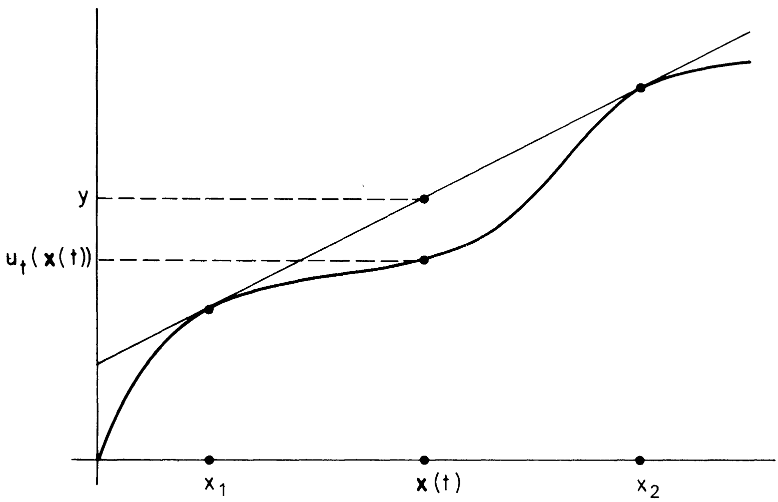

FIGURE 3

benefit from the split; this contradicts the efficiency of the value. The argument for arbitrary n is similar, except that it may be necessary to split dt into n traders $dt_1, \ldots, dt_n$. We are of course making strong use of the non-atomicity of the market in this step.

STEP 2: $u_t'(\boldsymbol{x}(t)) = u_s'(\boldsymbol{x}(s))$ *for all* s *and* t.

Since the demonstration is in any case unrigorous, we assume without twinges of conscience that we are in the interior,[14] i.e., that $\boldsymbol{x}(t) > 0$, $\boldsymbol{x}(s) > 0$. Let $n = 1$. If the assertion were false, say $u_t'(\boldsymbol{x}(t)) > u_s'(\boldsymbol{x}(s))$, then society could gain by moving $x(t)$ to the right and $x(s)$ to the left, in contradiction to the efficiency of the value (see Figure 4). The argument is similar for general n.

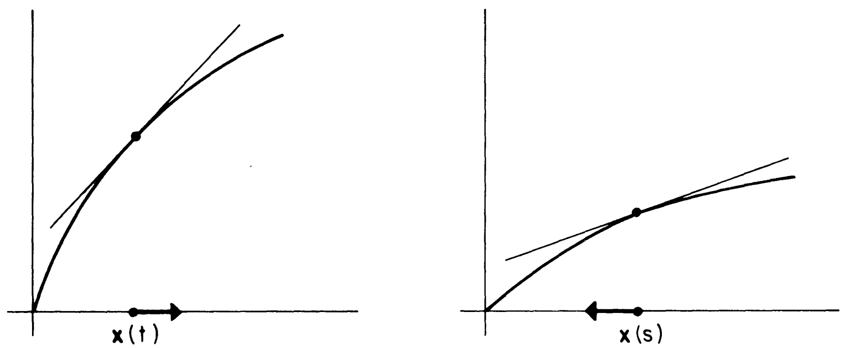

FIGURE 4

Set $p = u_t'(\boldsymbol{x}(t))$; by Step 3, p does not depend on t.

STEP 3: *For all x in Ω, $u_t(\boldsymbol{x}(t)) - u_t(x) \geqq p \cdot (\boldsymbol{x}(t) - x)$.*

From Step 2 it follows that the hyperplane with slope p through the graph of u_t at $x = \boldsymbol{x}(t)$ is tangent to the graph there; from Step 1 that this hyperplane supports the graph. This is precisely what is asserted here (see Figure 5).

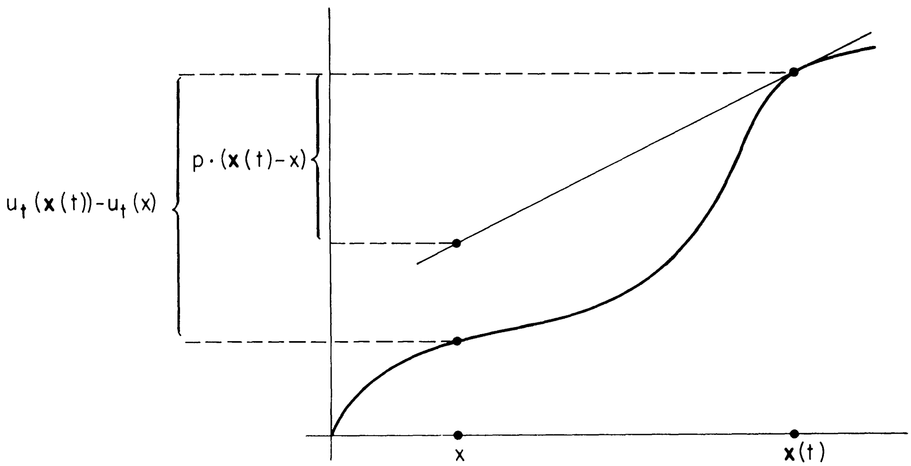

FIGURE 5

[14] Of course no such assumption is used in the rigorous proof. Even the statement of the step is false without the "interior" assumption, and must be replaced by a more complicated statement.

STEP 4: *$\boldsymbol{x}(t)$ is maximal with respect to the preference relation $\succ_t$ in* $\{x : p \cdot x \leqslant p \cdot \boldsymbol{x}(t)\}$.

Indeed, if $p \cdot x \leqslant p \cdot \boldsymbol{x}(t)$, then from Step 3 it follows that $u_t(x) \leqslant u_t(\boldsymbol{x}(t))$.

Step 4 asserts that the prices p are *efficiency prices*. When the u_t are concave, Steps 2, 3, and 4 are well-known even in the finite case; the p^i are the "shadow prices" in the maximization that defines $v_U(T)$. The non-atomicity enables the establishment of Step 1, and so the extension to non-concave u_t.

The following step is crucial:

STEP 5: $(\varphi v_U)(dt) = (u_t(\boldsymbol{x}(t)) + p \cdot (\boldsymbol{a}(t) - \boldsymbol{x}(t)))\mu(dt)$.

To demonstrate this, let us first calculate the marginal contribution $v_U(T) - v_U(T \backslash dt)$ of dt to society. Since $v_U(T)$ is achieved at x, trader dt will, after joining the rest of society, consume $\boldsymbol{x}(t)\mu(dt)$, whereas he has contributed $\boldsymbol{a}(t)\mu(dt)$. Therefore his net contribution of commodities to society is $(\boldsymbol{a}(t) - \boldsymbol{x}(t))\mu(dt)$. This is distributed among all other traders, each of whom will receive only an infinitesimal portion of this (in itself infinitesimal) bundle. Therefore the bundle *density* of each trader will change by only an infinitesimal amount, and so by Step 2, the utility of each trader will change by p times the increment in his bundle. Hence the total increment in utility to all members of T due to the distribution of $(\boldsymbol{a}(t) - \boldsymbol{x}(t))\mu(dt)$ among them is $p \cdot (\boldsymbol{a}(t) - \boldsymbol{x}(t))\mu(dt)$. But over and above his contribution of goods to society, dt himself will get utility out of his consumption of $\boldsymbol{x}(t)\mu(dt)$, and this additional utility, which is $u_t(\boldsymbol{x}(t))\mu(dt)$, also must be taken into account when calculating the marginal contribution of dt. We conclude that

$$v_U(T) - v_U(T \backslash dt) = (u_t(\boldsymbol{x}(t)) + p \cdot (\boldsymbol{a}(t) - \boldsymbol{x}(t)))\mu(dt). \tag{8.2}$$

Now let S be the set of traders up to dt in a random ordering of all the traders. Because of the many traders, the randomly chosen coalition S will almost surely be an almost perfect sample of the population of all traders (insofar as the distribution of utilities and initial bundles is concerned). Hence the contribution $v_U(S) - v_U(S \backslash dt)$ of dt to S is almost surely the same as his marginal contribution $v_U(T) - v_U(T \backslash dt)$ to all of society. Hence the expectation of his contribution to S—which is precisely $(\varphi v_U)(dt)$—is also the same as his marginal contribution. Step 5 thus follows from (8.2).

STEP 6: *Every value allocation is competitive.*

A value allocation $\boldsymbol{x}$ necessarily satisfies (8.1), since by (5.3) and the efficiency of the value (3.1) we have $\int_T u_t(\boldsymbol{x}(t))\mu(dt) = (\varphi v_U)(T) = v_U(T)$. From (5.4) and Step 5 we then obtain $p \cdot (\boldsymbol{a}(t) - \boldsymbol{x}(t)) = 0$ for all t. Thus $p \cdot \boldsymbol{a}(t) = p \cdot \boldsymbol{x}(t)$, and so from Step 4 it follows that $(p, \boldsymbol{x})$ is a competitive equilibrium.

Step 6 is identical with Proposition 5.1, and thus the demonstration of the latter is complete.

STEP 7: *If the preferences have concave utilities, then every competitive allocation is a value allocation.*

Let $\{w_t\}$ be a family of concave utilities, and let $(p, \boldsymbol{x})$ be a competitive equilibrium. From the concavity it follows that for all t and x, $w_t(\boldsymbol{x}(t)) - w_t(x) \geqslant w_t'(\boldsymbol{x}(t)) \cdot (\boldsymbol{x}(t) - x)$, and hence, as in Step 4 above, that the hyperplane through $\boldsymbol{x}(t)$ orthogonal to $w_t'(\boldsymbol{x}(t))$ supports the indifference surface of u_t through $\boldsymbol{x}(t)$. Hence $w_t'(\boldsymbol{x}(t))$ must be proportional to p. So if we define a family $U = \{u_t\}$ of different utilities for the same preferences by multiplying each w_t by an appropriate constant (which may depend on t), then we obtain $p = u_t'(\boldsymbol{x}(t))$ for all t. Hence $u_t(\boldsymbol{x}(t)) - u_t(x) \geqslant p \cdot (\boldsymbol{x}(t) - x)$ for all t. From this it follows that if $\boldsymbol{y}$ is another allocation, then $\int_T u_t(\boldsymbol{x}(t))\mu(dt) - \int_T u_t(\boldsymbol{y}(t))\mu(dt) \geqslant p \cdot \int (\boldsymbol{x} - \boldsymbol{y}) = p \cdot \int (\boldsymbol{a} - \boldsymbol{a}) = 0$, so that $v_U(T)$ is achieved at $\boldsymbol{x}$, i.e., (8.1) holds. Hence, by Step 5, $(\varphi v_U)(dt) = (u_t(\boldsymbol{x}(t)) + p \cdot (\boldsymbol{a}(t) - \boldsymbol{x}(t)))\mu(dt)$. Since $(p, \boldsymbol{x})$ is a competitive equilibrium, $p \cdot \boldsymbol{a}(t) = p \cdot \boldsymbol{x}(t)$, and hence $(\varphi v_U)(dt) = u_t(\boldsymbol{x}(t))\mu(dt)$. But this is precisely the definition (5.2) of value allocation, and so the proof of Step 7 is complete.

STEP 8: *Uniformly smooth preferences can be represented by concave utilities.*

For a time it was thought likely that all preferences with quasi-concave utilities could be represented also by concave utilities; that this is not the case was shown by examples of de Finetti [7] and Fenchel [**12**], which will be considered in more detail in Section 9 below. At this point we cannot "demonstrate" this step, but perhaps this matter will be clearer to the reader after he has read the discussion in Section 9.

We add that in independent work, Mas-Collel [**17**] and Kannai [**16**] have shown that all preferences having quasi-concave utilities can be approximated by preferences having concave utilities, so that Step 8 is "approximately" true even without the smoothness condition.

Proposition 5.2 is, of course, an immediate consequence of Steps 7 and 8. Since Proposition 5.1 has already been established (see Step 6), the demonstration of the Value Equivalence Theorem is complete. The most crucial steps in this demonstration are probably 5 and 8.

9. DISCUSSION OF SMOOTHNESS AND SOME COUNTER-EXAMPLES

This section is devoted to the differentiability and smoothness conditions for individuals; the uniformity conditions will be discussed in the next section.

If the utilities u_t are not differentiable, the TU game v_U will in general not have an asymptotic value,[15] so one cannot even begin to speak of a value allocation. Moreover, though φv_U may exist under certain circumstances even when the utilities are not differentiable,[16] value theory for the game v_U is in this case quite

[15] See [**5**, Example 33.9], especially the end of the discussion of that example.

[16] See Footnote 15.

undeveloped, and we would not be able to apply the basic tools from [**5**] that we need for the proof of our theorem.[17]

So much for the first-order differentiability. Next, we discuss smoothness, which is needed to ensure that the preferences have concave utilities;[18] the importance of this will soon become apparent.

Smoothness has three basic ingredients: quasi-concavity, positive Gaussian curvature, and positive gradients. We discuss the three ingredients separately.

Without quasi-concavity—e.g., for a preference order like that pictured in Figure 6—we will obviously not be able to find concave utilities (the preferred sets $\{x : u(x) \geqslant y\}$ are convex if u is concave). Moreover, it is then easy to find an example with a competitive allocation that is not a value allocation. Let all traders have the same preferences, and the same initial bundle $\boldsymbol{a}(t) = a$, as pictured in

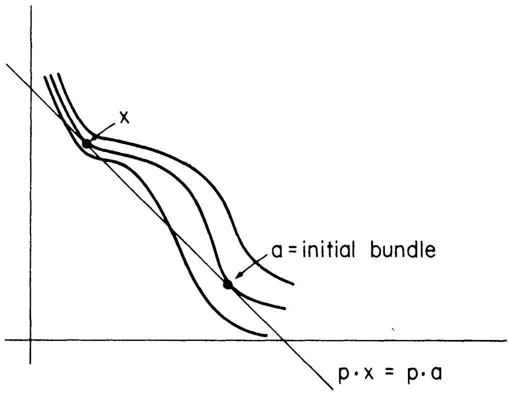

FIGURE 6

Figure 6. Then there is a unique competitive allocation, namely $\boldsymbol{a}$; the prices will of course be the p indicated in Figure 6. But now we will argue that $\boldsymbol{a}$ cannot be a value allocation. Indeed, if it were, then there would, in particular, have to exist a bounded differentiable family $U = \{u_t\}$ such that the maximum in the definition of $v_U(T)$ is achieved when $\boldsymbol{x} = \boldsymbol{a}$. By the argument of Step 1 of Section 7 (which can easily be made precise), it would follow that $(a, u_t(a))$ would have to be on the boundary of the convex hull of the graph of u_t. If that were the case, there would exist a supporting hyperplane to the graph through $(a, u_t(a))$; and it would necessarily pass through $(x, u_t(x))$. Since the hyperplane supports the graph and passes through both these points on the graph, it follows that $u_t'(a) = u_t'(x)$. But we have constructed the preferences so that though $u_t'(x)$ has the same direction as $u_t'(a)$, it must have a larger magnitude (the indifference curves are closer). Hence $\boldsymbol{a}$ cannot be a value allocation. Indeed, since every value allocation is competitive, and $\boldsymbol{a}$ is the only competitive allocation, there will be no value allocation at all.

[17] Specifically, Lemma 13.2. On the basis of the available evidence, it seems reasonable to conjecture that without differentiability, something like Proposition 5.1 will still go through, but Proposition 5.2 will not.

[18] More precisely, utilities that are concave over the relevant part of Ω.

The requirement of positive Gaussian curvature says that the indifference surfaces should never be too "flat"—at any point in any direction. Specifically, for x in Ω, consider the hyperplane H_x tangent at x to the indifference surface I_x through x. Let G_x be the line orthogonal to H_x at x (i.e., the line spanned by the gradient at x of a utility function), and let L be any line in H_x through x. We may take G_x and L as coordinate axes in the plane $P_{x,L}$ spanned by them. The situation is pictured in Figure 7. Let f be the function (of distance along L) whose graph

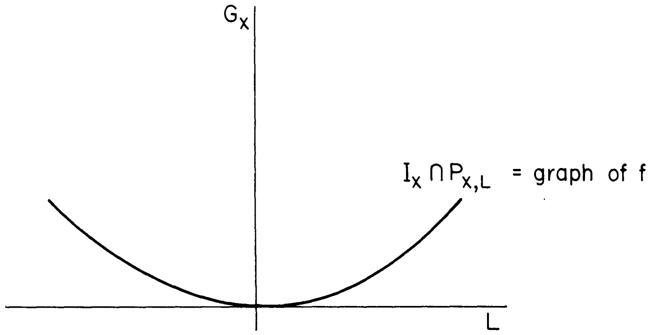

FIGURE 7

is the intersection of the indifference surface I_x with the plane $P_{x,L}$. Then the requirement of positive Gaussian curvature says[19] that the second derivative f'' be positive at the origin of $P_{x,L}$; i.e., that the indifference surface not be "too flat" at x in the direction L, and that this be true for all x and all L through x in H_x.

In economic terms, positive Gaussian curvature is equivalent[20] to finite elasticity of substitution.[21] Another familiar characterization is that the quadratic form $xu''x^*$ be positive definite when restricted to the plane $u' \cdot x = 0$.

The following example of an otherwise well-behaved preference relation with vanishing Gaussian curvature that has no concave utility is due to W. Fenchel [**12**]. Consider the relation pictured in Figure 8, with straight indifference curves that *are not parallel*.[22] If u is any C^1 utility for those preferences, the gradient of u at x has the same direction as the gradient at a; but it has a larger magnitude, since the indifference curves are closer at x. These facts enable us to apply an argument exactly as in our discussion of quasi-concavity, and to deduce that u cannot be concave; and moreover, that in the market M in which all the traders have the preference relations of Figure 8 and initial bundles $\boldsymbol{a}(t) = a$, the competitive allocation $\boldsymbol{a}$ is not a value allocation and in fact there is no value allocation.[23]

[19] Given quasi-concavity.

[20] See Footnote 19.

[21] I am indebted for this observation to C. C. von Weiszäcker.

[22] In Figure 8 the indifference curves, when extended, all pass through a single point outside (and to the left of) Ω. But this feature is not crucial; what is needed is only that the indifference curves be straight and closer at x than at y.

[23] In this case there is more than one competitive allocation $\boldsymbol{x}$, since $\boldsymbol{x}(t)$ may be essentially anywhere on the indifference curve of a as long as $\int \boldsymbol{x} = \int \boldsymbol{a}$. However, the argument that *none* of the competitive allocations can be value allocations—and hence that there is no value allocation—is not essentially different from the one we had before.

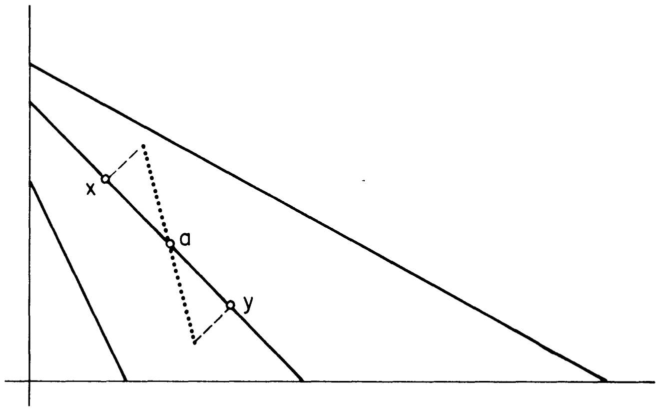

FIGURE 8

There is another argument for these facts that is of some interest. Let us think of Figure 8 as a contour map, the "height" being the differentiable utility u. The contour lines are closer at x than at y, which means the ground is steeper there. In particular, the ground rises faster along the dashed line perpendicular to the indifference curve at x than it falls along the parallel dashed line starting at y. So if one were to string a telegraph line along the dotted line and pull it taut, it would pass *over* the indifference curve containing x and y. This means that u cannot be concave. Moreover, in the market M defined in the previous paragraph, our telegraph line and an argument similar to that of Step 1 of Section 7 show that $\boldsymbol{a}$ cannot be a value allocation and in fact that there is no value allocation. Moreover, it is fairly clear that this argument goes through even under strict quasi-concavity,[24] as long as the indifference curves are still "very flat" at a, provided, of course, that the tangents to the indifference curves near a are "markedly non-parallel", e.g., that they all pass through the same point (outside of Ω).

We come now to the requirement that the gradients u' be positive. This prevents the "de Finetti phenomenon" [7]: the presence of two points x and a such that

$$(9.1) \qquad u(x) = u(a), \quad u'(a) = 0, \quad \text{and} \quad u'(x) > 0$$

(see Figure 9). If (9.1) holds for some C^1 utility u, it holds for all such utilities; what it means is that the indifference curves are infinitely more bunched up near x than near a. Under (9.1) u cannot be concave, because that, together with $u'(a) = 0$, would imply that the maximum of u along the line perpendicular to the indifference surface (dotted in the figure) is reached at a; this is impossible if the preferences are monotone.

Consider finally a market, all of whose traders have the same utility obeying (9.1), and the same initial bundle $\boldsymbol{a}(t) = a$. If the utilities are strictly quasi-concave. then the only competitive allocation is $\boldsymbol{x} = \boldsymbol{a}$. If $\boldsymbol{a}$ were a value allocation, there would exist a family $U = \{u_t\}$ of utilities such that $v_U(T)$ is achieved at $\boldsymbol{a}$. This would imply[25] that $(a, u_t(a))$ is on the boundary of the convex hull of the graph

[24] That is, $u(\alpha x + (1 - \alpha)y) > \min(u(x), u(y))$ for $\alpha \in (0, 1)$.

[25] Step 1 of Section 8.

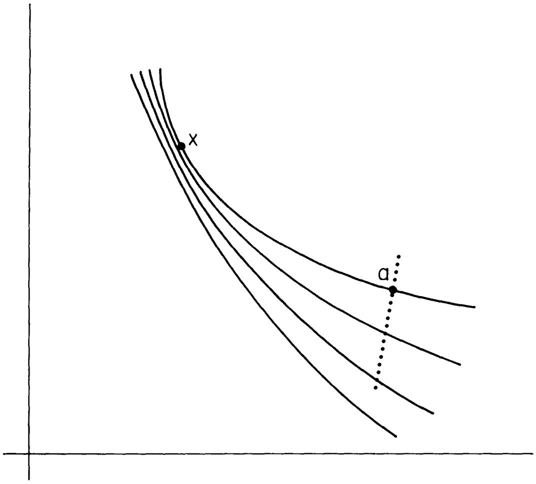

FIGURE 9

of u_t for each t. But since $u_t'(a) = 0$, this supporting hyperplane would be flat, and this would contradict the monotonicity of u_t. Thus $\boldsymbol{a}$ is not a value allocation and indeed, there is no value allocation.

10. DISCUSSION OF THE UNIFORMITY CONDITIONS

An individual preference relation is *smooth* [**10**] if it has a quasi-concave C^2 utility function with an everywhere strictly positive gradient and indifference curves that have everywhere positive Gaussian curvature. This implies that in each compact subset of Ω, the gradient u' is bounded and bounded away from 0, the Gaussian curvature is bounded away from 0, and the second derivative u'' is bounded. *Uniform* smoothness of a family $\{u_t\}$ requires that the bounds involved in these three statements be independent of t (though they may, of course, depend on the compact subset of Ω). It also requires[26] that the utilities u_t be bounded over all of Ω (i.e., not only compact subsets), uniformly in t.

Of these four requirements, one (about Gaussian curvatures) is stated directly in terms of preferences; but the others are stated in utility terms, and it would be desirable to see what they say about the preferences. The last one—about a uniform bound for the u_t—places no restriction on the preferences; if u_t are utilities satisfying the other restrictions but possibly not uniformly bounded, then $(u_t - u_t(0))/(1 + u_t - u_t(0))$ will satisfy *all* the requirements. The requirement relating to upper and lower positive bounds for the gradient u_t' has two aspects: that relating to the direction and that relating to the magnitude. That relating to the direction has an obvious interpretation in terms of the indifference surfaces (in the two-dimensional case, it simply says that the indifference curves have

[26] This requirement could be replaced by the far weaker requirement that $u_t(x) = o(\|x\|)$, integrably in t (see [**4**] or [**5**, Section 31, especially (31.2)]).

uniformly bounded slopes). To understand the requirement relating to the magnitudes of the gradients, note that whereas these magnitudes of course depend on the choice of a utility, the *ratio* between the magnitudes at two points x and y on the same indifference surface depends only on the preferences. This ratio is required to be bounded and bounded away from zero.

As for the uniform boundedness of the second derivatives, this is intuitively less clear, but it too can presumably be interpreted in terms of the preferences.

It is perhaps worthwhile to state that a sufficient condition for the family $\{u_t\}$ to be uniformly smooth is that (i) each of the u_t be smooth, (ii) the u_t be bounded over Ω, uniformly in t, and (iii) $\{u_t\}$ be contained in a compact subset of the space $\mathscr{U}$ of C^2 utility functions on Ω, endowed with the topology of C^2 uniform convergence on compact sets.[27]

Assumption 5.2, that $\boldsymbol{a}$ is uniformly bounded, is merely a normalization: if it fails, it is always possible to make it hold simply by replacing the underlying measure μ by the measure ν defined by $\nu(S) = \int_S \Sigma_{i=1}^n \boldsymbol{a}_i(t)\mu(dt)$. In the new representation, all initial bundle densities would be $\leqslant (1, \ldots, 1)$. But such a transformation would change also the utilities, and this would lead to complications in the statement of the uniformity conditions. Thus if we wanted to dispense with Assumption 5.2, we would have to replace the fixed compact sets in Conditions 4.2 and 4.3 by sets varying with t in such a way so that the maximal distance from the origin is a bounded multiple of $\Sigma_{i=1}^n \boldsymbol{a}^i(t)$. In other words, the "size" of a trader is essentially determined by the size of his initial bundle (not by his measure), and when speaking about bounds of various kinds we must really scale them to the size of the trader. It is in order to avoid these complications that we made the normalization in Assumption 5.2.

11. THE THEORY IN A CARDINAL CONTEXT

The definition of the market model in this paper is an *ordinal* one; that is, it involves only the preferences (and initial bundles) of the traders, and not any specific choice of utility functions u_t. In the definition of "value allocation", the u_t are not fixed, but are allowed to range over *all* appropriate utility functions that represent the given preferences. Hence the concept of value allocation is also an ordinal one; i.e., whether or not a given $\boldsymbol{x}$ is a value allocation depends only on the preferences (and initial bundles) of the traders.

A purely ordinal model of this kind is, however, not the traditional setting for a value theory. In this section we will consider the Shapley value in a model in which it should feel more at home—one that specifies the *cardinal* utilities of the traders, rather than just their ordinal preferences. By this we mean a model in which the traders have an intrinsic numerical scale on which the intensity of their preferences can be measured. Thus, it becomes meaningful to say not only that x

[27] A sub-basis for this topology consists of sets of the form $W(w, \varepsilon, C)$, defined for each w in $\mathscr{U}$, $\varepsilon > 0$, and compact subset C of Ω to be the set of all u in $\mathscr{U}$ such that for all x in C, $|u(x) - w(x)| < \varepsilon$, $\|u'(x) - w'(x)\| < \varepsilon$, and $\|u''(x) - w''(x)\| < \varepsilon$. It is weaker than the Whitney topology discussed in [**27**], so that if a set is compact in the Whitney topology, it is a fortiori compact in this topology.

is preferred to y, and y to z, as in an ordinal context; one can go further and say, for example, that y is relatively close to x on the preference scale (e.g., $\frac{9}{10}$ of the way between z and x). One can lend operational significance to this kind of statement by the method of von Neumann and Morgenstern [**28**], i.e., by referring to lotteries with prizes x, y, and z. But this is by no means the only possible interpretation. For example, cardinal utilities can be defined by the method of Debreu [**9**] in the case of independent commodities, or when the set of commodities can be partitioned into independent "factors" (i.e., sets of commodities); or one can define them by the method of "just noticeable differences" (cf. Fishburn [**13**, p. 18ff. and p. 80ff.]). If one wishes, one can also simply accept intensity of preferences as a fact of life that is not to be ignored, and so assume that cardinal utilities are given a priori. In any case, in our application it makes no difference how the cardinal utilities are derived, as long as they are in fact given for all traders.

Formally, call two functions u and v on Ω *equivalent* if there are real numbers α and β, with $\alpha > 0$, such that $u = \alpha v + \beta$; and define a *preference scale* on Ω to be an equivalence class of functions on Ω. A member of the equivalence class is called a *cardinal utility* for that preference scale. Preference scales play the same role in cardinal contexts that preference relations play in ordinal contexts. A cardinal utility is to a preference scale what a utility is to a preference relation; i.e., a representation by means of a numerical function.

It should be particularly noted that though specifying preference scales for all traders enables each trader to compare the intensities of his own preferences, it allows no interpersonal comparisons—no comparison of the intensities of preference of different traders. Our definition of preference scale embodies the familiar statement that "cardinal utilities are determined only up to an additive and a positive multiplicative constant". Thus if we are given cardinal utilities for all traders we can multiply them by arbitrary constants that are different for the different traders, without changing the preference scales, so that no consistent comparisons can be based on such utilities.

To stress the ordinal character of the markets and utilities defined in Section 5, we will henceforth in this section refer to them as *ordinal markets* and *ordinal utilities*.

Define a *cardinal market* in exactly the same way as an ordinal market, except that the preference relations $\succ_t$ are replaced by preference scales $\not{p}_t$. Adopt Assumptions 5.1 and 5.2; define *non-atomic*, *allocation*, and *competitive* allocation as in an ordinal market. Define a *cardinal value allocation* to be an allocation $\boldsymbol{x}$ such that there is a bounded differentiable family $U = \{u_t\}$ of cardinal utilities for the preference scales $\not{p}_t$ for which (5.3) holds[28] for all S in $\mathscr{C}$, where the game v_U is defined by (5.1). All this is the same as in Section 5, except that we have replaced the preference relations by preference scales.

There are of course relationships between the ordinal and the cardinal concepts. A preference scale $\not{p}$ and a preference relation $\succ$ are called *associated* if the cardinal utilities for $\not{p}$ are ordinal utilities for $\succ$. There is exactly one preference relation

[28] Or equivalently, (5.2) holds for all t in T.

associated with each preference scale, but generally many scales with each relation. A cardinal and an ordinal market are *associated* if the ordinal market is obtained from the cardinal market by replacing each of the scales $\not{p}_t$ by the associated relation $\succ_t$. It is easy to see that the value allocations in an ordinal market are precisely those allocations $\boldsymbol{x}$ that are cardinal value allocations in some associated cardinal market.

As an illustration, consider a cardinal market $\mathcal{M}_c$ with two traders, say $T = \{1, 2\}$. Let u_1 and u_2 be cardinal utilities for $\not{p}_1$ and $\not{p}_2$, and let us assume that they are strictly concave. An *outcome* can be viewed as a pair $(u_1(\boldsymbol{y}(1)), u_2(\boldsymbol{y}(2)))$, where $\boldsymbol{y}$ is an allocation; the set X of all outcomes is compact and convex. There is exactly one cardinal value allocation $\boldsymbol{x}$, and it is such that $(u_1(\boldsymbol{x}(1)), u_2(\boldsymbol{x}(2)))$ is the solution[29] to the "Bargaining Problem" of Nash [**18**], with outcome set X and conflict payoff $(u_1(\boldsymbol{a}(1)), u_2(\boldsymbol{a}(2)))$.

Next, consider the ordinal market $\mathcal{M}_0$ associated with $\mathcal{M}_c$. From an argument of Shapley [**23**, Sec. 3], it follows that *every* Pareto optimal allocation $\boldsymbol{x}$ with $\boldsymbol{x}(1) \succ_1 \boldsymbol{a}(1)$ and $\boldsymbol{x}(2) \succ_2 \boldsymbol{a}(2)$ is a value allocation in $\mathcal{M}_0$; in fact, every such $\boldsymbol{x}$ is the *unique* cardinal value allocation in some cardinal market associated with $\mathcal{M}_0$. Thus, in two-trader markets, the cardinal value allocations depend strongly on the preference scales of the traders, not only on their preference relations; which is of course what one would expect. A priori, moreover, there is every reason to expect that the situation is similar also when there are more than two traders, and also in the case of continuous markets. But exactly here we are in for a surprise.

PROPOSITION 11.1: *In a non-atomic cardinal market, assume that there is a bounded differentiable family of concave cardinal utilities for the preference scales. Then the set of cardinal value allocations coincides with the set of competitive allocations.*

The surprising aspect of this is that it asserts equality between the ordinal concept of "competitive allocation" and the cardinal concept of "cardinal value allocation." Non-atomicity turns the concept of "cardinal value allocation" into an ordinal concept—it does not depend on the preference scales, but only on the preference relations. It is to be noted that this is so only when we confine ourselves to preference scales with concave cardinal utilities; even then, though, it is a startling and somewhat mysterious phenomenon.

On the intuitive, unrigorous level, we can demonstrate Proposition 11.1 by the methods of Section 8—in particular, it follows from Step 6 and the demonstration of Step 7. A rigorous proof will be given in Section 14.

Like the Value Equivalence Theorem, Proposition 11.1 (which might be called the "Cardinal Value Equivalence Theorem") depends strongly on the non-atomicity: for finite markets, it is false in both directions. Indeed, we can easily find a two-trader market $\mathcal{M}_c$ of the type discussed above which has a unique

[29] That is, the maximum of $(z_1 - u_1(\boldsymbol{a}(1)))(z_2 - u_2(\boldsymbol{a}(2)))$ over $z \in X$ is achieved when $z = (u_1(\boldsymbol{x}(1)), u_2(\boldsymbol{x}(2)))$.

competitive allocation and a unique cardinal value allocation, which are different from each other.

The ordinal market $\mathcal{M}_0$ associated with such an $\mathcal{M}_c$ also shows that the Value Equivalence Theorem is false in finite markets, since in $\mathcal{M}_0$ there are many value allocations that are not competitive. It is also false in the converse direction: there are finite markets with competitive allocations that are not value allocations. For an example, take $T = \{1, 2, 3\}$, $n = 2$, $u_t(x) = \sqrt{x^1 + 1} + \sqrt{x^2 + 1}$ for all t, $\boldsymbol{a}(1) = (2, 0)$, $\boldsymbol{a}(2) = (1, 1)$, $\boldsymbol{a}(3) = (0, 2)$. There is a unique competitive allocation $\boldsymbol{x}$, given by $\boldsymbol{x}(1) = \boldsymbol{x}(2) = \boldsymbol{x}(3) = (1, 1)$. Note that $\boldsymbol{x}(2) = \boldsymbol{a}(2)$, so that at $\boldsymbol{x}$, trader 2 gets only his "personal minimum" (what he can guarantee to himself without trading). But any value allocation must yield trader 2 more than his personal minimum, since for any choice of utility functions, he contributes at least his personal minimum in all orderings of the players, and contributes more than his personal minimum in the orderings (1, 2, 3) and (3, 2, 1). Therefore $\boldsymbol{x}$ is not a value allocation.

In game theoretic terms, a cardinal market is an example of a *non-transferable utility (NTU) game*,[30] or what is more commonly called a "game without side payments" (see [**2** or **3**]). Shapley [**23**] has adapted the value notion to NTU games; our definition of "cardinal value allocation" is essentially a specialization to markets of Shapley's NTU value. There is absolutely no reason to expect the NTU value to be invariant under monotonic concave (but non-linear) transformations of the utilities; but Proposition 11.1 asserts that this is precisely what happens in a large class of NTU games, namely the non-atomic cardinal markets. It appears that the game theoretic condition underlying this phenomenon is the "homogeneity" of non-atomic markets: the payoffs available to the members of a coalition depend only on its composition, not on its size; and one may vary the size of a coalition at will, almost without affecting its composition.[31]

12. DISCUSSION OF THE LITERATURE

Though the material of this paper has many roots in previous work, we will here only touch on some of the highlights. Perhaps the first general theorem connecting the value and the competitive equilibrium was established by Shapley [**22**], using the so-called "transferable utility (TU) markets"[32] in an asymptotic model with a fixed finite number of types of traders, in which the number of traders of each type is allowed to tend to ∞. In a TU market with concave differentiable utilities there is a unique competitive payoff,[33] and Shapley proved that the values

[30] NTU games are generally defined in terms of a specific choice of cardinal utilities, not in terms of preference scales. However, since all concepts associated with such games are usually required to be "invariant" under monotonic linear transformations of the utilities, we are essentially dealing with games defined in terms of preference scales.

[31] Cf. [**5**, Chapter V]. A more familiar term for homogeneity is "constant returns to scale," as applied to coalitions rather than production processes; increasing or decreasing the scale of a coalition will not affect the possibilities open to its members.

[32] Cf. Footnote 9.

[33] That is, payoff corresponding to a competitive equilibrium.

of the TU games corresponding to the finite markets converge to this payoff. See also Shapley and Shubik [24] for a discussion of this work.

Recently, Aumann and Shapley [5] studied the value problem in the context of a TU non-atomic market. Assuming differentiability but not concavity, they showed that a TU non-atomic market has a unique competitive payoff, that the corresponding TU game has a value, and that the two coincide. No assumption of finite number of types was used.

In a parallel development Champsaur [6] studied the value problem in an asymptotic finite-type framework, but using general Walrasian markets (the kind studied here) rather than TU markets. He considered a special class of cardinal value allocations, namely those obtained by assigning to traders of the same type not only the same preference scale, but actually the same cardinal utility.[34] In that case, traders of the same type get the same utility payoffs; thus if m is the number of types and k the number of traders of each type, then for each k we get a vector $w_k \in E^m$ representing the utility payoffs of the various types. Champsaur [6] showed that as $k \to \infty$, every limit point of $\{w_k\}$ must be a competitive payoff. However, he did not prove the converse, that every competitive payoff is such a limit point. Concavity was assumed, but not differentiability.

Thus the situation can be summed up in Table I.

TABLE I

	TU	Walrasian
Asymptotic, finitely many types	Shapley [22]	Champsaur [6]
Non-atomic continuum	Aumann and Shapley [5]	This paper

Finally, we mention the important paper of Negishi [19], who was the first to use TU markets to prove a theorem about NTU markets; many of the underlying ideas of the current paper will be seen to be closely related to those of Negishi.

13. PROOF OF PROPOSITION 5.1

We start out by recalling a number of definitions, results, and notations from Aumann-Shapley [5].

Let us be given a non-atomic market as described in Section 5, and a family $U = \{u_t\}$ of utilities. Define v_U by (5.1).

[34] No such assumption is made in our treatment. Every trader may be of a different type; and even if there are many traders of the same type, they may be assigned different cardinal utilities, and in the ordinal case, even different preference scales. Nevertheless, there is implicit in our model a situation that is in principle similar to that of Champsaur. No matter how one assigns the cardinal utilities, one can, because of the non-atomicity, partition practically all the traders (i.e., all but a set of arbitrarily small measure) into a finite number of sets of equal measure, in each of which all the traders will have *approximately* the same cardinal utilities and initial bundles. In some sense, it is this that is responsible for the value equivalence phenomenon. (Compare the discussion of the core equivalence phenomenon at the top of [1, p. 49]; there the Debreu–Scarf asymptotic model plays a role similar to that played by the Champsaur model here.)

LEMMA 13.1: *If the family U is bounded differentiable, then the maximum in the definition of $v_U(S)$ is attained for all S, and the asymptotic value φv_U of v_U exists.*

Proof: This is an immediate consequence of [**5**, Proposition 31.7], which asserts these conclusions (among others) under conditions on U that are considerably weaker than bounded differentiability.

If $\boldsymbol{x}$ is an allocation, write $u(\boldsymbol{x})$ for the function on T whose value at t is $u_t(\boldsymbol{x}(t))$. A *transferable utility competitive equilibrium* (TUCE) for U is a pair $(\boldsymbol{x}, p)$, where $\boldsymbol{x}$ is an allocation and $p \in \Omega$, such that for all $t \in T$, $u_t(x) - p \cdot (x - \boldsymbol{a}(t))$ attains its maximum (over $x \in \Omega$) at $x = \boldsymbol{x}(t)$.

LEMMA 13.2: *If $(\boldsymbol{x}, p)$ is any TUCE for U, then for all $S \in \mathscr{C}$,*

$$\int_S (u(\boldsymbol{x}) - p \cdot (\boldsymbol{x} - \boldsymbol{a})) = (\varphi v_U)(S).$$

Proof: This is an immediate consequence of [**5**, Proposition 32.3] (see the remark following the statement of the proposition), which asserts this conclusion (as well as some others) under Assumption 5.1 and conditions on U that are considerably weaker than those that define bounded differentiability.

Write $x > y$ if $x \geqq y$ and $x \neq y$. The function u_t is called *increasing* if $x > y$ implies $u_t(x) > u_t(y)$.

LEMMA 13.3: *Let $(\boldsymbol{x}, p)$ be a TUCE for U, such that for almost all t in T, $p \cdot (\boldsymbol{x}(t) - \boldsymbol{a}(t)) = 0$. Assume that the u_t are increasing. Then $\boldsymbol{x}$ is a competitive allocation.*

PROOF: From the fact that the u_t are increasing it follows that $p \neq 0$. Since $\boldsymbol{x}$ is by definition an allocation, we must therefore only show that for almost all t, the function u_t is maximized over the budget set $\{x \in \Omega : p \cdot x \leqq p \cdot \boldsymbol{a}(t)\}$ by $x = \boldsymbol{x}(t)$. Suppose, indeed, that it were not. Then for a set of t's of positive measure there is a y in Ω such that $u_t(y) > u_t(\boldsymbol{x}(t))$ and $p \cdot y \leqq p \cdot \boldsymbol{a}(t)$. For almost all of those t's we will have $u_t(y) - p \cdot (y - \boldsymbol{a}(t)) \geqq u_t(y) > u_t(\boldsymbol{x}(t)) = u_t(\boldsymbol{x}(t) - p \cdot (\boldsymbol{x}(t) - \boldsymbol{a}(t))$, in contradiction to the assumption that $(\boldsymbol{x}, p)$ is a TUCE.

PROOF OF PROPOSITION 5.1: Let $\boldsymbol{x}$ be a value allocation, and let U be the bounded differentiable family of utilities for the market such that for all $S \in \mathscr{C}$,

$$(13.1) \qquad (\varphi v_U)(S) = \int_S u(\boldsymbol{x}).$$

In particular, $v_U(T) = (\varphi v_U)(T) = \int_T u(\boldsymbol{x})$, i.e., the maximum in the definition of $v_U(T)$ is attained at $\boldsymbol{x}$. Therefore by [**5**, Proposition 32.1], there is a p such that $(\boldsymbol{x}, p)$ is a TUCE with respect to U. Hence, by Lemma 13.2, for all $S \in \mathscr{C}$ we have

$$(13.2) \qquad (\varphi v_U)(S) = \int_S (u(\boldsymbol{x}) - p \cdot (\boldsymbol{x} - \boldsymbol{a})).$$

Combining (13.1) and (13.2), we see that $\int_S p\cdot(\boldsymbol{x}-\boldsymbol{a})=0$ for all $S\in\mathscr{C}$. Hence $p\cdot(\boldsymbol{x}(t)-\boldsymbol{a}(t))=0$ almost everywhere. But members of a bounded differentiable family are by definition increasing, since $u_t'>0$. Hence by Lemma 13.3, $\boldsymbol{x}$ is competitive. Thus the proof of Proposition 8.1 is complete.

14. TOWARD THE PROOF OF PROPOSITION 5.2

In this section we prove Proposition 5.2 under Hypothesis (ii), and also Proposition 11.1.

Let $x\in\Omega$. A continuously differentiable function u on Ω is called *concave over Ω at x* (or simply *concave at x*) if for all y in Ω, $u(y)-u(x)\leqslant u'(x)\cdot(y-x)$. It should be noted that this concept is not a local one; it depends on the behavior of u on all of Ω, not only near x.

REMARK 14.1: *A concave continuously differentiable function u on Ω is concave over Ω at each x in Ω.*

PROOF: First let x be the interior of Ω. For $0\leqslant\theta\leqslant 1$, define $f(\theta)=u(\theta y+(1-\theta)x)$. Then f is concave and continuously differentiable on $[0,1]$ and hence $u(y)-u(x)=f(1)-f(0)\leqslant f'(0)=u'(x)\cdot(y-x)$. If x is on the boundary of Ω, one obtains the result by a limiting argument from the interior case.

Let b in Ω be a uniform bound on the initial allocation $\boldsymbol{a}$, i.e., $\boldsymbol{a}(t)<b$ for all t; such a bound exists by Assumption 5.2.

For $x\in E^n$, set $\|x\|=\max_i|x^i|$.

LEMMA 14.1: *Let $\boldsymbol{x}$ be a competitive allocation, with prices p. Let $\{w_t\}$ be a bounded differentiable family of utilities such that the w_t are concave at each x with $\|x\|\leqslant \max_i b^i/\min_i p^i$. Then there is a measurable function α on T, bounded and bounded away from 0, such that for the family $U=\{u_t\}$ of utilities defined by $u_t=\alpha(t)w_t$, we have $(\varphi v_U)(S)=\int_S u(\boldsymbol{x})$ for all $S\in\mathscr{C}$. In particular, $\boldsymbol{x}$ is a value allocation.*

PROOF: We will find an α such that $(\boldsymbol{x},p)$ is a TUCE for U, i.e., such that for almost all t in T, we have that for all y in Ω,

$$u_t(y)-u_t(\boldsymbol{x}(t))\leqslant p\cdot(y-\boldsymbol{x}(t)). \tag{14.1}$$

Since $\boldsymbol{x}$ is a competitive allocation with prices p, and the u_t are increasing, it follows that $p>0$, and that

$$p\cdot\boldsymbol{a}(t)=p\cdot\boldsymbol{x}(t)\quad\text{for almost all } t. \tag{14.2}$$

Hence for almost all t we have that for all y in Ω,

$$w_t(y)>w_t(\boldsymbol{x}(t))\Rightarrow p\cdot y>p\cdot\boldsymbol{x}(t). \tag{14.3}$$

For the time being, consider a fixed t satisfying (14.3) for all y in Ω; let $w = w_t$ and $x = \boldsymbol{x}(t)$. Note that for each i, $p^i x^i \leqslant p \cdot x = p \cdot \boldsymbol{a}(t) \leqslant p \cdot b \leqslant \max_i b^i$. Hence

$$(14.4) \qquad \|x\| \leqslant \frac{\max_i b^i}{\min_i p^i},$$

and so w is concave at x. Assume first that $x \neq 0$; let $j = \boldsymbol{j}(t)$ be such that $x^j > 0$. Set

$$(14.5) \qquad \alpha = \boldsymbol{\alpha}(t) = \frac{p^j}{w^j(x)},$$

where w^j is the jth derivative of w. We wish to prove (14.1). Without loss of generality assume $j = 1$.

We first show that for all i,

$$(14.6) \qquad p^i \geqslant \alpha w^i(x).$$

Indeed, suppose $p^i < \alpha w^i(x)$. By (14.5) and $j = 1$, it follows that $p^i/p^1 < w^i(x)/w^1(x)$. Let θ be such that

$$(14.7) \qquad \frac{p^i}{p^1} < \theta < \frac{w^i(x)}{w^1(x)},$$

and consider a point of the form $x_s = x + (-\theta s, 0, \ldots, 0, s, 0, \ldots, 0)$, where the term s occurs in the ith coordinate; if $s > 0$ is sufficiently small, then $x_s \in \Omega$, since $x^1 > 0$. Letting $s \to 0+$, we find $dw(x_s)/ds = -\theta w^1(x) + w^i(x) > 0$, by (14.7). Hence for $s > 0$ sufficiently small, $w(x_s) > w(x)$. But then from (14.3) it follows that $p \cdot x_t \geqslant p \cdot x$, i.e., $-p^1\theta s + p^i s > 0$, contradicting (14.7). This proves (14.6).

Next, let i be such that $x^i > 0$. Then we claim that

$$(14.8) \qquad p_i \leqslant \alpha w^i(x).$$

Indeed, suppose $p^i > \alpha w^i(x)$. Reasoning as in the proof of (14.6), we find a θ such that $(p^i/p^1) > \theta > (w^i(x)/w^1(x))$. Letting $x_s = x + (\theta s, 0, \ldots, 0, -s, 0, \ldots, 0)$, we note that $x_s \in \Omega$ for small s, since $x^i > 0$. Consideration of $dw(x_s)/ds$ then leads to a contradiction as before, so (14.8) is proved.

Let $L = \{i : x^i > 0\}$ and $M = \{i : x^i = 0\}$. From (14.6) and (14.8) it follows that $p^i = \alpha w^i(x)$ for $i \in L$ and $p^i \geqslant \alpha w^i(x)$ for $i \in M$. If $y \in \Omega$, then for $i \in M$ we have $y^i - x^i = y^i \geqslant 0$; hence $\alpha w^i(x)(y^i - x^i) \leqslant p^i(y^i - x^i)$, with equality holding for $i \in L$. From the concavity of w at f it then follows that $u_t(y) - u_t(x) = \alpha(w(y) - w(x)) \leqslant \alpha w'(x) \cdot (y - x) \leqslant p \cdot (y - x)$, which is what (14.1) asserts. Thus $\boldsymbol{\alpha}(t)$ is defined and (14.1) is established when $\boldsymbol{x}(t) \neq 0$.

Suppose next that $x = \boldsymbol{x}(t) = 0$. Set $\alpha = \boldsymbol{\alpha}(t) = \min_i p^i/\max_i w^i(0)$. The concavity of w at 0 then gives $u_t(y) - u_t(0) = \alpha(w(y) - w(0)) \leqslant \alpha w'(0) \cdot y \leqslant p \cdot y$, which is again what (14.1) asserts.

The measurability of $\boldsymbol{\alpha}$ is easily verified. To show that $\boldsymbol{\alpha}$ is bounded and bounded away from 0, note that p is a constant > 0, and w_t' is by assumption bounded and bounded away from 0 in compact sets; in particular, therefore, it is bounded and bounded away from 0 in the compact set

$$\left\{y \in \Omega : \|y\| \leqslant \frac{\max_i b^i}{\min_i p^i}\right\},$$

to which, by (14.4), all the $\boldsymbol{x}(t)$ belong.

Thus we have shown that U is bounded differentiable and $(\boldsymbol{x}, p)$ is a TUCE. From Lemma 13.2 and (14.2), we then deduce that for all $S \in \mathscr{C}$, $(\varphi v_U)(S) = \int_S (u(\boldsymbol{x}) - p \cdot (\boldsymbol{x} - \boldsymbol{a})) = \int_S u(\boldsymbol{x})$. This completes the proof of Lemma 14.1.

From Remark 14.1 and Lemma 14.1 it follows that if there exists a bounded differentiable family of concave utilities for the preferences, then every competitive allocation is a value allocation. This yields Proposition 5.2 under Hypothesis (ii), and together with Proposition 5.1 (proved in the previous section), also Proposition 11.1.

15. PROOF OF THE VALUE EQUIVALENCE THEOREM

In view of Proposition 5.1 and Lemma 14.1, it remains only to prove that if the preferences are uniformly smooth, then we can find a bounded differentiable family of utilities that is concave over Ω in an arbitrarily large set. Specifically, we will prove

LEMMA 15.1: *Let $U = \{u_t\}$ be a uniformly smooth family of utilities for the given preferences. Then for every $\gamma > 0$ there exists a bounded differentiable family $\{w_t\}$ of utilities such that the w_t are concave over Ω at each x with $\|x\| \leqslant \gamma$.*

REMARK: Necessary and sufficient conditions for a preference relation to possess a concave utility have been obtained by Fenchel [**12**]. His conditions imply, as a special case, that under appropriate conditions, smooth preferences have concave utilities. Here, however, we have found it more convenient not to use his result, but to prove what is needed "from scratch." This is done for a number of reasons: First, we are interested only in a comparatively simple sufficient condition; whereas the mere statement of Fenchel's necessary and sufficient condition is complex and lengthy. Second, we need a uniform result for a family of preferences, whereas Fenchel's result applies to a single preference relation; therefore even if we used his result we would have to reprove it in the "uniformized" version. Finally, his conditions do not apply exactly here; for example, he considers an open rather than a closed domain of definition. Nevertheless, we will lean heavily on his methods, and there is no question that the principle that "smooth preferences have concave utilities" is to be attributed to Fenchel.

PROOF: Let $u_t^i = \partial u_t / \partial x_i$. Since u_t is bounded differentiable, u_t' is uniformly bounded and uniformly positive in compact sets. Hence there is a $\delta > 0$ such that for all t and all x in Ω with $\|x\| \leqslant \gamma$, we have

$$(15.1) \qquad \min_i \left(\frac{u_t^i}{\sum_{i=1}^n u_t^i} \right) \geqslant \delta.$$

Obviously, $\delta \leqslant 1/n$. Let $\Gamma = \{x \in \Omega : x \leqslant 2\gamma/\delta\}$, and let Δ be the interior of Γ. Set $h(\theta) = -e^{-\xi\theta}$ where $\xi > 0$ is a parameter that will be fixed later. We first wish to show that when ξ is sufficiently large, $h \circ u_t$ is concave on Δ for all t (i.e., $h \circ u_t | \Delta$ is concave for all t). For simplicity of notation, we will for the time being suppress the subscript t, i.e., we will write $u_t = u$.

It is well known (see, for example, [**11**, p. 51]) that a C^2 function w on a convex open set is concave if and only if the matrix $w'' = w''(x)$ of its second partial derivatives is negative semidefinite for each x, i.e., $yw''y^* \leqslant 0$ for all y in E^n, where $*$ denotes "transpose." A straightforward calculation yields $(h \circ u)'' = \xi e^{-\xi u}(u'' - \xi u'^* u')$, where u' is a row vector. Thus we must show that $u'' - \xi u'^* u'$ is negative semidefinite for appropriate ξ. We will in fact show that for ξ sufficiently large,

$$(15.2) \qquad u'' - \xi u'^* u' \quad \text{is negative definite at each } x \text{ in } \Delta,$$

i.e., that $y(u'' - \xi u'^* u')y^* < 0$ for all $y \neq 0$.

Let $H (= H_{x,t})$ be the hyperplane $\{y : y \cdot u' = 0\}$. Then we assert

$$(15.3) \qquad y \in H \Rightarrow yu''y^* \leqslant 0.$$

Indeed, for $y \in H$, consider the expression $u(x + \theta y)$ as a function of the real variable θ; it is defined for θ in some neighborhood of 0, since x is in the interior of Ω. The point $x + \theta y$ is in the hyperplane through x orthogonal to u'; by its quasiconcavity, u achieves its maximum in this hyperplane at x. Hence $u(x + \theta y)$, as a function of θ, achieves its maximum at 0, and so

$$0 \geqslant \frac{d^2}{d\theta^2} u(x + \theta y) = yu''y^*;$$

this proves (15.3).

Next we assert

$$(15.4) \qquad y \in H \Rightarrow yu''y^* < 0, \quad \text{or} \quad y = 0;$$

in other words, not only is u'' negative semidefinite on H, which is what (15.3) asserts, but it is in fact negative definite on H. Suppose it were not; let $y_0 \in H$ be such that $y_0 \neq 0$ and $y_0 u'' y_0^* = 0$ (the possibility $y_0 u'' y_0^* > 0$ is ruled out by (15.3)). By (15.3), $yu''y^*$ attains its maximum subject to $yu'^* = 0$ at $y = y_0$. Therefore there is a Lagrange multiplier λ such that $2u''y_0^* + \lambda u'^* = 0$ and $u'y_0^* = 0$. Hence the system

$$\begin{pmatrix} u'' & u'^* \\ u' & 0 \end{pmatrix} z^* = 0$$

of linear equations in $z = (z^1, \ldots, z^{n+1})$ has the distinct solutions $z = 0$ and

$z = (y_0, \lambda/2)$, and so the determinant of the matrix defining the system vanishes. But this violates the positivity of the Gaussian curvature (see Condition 4.2 and Equations (4.1) and (4.2)), so (15.4) is proved.[35]

According to a known theorem,[36] (15.4) implies (15.2) for sufficiently large ξ; but we must still show that the choice of ξ may be made independent of x in Δ, and of t.

Let $Q_\xi = u'' - \xi u'^* u'$, let $S^{n-1} = \{y \in E^n : \|y\| = 1\}$, and let $g(\xi)$ be the maximum of $yQ_\xi y^*$ over all y in S^{n-1}. Clearly g is non-increasing. Moreover, since $yQ_\xi y^*$ is continuous in y and ξ, and S^{n-1} is compact, it follows that g is continuous. Hence if ξ_0 is the infimum of all ξ such that $g(\xi) < 0$, then $g(\xi_0) = 0$; in other words, $yQ_\xi y^*$ is negative definite for $\xi > \xi_0$, whereas for $\xi = \xi_0$, it is negative semidefinite but not negative definite. Now it is well known that any quadratic form Q can be diagonalized, i.e., there is a non-singular matrix F such that FQF^* is a diagonal matrix; if Q is negative definite then all the diagonal entries are negative, and it follows that the determinant $|Q|$ does not vanish. If Q is negative semidefinite but not definite, then at least one of the diagonal entries vanishes and hence $|Q| = 0$. Thus $|Q_\xi| \neq 0$ for $\xi > \xi_0$ and $|Q_\xi| = 0$ for $\xi = \xi_0$, i.e.,

$$\xi_0 = \max\{\xi : |u'' - \xi u'^* u'| = 0\}. \tag{15.5}$$

Now letting i and j run from 1 to n, we have

$$|Q_\xi| = |u'' - \xi u'^* u'| = |u^{ij} - \xi u^i u^j| = \begin{vmatrix} u^{ij} - \xi u^i u^j & u^i \\ 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} u^{ij} & u^i \\ u^j & 1 \end{vmatrix} = |u^{ij}| + \xi \begin{vmatrix} u^{ij} & u^i \\ u^j & 0 \end{vmatrix} = |u''| + \xi u^H.$$

From this and (15.5) it follows that $\xi_0 = -|u''|/u^H$. Now making the dependence of u on t explicit once more, and applying (4.2), we obtain

$$\xi_0 = \frac{|u_t''| \, \|u_t'\|_2^{n+1}}{u_t^c}. \tag{15.6}$$

By Conditions 4.2 and 4.3, and the bounded differentiability of the u_t, there is a uniform bound for the right side of (15.6) over all x in Δ and all t. If $\xi > 0$ is chosen greater than that bound, then $u_t'' - \xi u_t'^* u_t'$ is negative definite for all x in Δ and all t, and hence $h \circ u_t$ is concave on Δ. From a simple continuity argument it then follows that $h \circ u_t$ is concave also on the closure Γ of Δ.

It remains to demonstrate the concavity over Ω, for $\|x\| \leqq \gamma$. Let r be a real function of a real variable θ with the following properties:

PROPERTY 15.1: *$r \in C^2$ and for all θ, $r'(\theta) > 0$.*

[35] I am indebted to Kenneth Arrow for this elegant proof of (15.4), which replaces a more awkward proof, using an explicit diagonalization, that I previously had.

[36] For an elegant proof in four lines, see Debreu [**8**, p. 296, Theorem 3].

PROPERTY 15.2: $r(\theta) \leqslant min\ (\theta, 1)$.

PROPERTY 15.3: $r(\theta) = \theta$ *for* $\theta \leqslant 0$.

For each t, set $v_t = h \circ u_t$, and define q_t by

$$q_t(\theta) = (\beta_t - \alpha_t)r((\theta - \alpha_t)/(\beta_t - \alpha_t)) + \alpha_t,$$

where $e = (1, \ldots, 1)$, $\beta_t = v_t(2\gamma e)$, and $\alpha_t = v_t(\gamma e)$. Properties 15.1 through 15.3 then yield:

(15.7) $q_t \in C^2$ and for all θ, $q_t'(\theta) > 0$.

(15.8) $q_t(\theta) \leqslant \min(\theta, v_t(2\gamma e))$.

(15.9) $q_t(\theta) = \theta$ for $\theta \leqslant v_t(\gamma e)$.

Now define $w_t = q_t \circ v_t$. We wish to demonstrate that w_t is concave on $\{x \in \Omega: \|x\| \leqslant \gamma\}$ over Ω. This means that for all x with $\|x\| \leqslant \gamma$, and for all y in Ω,

(15.10) $w_t(y) \leqslant w_t(x) + w_t'(y) \cdot (y - x)$.

To prove (15.10), we again suppress the subscript t, writing w for w_t, v for v_t, and u for u_t. First note that because $\|x\| \leqslant \gamma$, it follows from (15.9) that $w(x) = v(x)$ and $w'(x) = v'(x)$. If $y \in \Gamma$, then from (15.8) and the concavity of v on Γ, we get $w(y) = q(v(y)) \leqslant v(y) \leqslant v(x) + v'(x) \cdot (y - x) = w(x) + w'(x) \cdot (y - x)$, proving (15.10) in this case. If $y \notin \Gamma$, then for at least one coordinate i, we have $y^i \geqslant 2\gamma/\delta$. Hence by (15.1),

$$\frac{u'(x)}{\sum_{i=1}^{n} u^i(x)} \cdot y \geqslant 2\gamma = \frac{u'(x)}{\sum_{i=1}^{n} u^i(x)} \cdot 2\gamma e.$$

Hence $u'(x) \cdot y \geqslant u'(x) \cdot 2\gamma e$, and so $v'(x) \cdot y \geqslant v'(x) \cdot 2\gamma e$ (since $v'(x) = h'(u(x))u'(x)$ and $h' > 0$ always). Combining this with (15.9) and the concavity of v on Γ we obtain $w(y) = q(v(y)) \leqslant v(2\gamma e) \leqslant v(x) + v'(x) \cdot (2\gamma e - x) \leqslant v(x) + v'(x) \cdot (y - x)$, as was to be shown. Thus (15.10) is established.

Since $\{w_t\}$ is easily seen to be bounded differentiable, the proof of Lemma 15.1 —and with it of the Value Equivalence Theorem—is complete.

The Hebrew University of Jerusalem

APPENDIX

MARGINAL AND "TRUE" CONTRIBUTIONS

In the introduction we stated that "the marginal contribution of a trader to society is almost surely the same as his 'true' contribution." This assertion can be demonstrated on two levels: intuitively, and in a rigorous fashion. The intuitive demonstration was already given in Step 5 of Section 7, where we showed that if the coalition S is randomly chosen, then the contribution $v_U(S) - v_U(S \backslash dt)$ of dt to S is almost surely the same as his marginal contribution $v_U(T) - v_U(T\backslash dt)$. We will now state this assertion precisely, and prove it rigorously.

Let us be given a non-atomic market, and let $U = \{u_t\}$ be a bounded differentiable family of utilities for the preferences. Set $v = v_U$. Let $\mathscr{P} = (\Pi_1, \Pi_2, \ldots)$ be a separating sequence of partitions of T into measureable sets, such that each Π_m refines Π_{m-1}. For each m, let $\underset{\sim}{\mathscr{R}}_m$ be a random order[37] on Π_m. For each B in Π_m, denote by $\underset{\sim}{Q} = \underset{\sim}{Q}^m_B$ the union of the members of Π_m up to and including B in the order $\underset{\sim}{\mathscr{R}}_m$. For each S with $B \subset S$ let $\Delta_B v(S) = v(S) - v(S\backslash B)$; $\Delta_B v(S)$ is the *contribution of B to S.*

PROPOSITION A.1: *For every $\varepsilon > 0$ there is an m_0 such that if $m > m_0$ then for every $B \in \Pi_m$ we have*

(A.1) $$|\Delta_B v(T) - (\varphi v)(B)| < \varepsilon\mu(B)$$

and

(A.2) $$prob\ \{|\Delta_B v(\underset{\sim}{Q}) - (\varphi v)(B)| < \varepsilon\mu(B)\} > 1 - \varepsilon.$$

This says that in a random order on a sufficiently fine partition Π of the player space, with high probability the incremental worth ("true" contribution) of an element B of Π is $(\varphi v)(B) + o(\mu(B))$, and also that the marginal contribution is $(\varphi v)(B) + o(\mu(B))$. In particular, the "true" and marginal contribution differ by a term that is $o(\mu(B))$. Note that $(\varphi v)(B)$ is in general of the order of magnitude of $\mu(B)$ (see Lemma 13.2).

We will make extensive and free use of the notation, terminology, and results of [**5**]. The reader should note that T in this paper plays the same role as I in [**5**], and that $u_t(x)$ is also denoted $u(x, t)$ in [**5**]; also that it is assumed that $u_t(0) = 0$ for all t, an assumption we can make without loss of generality.

There is one respect in which our notation will differ from that of [**5**]. In [**5**], both random variables and functions of t were denoted by bold face letters; no confusion could result, since these two kinds of objects were never considered at the same time. Here confusion might result, and so we shall indicate random variables by a wiggly underline, and functions of t by bold face.

Before proceeding with the proof, we shall introduce some auxiliary definitions that will enable us to avoid unnecessary repetition of tiresome "epsilon-delta" phraseology. Let $F = F^m_B$ be a sentence depending on m and on the choice of an element B of Π_m (an example is Formula (A.2)). We shall say that F *eventually* holds if there is an m_0 such that F^m_B holds for $m > m_0$ and all $B \in \Pi_m$. If $\underset{\sim}{f} = \underset{\sim}{f}^m_B$ is a random variable depending on m and the choice of a B in Π_m (for example $v(\underset{\sim}{Q})$), then $\underset{\sim}{f}$ is *eventually probably small* if for each $\varepsilon > 0$, prob $\{|\underset{\sim}{f}| < \varepsilon\} > 1 - \varepsilon$ eventually holds (if $\underset{\sim}{f}$ is multidimensional, replace $|\underset{\sim}{f}|$ by $\|\underset{\sim}{f}\|$). If $\underset{\sim}{f} - \underset{\sim}{g}$ is eventually probably small, we shall say that $\underset{\sim}{f}$ is *eventually probably close* to $\underset{\sim}{g}$. Proposition A.1 may be paraphrased by saying that (A.1) holds and that $(\Delta_B v(\underset{\sim}{Q}) - \Delta_B v(T))/\mu(B)$ is eventually probably small.

A number of lemmas will be required. Let $r = r_m = |\Pi_m|$ and let $\underset{\sim}{h} = \underset{\sim}{h}^m_B$ be the number of elements of Π_m included in $\underset{\sim}{Q}$, i.e., the number of elements up to and including B in the order $\underset{\sim}{\mathscr{R}}_m$.

LEMMA A.1: *Let $\xi \in NA^+$. Then $\xi(\underset{\sim}{Q})$ is eventually probably close to $(\underset{\sim}{h}/r)\xi(T)$.*

PROOF: Set $\delta^m = \max\{\xi(A) : A \in \Pi_m\}$. Since $\underset{\sim}{h}$ takes on each integer value between 1 and r with the same probability, we have

$$E\left(\xi(\underset{\sim}{Q}\backslash B) - \frac{\underset{\sim}{h}}{r}\xi(T\backslash B)\right)^2 = \frac{1}{r}\sum_{h=1}^{r} E\left(\left(\xi(\underset{\sim}{Q}\backslash B) - \frac{h}{r}\xi(T\backslash B)\right)^2 \mid \underset{\sim}{h} = h\right).$$

Referring to the proof of [**5**, Lemma 18.7], we find that each of the summands in the last sum is $\leqslant \delta^m$, and that $\delta^m \to 0$. Hence $E(\xi(\underset{\sim}{Q}\backslash B) - (\underset{\sim}{h}/r)\xi(T\backslash B))^2 \leqslant \delta^m \to 0$. Since $\xi(B) \leqslant \delta^m \to 0$, the lemma follows.

LEMMA A.2: *Let $\xi \in NA^+$, $\xi(T) > 0$. Then for each $\theta > 0$, eventually prob $\{\xi(\underset{\sim}{Q}) > \theta\xi(T)\} > 1 - 2\theta$.*

PROOF: Since $\underset{\sim}{h}$ takes on each integer value between 1 and r with equal probability, it follows that for every $\theta > 0$, prob $\{\underset{\sim}{h}/r > \theta\} \to 1 - \theta$ as $m \to \infty$. Lemma A.2 then follows from Lemma A.1.

LEMMA A.3: *$u'_{\underset{\sim}{Q}}(\int_{\underset{\sim}{Q}} \mathbf{a})$ is eventually probably close to $u'_T(\int \mathbf{a})$.*

[37] A random variable whose values are orders on Π_m, where each of the $|\Pi_m|!$ possible orders is assigned the same probability.

PROOF: Let $\varepsilon > 0$ be given. Set $\alpha = \Sigma \int \boldsymbol{a}$. By [5, Proposition 38.14], there is a $\delta > 0$ such that for all S with $\mu(S) \geqslant \varepsilon$, all b in $A(\varepsilon, \alpha)$, and all δ-approximations $\hat{u}$ to u, we have

(A.3) $$\|\hat{u}_S'(b) - u_S'(b)\| < \varepsilon.$$

Let $\hat{u}$ be a δ-approximation to u that is of finite type [5, Proposition 35.6]. Define an n-dimensional vector measure ζ by $\zeta(S) = \int_S \boldsymbol{a}$. Then

(A.4) $$\left\| u_{\underset{\sim}{Q}}'\left(\int_{\underset{\sim}{Q}} \boldsymbol{a}\right) - u_T'\left(\int \boldsymbol{a}\right)\right\| \leqslant \|u_{\underset{\sim}{Q}}'(\zeta(\underset{\sim}{Q})) - \hat{u}_{\underset{\sim}{Q}}'(\zeta(\underset{\sim}{Q}))\| + \|\hat{u}_{\underset{\sim}{Q}}'(\zeta(\underset{\sim}{Q})) - \hat{u}_T'(\zeta(T))\|$$
$$+ \|\hat{u}_T'(\zeta(T)) - u_T'(\zeta(T))\| = \underset{\sim}{D}_1 + \underset{\sim}{D}_2 + D_3.$$

Applying Lemma A.2 (with $\xi = \mu$ and $\theta = \varepsilon$) and (A.3), we find that eventually

(A.5) $$\text{prob}\,\{\underset{\sim}{D}_1 < \varepsilon\} > 1 - 2\varepsilon$$

and

(A.6) $$D_3 < \varepsilon.$$

To estimate $\underset{\sim}{D}_2$, let k be the number of different "types" in $\hat{u}$, i.e., the number of different $\hat{u}_t$. Denote the nonnegative orthant of E^k by Ξ. Then there is a continuous function g on $\Xi \times \Omega$ and a k-dimensional vector η of NA^+ measures on T, such that

(A.7) $$\hat{u}_S(z) = g(\eta(S), z)$$

for all $S \in \mathscr{C}$ and $z \in \Omega$, $\eta(T) > 0$, and g is homogeneous of degree 1 on $\Xi \times \Omega$ and continuously differentiable in the interior of $\Xi \times \Omega$ [5, (39.17), Propositions 39.8 and 39.13, and Lemma 39.9]. Set $q = n + k$, $\varepsilon_1 = \varepsilon/3q$, and $v = (\eta, \zeta)$. Since $\varepsilon_1 v(T) > 0$, we may find a compact neighborhood U of the straight line segment connecting $\varepsilon_1 v(T)$ to $v(T)$, such that g' is uniformly continuous in U. Hence there is a δ_1, with $0 < \delta_1 < \varepsilon_1$, such that

(A.8) $$x, y \in U \quad \text{and} \quad \|x - y\| < \delta_1 \Rightarrow \|g'(x) - g'(y)\| < \varepsilon.$$

Now by Lemma A.2, eventually prob $\{v(\underset{\sim}{Q}) > \varepsilon_1 v(T)\} > 1 - 2q\varepsilon_1$, and by Lemma A.1, $v(\underset{\sim}{Q})$ is eventually probably close to $(\underset{\sim}{h}/r)v(T)$. Hence by (A.8), eventually prob $\{\|g'(v(\underset{\sim}{Q})) - g'(\underset{\sim}{h}/rv(T))\| < \varepsilon\} > 1 - 3q\varepsilon_1 = 1 - \varepsilon$. But $g'((\underset{\sim}{h}/r)v(T)) = g'(v(T))$, because g is homogeneous. Hence by (A.7), eventually prob $\{\|\hat{u}_{\underset{\sim}{Q}}'(\zeta(\underset{\sim}{Q})) - \hat{u}_T'(\zeta(T))\| < \varepsilon\} > 1 - \varepsilon$. Combining this with (A.5) and (A.6), we find that eventually prob $\{\underset{\sim}{D}_1 + \underset{\sim}{D}_2 + D_3 < 3\varepsilon\} > 1 - 3\varepsilon$, and hence by (A.4), the proof of Lemma A.3 is complete.

LEMMA A.4: *For each $\varepsilon > 0$ there is a bounded function $\boldsymbol{\eta}$ such that if $\mu(S) \geqslant \varepsilon$, b in Ω satisfies $\Sigma b \leqslant \Sigma \int \boldsymbol{a}$, and $u_S(b)$ is attained at $\boldsymbol{y}$, then $\boldsymbol{y}(s) \leqslant \boldsymbol{\eta}(s)e$ for almost all $s \in S$.*

PROOF: First note that in [5, Lemma 37.1] we may choose $\boldsymbol{\eta}_\delta$ bounded; indeed, if the uniform bound on u is γ, then we may choose $\boldsymbol{\eta}_\delta(s) \equiv \max(\gamma/\delta, 1/\delta^2)$. Lemma A.4 then follows immediately from [5, Lemma 37.8], when one refers to the first paragraph of the proof of that lemma.

LEMMA A.5: *Let $\xi \in NA^+$. Then max $\{\xi(A) : A \in \Pi_m\} \to 0$ as $m \to \infty$.*

PROOF: This follows from the fact that $\mathscr{P}$ is separating. For details, refer to the proof of [5, Lemma 18.7].

COROLLARY A.1: *Let $v(\underset{\sim}{Q})$ be achieved at $\underset{\sim}{y}$. Then $\int_B \underset{\sim}{y}$ is eventually probably small.*

PROOF: Let $\varepsilon > 0$ be given, and let $\boldsymbol{\eta}$ correspond to ε in accordance with Lemma A.4. Let γ be a common bound for $\boldsymbol{\eta}$ and for $\Sigma \boldsymbol{a}$ (see Assumption 5.2). If $\mu(\underset{\sim}{Q}) \leqslant \varepsilon$, then $\int_B \underset{\sim}{y} \leqslant \int_{\underset{\sim}{Q}} \underset{\sim}{y} = \int_{\underset{\sim}{Q}} \boldsymbol{a} \leqslant \gamma\varepsilon e$. If $\mu(\underset{\sim}{Q}) \geqslant \varepsilon$, then by Lemma 1.4, $\int_B \underset{\sim}{y} \leqslant \gamma\mu(B)e$. But by Lemma A.5, $\mu(B)$ is eventually smaller than ε. Thus the Corollary is proved.

PROOF OF PROPOSITION A.1: First we demonstrate that eventually (A.1) holds. Suppose $v(T)$ is attained at $\boldsymbol{x}$. From [5, Propositions 32.3 and 38.5] it follows that

(A.9) $$(\varphi v)(B) = \int_B \left[u(\boldsymbol{x}) + u_T'\left(\int \boldsymbol{a}\right)(\boldsymbol{a} - \boldsymbol{x})\right].$$

From [**5**, Lemma 40.1], it follows that

$$\text{(A.10)} \qquad \Delta_B v(T) = v(T) - v(T\backslash B) = \int_B [u(\boldsymbol{x}) + u'_{T\backslash B}(c)(\boldsymbol{a} - \boldsymbol{x})],$$

where c is on the straight line segment connecting $\int_{T\backslash B}\boldsymbol{x}$ to $\int_{T\backslash B}\boldsymbol{a}$, i.e., (since $\int \boldsymbol{x} = \int \boldsymbol{a}$) $c = \int \boldsymbol{a} - \int_B(\theta\boldsymbol{x} + (1-\theta)\boldsymbol{a})$, where $0 \leqslant \theta \leqslant 1$. Set $b = \int \boldsymbol{a} - \int_B \boldsymbol{x}$. Then $b - c = (1-\theta)\int_B(\boldsymbol{a} - \boldsymbol{x})$, and hence from [**5**, Proposition 38.7] and Lemma A.5, it follows that for each $\varepsilon_1 > 0$, eventually

$$\text{(A.11)} \qquad \|u'_{T\backslash B}(b) - u'_{T\backslash B}(c)\| < \varepsilon_1.$$

Again using [**5**, Lemma 40.1], we find

$$\text{(A.12)} \qquad u'_{T\backslash B}(b) = u'_T\left(\int \boldsymbol{a}\right),$$

on condition that $\int_{T\backslash B}\boldsymbol{a} > 0$; but this follows from Assumption 5.1 and Lemma A.5 for sufficiently large m, so that (A.12) eventually holds. Formulas (A.9), (A.10), (A.11), and (A.12) yield eventually $|\Delta_B v(T) - (\varphi v)(B)| \leqslant \varepsilon_1 \Sigma \int_B (\boldsymbol{a} + \boldsymbol{x})$; together with Assumption 5.2 and Lemma A.4, this yields that (A.1) holds eventually.

The proof of (A.2) is quite similar. Suppose $v(\underset{\sim}{Q})$ is achieved at $\underset{\sim}{y}$; [**5**, Lemma 40.1] yields

$$\text{(A.13)} \qquad \Delta_B v(\underset{\sim}{Q}) = v(\underset{\sim}{Q}) - v(\underset{\sim}{Q}\backslash B) = \int_B [u(\underset{\sim}{y}) + u'_{\underset{\sim}{Q}}(\underset{\sim}{c})(\boldsymbol{a} - \underset{\sim}{y})],$$

where $\underset{\sim}{c} = \int_{\underset{\sim}{Q}}\boldsymbol{a} + \underset{\sim}{\theta}\int_B(\boldsymbol{a} - \underset{\sim}{y})$ and $0 \leqslant \underset{\sim}{\theta} \leqslant 1$. Set $\underset{\sim}{b} = \int_{\underset{\sim}{Q}}\boldsymbol{a} - \int_B \underset{\sim}{y}$. Then $\boldsymbol{b} - \boldsymbol{c} = (1-\theta)\int_B(\boldsymbol{a} - \underset{\sim}{y})$, and hence from [**5**, Proposition 38.7], Corollary A.1, and Lemma A.2 (with $\xi = \mu$ and θ small), it follows that $u'_{\underset{\sim}{Q}\backslash B}(\underset{\sim}{b})$ is eventually probably close to $u'_{\underset{\sim}{Q}\backslash B}(\underset{\sim}{c})$. By [**5**, Lemma 40.1], $u'_{\underset{\sim}{Q}\backslash B}(\underset{\sim}{b}) = u'_{\underset{\sim}{Q}}(\int_{\underset{\sim}{Q}}\boldsymbol{a})$ whenever $\int_{\underset{\sim}{Q}\backslash B}\boldsymbol{a} > 0$. By Lemma A.2, for each $\varepsilon_2 > 0$, eventually prob $\{\int_{\underset{\sim}{Q}\backslash B}\boldsymbol{a} > 0\} > 1 - \varepsilon_2$. Hence $u'_{\underset{\sim}{Q}\backslash B}(b)$ is eventually probably close to $u'_{\underset{\sim}{Q}}(\int_{\underset{\sim}{Q}}\boldsymbol{a})$, and by Lemma A.3, this is eventually close to $u'_T(\int\boldsymbol{a})$. Hence

$$\text{(A.14)} \qquad u_{\underset{\sim}{Q}}\left(\int_{\underset{\sim}{Q}}\boldsymbol{a}\right) \textit{ and } u'_{\underset{\sim}{Q}\backslash B}(\underset{\sim}{c}) \textit{ are both eventually probably close to } u'_T\left(\int\boldsymbol{a}\right).$$

Now by [**5**, Proposition 38.5], we have $u_t(\underset{\sim}{y}(t)) - u_t(\boldsymbol{x}(t)) \leqslant u'_T(\int\boldsymbol{a})(\underset{\sim}{y}(t) - \boldsymbol{x}(t))$. Hence by (A.9) and (A.13),

$$\text{(A.15)} \qquad \Delta_B v(\underset{\sim}{Q}) - (\varphi v)(B) \leqslant \left(u'_{\underset{\sim}{Q}}(\underset{\sim}{c}) - u'_T\left(\int\boldsymbol{a}\right)\right)\int_B(\boldsymbol{a} - \underset{\sim}{y}).$$

Again by [**5**, Proposition 38.5], $u_t(\boldsymbol{x}(t)) - u_t(\underset{\sim}{y}(t)) \leqslant u'_{\underset{\sim}{Q}} \;{}_{\underset{\sim}{Q}}\,\boldsymbol{a})(\boldsymbol{x}(t) - \underset{\sim}{y}(t))$, and hence by (A.9) and (A.13)

$$\text{(A.16)} \qquad (\varphi v)(B) - \Delta_B v(\underset{\sim}{Q}) \leqslant \left(u'_{\underset{\sim}{Q}}\left(\int_{\underset{\sim}{Q}}\boldsymbol{a}\right) - u'_T\left(\int\boldsymbol{a}\right)\right)\int_B(\boldsymbol{x} - \underset{\sim}{y}) - \left(u'_T\left(\int\boldsymbol{a}\right) - u_{\underset{\sim}{Q}}(\underset{\sim}{c})\right)\int_B(\underset{\sim}{y} - \boldsymbol{a}).$$

Now let $\varepsilon_3 > 0$ be given, and let $\boldsymbol{\eta}$ correspond to ε_3 in accordance with Lemma A.4. Let γ be a common bound for $\boldsymbol{\eta}$ and $\Sigma\,\boldsymbol{a}$ (see Assumption 5.2). By Lemma A.2, eventually $\mu(\underset{\sim}{Q}) \geqslant \varepsilon_3$ with probability $\geqslant 1 - 2\varepsilon_3$. Set $\varepsilon_4 = \varepsilon_3/2\gamma m$. Hence by (A.14), (A.15), and (A.16), eventually, with probability $\geqslant 1 - 2\varepsilon_3$, $|\Delta_B v(\underset{\sim}{Q}) - (\varphi v)(B)| \leqslant 4\varepsilon_4\gamma m\mu(B) = 2\varepsilon_3\mu(B)$. This completes the proof of Proposition A.1.

REFERENCES

[**1**] Aumann, R.: "Markets with a Continuum of Traders," *Econometrica*, 32 (1964), 39–50.

[**2**] ———: "A Survey of Cooperative Games Without Side Payments," in *Essays in Mathematical Economics in Honor of Oskar Morgenstern*, ed. by M. Shubik. Princeton, N.J.: Princeton University Press, 1967, 3–27.

[**3**] Aumann, R., and B. Peleg: "von Neumann-Morgenstern Solutions to Cooperative Games Without Side Payments," *Bulletin of the American Mathematical Society*, 66 (1960), 173–179.

[**4**] Aumann, R., and M. Perles: "A Variational Problem Arising in Economics," *Journal of Mathematical Analysis and Applications*, 11 (1965), 488–503.

[**5**] Aumann, R., and L. Shapley: *Values of Non-Atomic Games*. Princeton, N.J.: Princeton University Press, 1971.

[**6**] Champsaur, P.: "Cooperation vs. Competition," unpublished manuscript, 1971.

[7] DE FINETTI, B.: "Sulle Stratificazioni Convesse," *Annali di Matematica*, 4 (1949), 173–183.
[8] DEBREU, G.: "Definite and Semi-Definite Quadratic Forms," *Econometrica*, 20 (1952), 295–299.
[9] ———: "Topological Methods in Cardinal Utility Theory," in *Mathematical Methods in the Social Sciences*, ed. by K. Arrow, S. Karlin, and P. Suppes. Stanford: Stanford University Press, 1959, 16–26.
[10] ———: "Smooth Preferences," *Econometrica*, 40 (1972), 603–616.
[11] EGGLESTON, H. G.: *Convexity*. Cambridge: Cambridge University Press, 1958.
[12] FENCHEL, W.: "Uber konvexe Funktionen mit vorgeschriebenen Niveaumannigfaltigkeiten," *Mathematische Zeitschrift*, 63 (1956), 496–506.
[13] FISHBURN, P.: *Utility Theory for Decision Making*, New York: Wiley, 1970.
[14] HILDENBRAND, W.: *Core and Equilibria of Large Economies*. Princeton, N.J.: Princeton University Press, 1974.
[15] KANNAI, Y.: "Values of Games with a Continuum of Players," *Israel Journal of Mathematics*, 4 (1966), 54–58.
[16] ———: "Approximation of Convex Preferences," *Journal of Mathematical Economics*, 1 (1974), 101–106.
[17] MAS-COLLEL, A.: "Continuous and Smooth Consumers: Approximation Theorems," *Journal of Economic Theory*, 8 (1974), 305–336.
[18] NASH, J. F.: "The Bargaining Problem," *Econometrica*, 18 (1950), 152–162.
[19] NEGISHI, T.: "Welfare Economics and Existence of an Equilibrium for a Competitive Economy," *Metroeconomica*, 12 (1960), 92–97.
[20] RAWLS, J.: *A Theory of Justice*. Cambridge, Mass.: Harvard University Press, 1971.
[21] SHAPLEY, L.: "A Value for *n*-Person Games," in *Contributions to the Theory of Games, Vol. II*, ed. by H. W. Kuhn and A. W. Tucker, Princeton, N.J.: Princeton University Press, 1953, 307–317.
[22] ———: "Values of Large Games VII: A General Exchange Economy with Money," Rand RM-4248-PR, 1964.
[23] ———: "Utility Comparison and the Theory of Games," in *La Decision*, ed. by G. Th. Guilbaud. Paris: Editions du CNRS, 1969.
[24] SHAPLEY, L., AND M. SHUBIK: "Pure Competition, Coalitional Power, and Fair Division," *International Economic Review*, 3 (1969), 337–362.
[25] SHESHINSKI, E.: "An Example of Income Tax Schedules which are Optimal for the Maximin Criterion," forthcoming in *International Economic Review*.
[26] SHITOVITZ, B.: "Oligopoly in Markets with a Continuum of Traders," *Econometrica*, 41 (1973), 467–501.
[27] SMALE, S.: "Global Analysis and Economics IIA: Extension of a Theorem of Debreu," *Journal of Mathematical Economics*, 1 (1974), 1–14.
[28] VON NEUMANN, J., AND O. MORGENSTERN: *Theory of Games and Economic Behavior*. Princeton, N.J.: Princeton University Press, 1944.
[29] WHITNEY, H.: "Functions Differentiable on Boundaries of Regions," *Annals of Mathematics*, 35 (1934), 482–485.