Econometrica, Vol. 54, No. 6 (November, 1986), 1271-1318

VALUES OF MARKETS WITH SATIATION OR FIXED PRICES

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To Gerard Debreu on his sixty fifth birthday, with admiration and affection.

In markets with satiation, competitive equilibria may fail to exist, because no matter what the prices are, the satiation points of some traders may be in the interiors of their budget sets. Thus some traders will be using less than the maximum budget available to them, creating a total budget excess. This suggests a revision of the equilibrium concept that allows the budget excess to be divided among all the traders, as dividends. Each trader's budget is then the sum of his dividend and the market value of his endowment. A given system of dividends and prices defines a dividend equilibrium if it generates equal supply and demand.

This in itself is not satisfactory because it is too broad: Every Pareto optimal allocation is sustained by some system of dividends and prices. However, the Shapley value yields much more specific information. We prove that, when there are many individually insignificant agents, every Shapley value allocation is generated by a system of dividends and prices in which all dividends are nonnegative and depend only on the net trade sets of the agents, not on their utilities. Moreover, the dependence is monotonic; the larger the net trade set, the higher the dividend.

The same result holds for markets with fixed prices, which can be analyzed formally as a special case of markets with satiation.

On a more technical level, our analysis has some unusual features. We use a finite-type asymptotic model, rather than a nonatomic continuum. Surprisingly, the results are qualitatively different. (The continuum is too rough a tool for our problem, and leads to inconclusive results.) Also, small coalitions play a critical role in our analysis. (We are led to equations in which the first-order terms cancel; the second-order terms, which take events of small probability into account, become decisive.)

KEYWORDS: Coupons equilibrium, exchange economy, fixed prices, game theory, satiation, Shapley value, unemployment.

1. INTRODUCTION

PURE EXCHANGE ECONOMIES, or *markets*, in which the preferences satisfy conditions of monotonicity and nonsatiation have been studied thoroughly in the past. In this paper we investigate the opposite situation: the utility functions need not be monotonic, and do have absolute maxima. The resulting theory has significant new qualitative features.

This study is not motivated by an abstract desire to remove as many assumptions as possible. It originated in the analysis of *fixed price* economies, which have been used extensively in the past decade[2] to model market failures such as unemployment. In such economies, all trade is restricted to take place at

[1] The work of R. J. Aumann was supported by CORE at Université Catholique de Louvain, by the Institute for Advanced Studies at the Hebrew University of Jerusalem, and by the Institute for Mathematical Studies in the Social Sciences (Economics) at Stanford University under a grant from the U.S. National Science Foundation. This work is part of the Projet d'Action Concertée on "Applications of Economic Decision Theory" sponsored by the Belgian Government under Contract No. 80/85-12. We are grateful to Jean-François Mertens for carefully reading the manuscript and suggesting significant improvements.

[2] See, e.g., the survey by Drazen (1980).

exogenously fixed prices $\bar{p}$. In effect, this limits each trader t to his *fixed price hyperplane*, i.e. the set of all bundles x in his original consumption set for which $\bar{p} \cdot x = \bar{p} \cdot e(t)$, where $e(t)$ is t's endowment; under the usual assumptions, t's utility has an absolute maximum on this set, and is not monotonic there.[3]

In general, price rigidities prevent a market from "clearing" (i.e., supply from matching demand); various quantity constraints or rationing schemes have been proposed to bring the situation back into equilibrium. In the more traditional markets, without fixed prices, there is a close relationship between competitive equilibria and game theoretic concepts such as the core[4] and the Shapley value;[5] thus one may expect game theory also to be helpful in suggesting equilibria for fixed price economies. It turns out that the core is not well suited to this purpose (see Section 11). But we shall find that the Shapley value allocations in fixed price economies correspond to a natural extension of competitive equilibria, closely related to the concept of coupons equilibrium defined by Drèze and Müller (1980).

To describe our results, let us return to the more general context of markets with satiation. The reason that competitive equilibria may fail to exist in such markets is that no matter what the prices[6] are, the satiation points of some traders may be in the interiors of their budget sets.[7] Thus some traders will be using less than the maximum budget available to them, creating a total budget excess. This suggests a revision of the equilibrium concept that allows the budget excess to be divided among all the traders, say as *dividends*: Each trader's budget is then the sum of his dividend and the market value of his endowment at the market prices. A given system of dividends and prices is in equilibrium if it generates equal supply and demand.

This in itself is not satisfactory because it is too broad: Drèze and Müller showed that the fundamental proposition of welfare economics continues to apply here, i.e., that *every* Pareto optimal allocation is generated by some system of dividends and prices. However, the Shapley value yields much more specific information. Our main result says that when there are many individually insignificant agents, every Shapley value allocation is generated by a system of dividends and prices in which all dividends are nonnegative and depend only on the net trade sets[8] of the agents, not on their utilities. Thus the income allocated to each agent—over and above the market value of his endowment—depends only on his trading *opportunities*; on what he is *able* to offer, not on what he wants to offer. Moreover, the dependence is monotonic; the larger the net trade set, the higher the dividend.

[3] Indeed, monotonicity is meaningless in this context, since there is no natural partial order on the fixed price hyperplane.

[4] Cf., e.g., Hildenbrand (1982).

[5] Cf., e.g., Aumann (1975) or Hart (1977b).

[6] We are here discussing endogenous market prices q, which should not be confused with the exogenous prices $\bar{p}$ in fixed price economies. See Section 10.

[7] See Section 3 for an example.

[8] The net trade set of agent t is $C(t) - e(t)$, where $C(t)$ is his consumption set, and $e(t)$ his endowment.

Two brief illustrations may clarify this point. When a bond issue is oversubscribed, bonds are normally rationed to the subscribers in proportion to the amount requested. Under complete information, this procedure has no equilibrium; the subscribers will always request more than they really want, this will be taken into account by the other subscribers, and so on. But in the rationing scheme implied by the Shapley value, the maximum that a subscriber may buy is based not on what he requests, which is subject to manipulation, but on what he *could* buy; on his net worth, say.

The second illustration deals with unemployment in a fixed wage context. Various rationing schemes that involve cutting down on working hours have been proposed. In the scheme suggested by the Shapley value, the maximum work week for any particular worker would depend on how much time he has. Thus a youngster who must by law attend school, or a kidney patient undergoing time-consuming dialysis, would be assigned a quota smaller than the average, *even though he might be able to fill the average quota.*

Economic models have two basic components, the objective and the subjective. The first consists of the physical opportunities or *abilities* of the agents: resources, technologies, constraints on consumption, and so on. The second consists of the utilities or *preferences.* In a market, the objective component is completely described by the net trade sets of the agents. Outcomes of economic models usually depend on both components, often quite intricately.

Competitive equilibria "decouple" the two components. Each agent optimizes over an endogenous choice set, his budget set; in equilibrium, the choices mesh, they "clear" the market. The optimization, of course, is subjective; it depends on the agent's preferences. But the choice set itself does not; it depends only on his net trade set, i.e., on purely objective factors. Our result implies that the dividend equilibria to which the Shapley value leads also decouple in this way.

On a more technical level, our analysis has several unusual features. Though we are dealing with a large number of individually insignificant agents, we do not model it with a nonatomic continuum; rather, we use a finite-type asymptotic model of the Debreu and Scarf (1963) genre. Asymptotic and continuous results may differ in various ways,[9] but usually, the results are qualitatively similar. Here they are not. The continuum is too rough a tool; it obliterates the fine structure of the problem, and so leads to inconclusive results. The matter will be discussed further in Section 11.2.

Another unusual feature, related to the first, is the critical importance of small coalitions. The Shapley value of a player is the expectation of his "contribution to Society" when the players are ordered at random; the probability that he is second or third in the order is small, and is usually ignored. Here, we are led to equations in which the first-order terms cancel, and the second-order terms, which take events of small probability into account, become decisive. When there is an excess supply of labor, the length of the work week allocated to a given worker

[9] E.g. in ease and transparency of the formulation, in the generality of the results, in the methods of proof, and in the discussion of errors and rates of convergence. Compare Aumann and Shapley (1974, Section 34, 208–210).

depends on his expected contribution when he arrives on the scene; unless he is very early, this is negligible.

The plan of the paper is as follows. In Sections 2–5, we present the model and state our main result; it is proved in Sections 7–9. Section 10 contains the application to fixed prices, and Section 11 is devoted to a discussion of some alternative approaches. In Section 12 we state some additional results of a more quantitative nature, including properties of the dividends that go beyond mere monotonicity. These results enable the calculation of some numerical examples in Section 13. Finally, Section 14 discusses open problems. Appendices A and B establish two mathematical propositions that are needed for the proof of the main result. Appendix C contains the proofs of the results stated in Section 12.

Since the proof of the main theorem is rather complex, we offer three aids to its understanding. Section 6 contains a summary of the underlying economic ideas. Section 9.2 gives a brief outline of the mathematical ideas. Finally, the flow charts in Appendix D provide an overview of the relationships between the various lemmas and propositions constituting the proof.

2. MARKETS WITH SATIATION

A (finite) *market with satiation* is defined by:

(2.1) a finite set T (the *trader space*);

(2.2) a positive integer d (the number of *commodities*);

(2.3) for each trader t, a compact convex subset X_t of R^d, whose interior is nonempty and contains the origin 0 (t's *net trade set*); and

(2.4) for each trader t, a concave continuous function u_t on X_t (t's *utility function*).

Because X_t is compact, the continuous function u_t must attain its maximum; denote by B_t the set of all points in X_t at which the maximum is attained (the *satiation* or *bliss* set of trader t), and note that it is compact and convex. To avoid trivialities, assume $0 \notin B_t$, i.e., the initial bundle never satiates.

A few matters of terminology and notation: the inner product of two vectors q and x is denoted $q \cdot x$; "w.r.t." means "with respect to" and "w.l.o.g." means "without loss of generality"; R^k, R^k_+, and R^k_{++} denote, respectively, Euclidean k-space, its (closed) nonnegative orthant, and its (open) strictly positive orthant; int and bd denote "interior" and "boundary" respectively.

3. DIVIDEND EQUILIBRIA

An *allocation* in a market with satiation is a vector $x = (x_t)_{t \in T}$, where x_t is a feasible net trade for trader t (i.e., $x_t \in X_t$), and $\sum_{t \in T} x_t = 0$. A *price vector* is any member of R^d other than 0; since utilities need not be monotonic, one cannot confine oneself to nonnegative prices. The classical notion of *competitive equilibrium* is defined for markets with satiation just as it is for ordinary markets: it consists of a price vector q and an allocation $\boldsymbol{x}$ such that for all t, x_t maximizes u_t over the *budget set* $\{x \in X_t\colon q \cdot x \leq 0\}$.

Competitive equilibria do not in general exist in markets with satiation. Consider, for example,[10] a market with two agents, 1 and 2, and one commodity; suppose that the satiation points are on opposite sides of the origin, e.g., $u_1(x) = -(x-1)^2$, $u_2(x) = -(x+2)^2$. W.l.o.g. the price vector is ± 1; in either case one agent receives 0 and the other his satiation point, and these do not sum to 0.

The example is not due to any pathologies associated with the low dimension.[11] Consider a market with two commodities and three agents having the same net trade set, and with indifference maps as illustrated in Figure 1. No matter what the price vector is, the satiation point of at least one trader must be in the interior of the budget set,[12] so that his utility will be maximized there over the budget

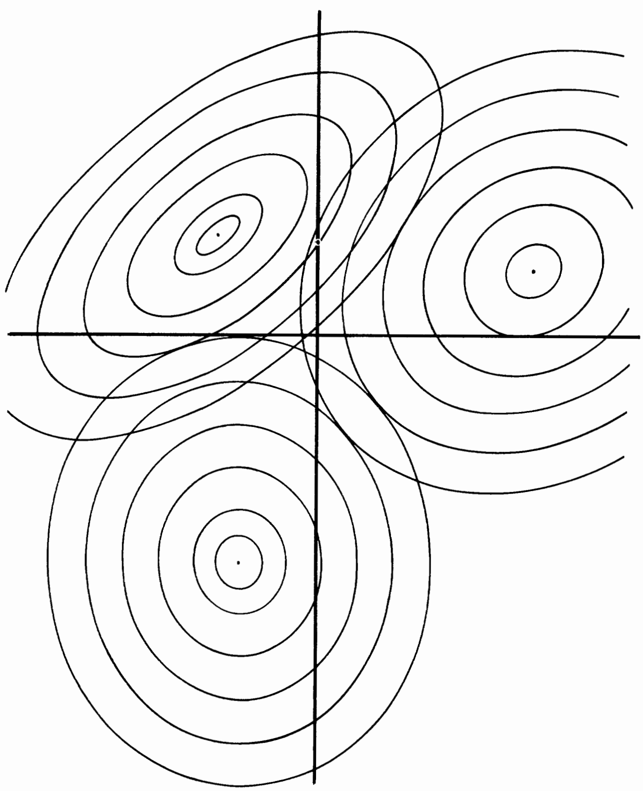

FIGURE 1

[10] This example appears in Drèze and Müller (1980).
[11] Such as the disconnectedness of the set of price vectors.
[12] This holds as long as 0 is in the interior of the convex hull of the three satiation points.

set; whereas the utilities of the remaining agents will be maximized on the budget line. The resulting three points cannot sum to 0.

What is happening is that at least one trader creates a surplus by refusing to make use of his entire budget; but the definition of competitive equilibrium does not permit the other agents to use this surplus, so an imbalance results.[13]

To overcome this problem, define a *dividend* to be a vector $\boldsymbol{c}=(c_t)_{t\in T}$ whose components c_t are real numbers. A *dividend equilibrium* consists of a price vector q, a dividend $\boldsymbol{c}$, and an allocation $\boldsymbol{x}$, such that for all t, x_t maximizes u_t over the *dividend budget set*

$$\{x \in X_t : q \cdot x \leq c_t\}.$$

A dividend may be thought of as a cash allowance added to the budget of each trader; its function is to distribute among the unsatiated agents the surplus created by the failure of the satiated agents to use their entire budget.

A dividend $\boldsymbol{c}$ is *nonnegative* if all the c_t are nonnegative; *monotonic in the net trade sets*, if $X_t \supset X_s$ implies $c_t \geq c_s$. Occasionally, the word dividend will also be used for a component c_t of $\boldsymbol{c}$.

4. VALUE ALLOCATIONS

A *comparison vector* on the trader space T is a vector $\boldsymbol{\lambda}=(\lambda_t)_{t\in T}$, where each λ_t is a positive real number. For each comparison vector $\boldsymbol{\lambda}$ and coalition[14] S, define

$$v_{\boldsymbol{\lambda}}(S) := \operatorname{Max}\left\{\sum_{t\in S} \lambda_t u_t(x_t) : \sum_{t\in S} x_t = 0 \text{ and } x_t \in X_t \text{ for all } t \text{ in } S\right\}. \tag{4.1}$$

In words, $v_{\boldsymbol{\lambda}}(S)$ is the maximum total utility that S can get for itself by redistributing its endowment among its members, when the utilities u_t are weighted by the λ_t. A *value allocation* (in a given market with satiation) is an allocation $\boldsymbol{x}$ for which there exists a comparison vector $\boldsymbol{\lambda}$ such that for all traders t,

$$(\phi v_{\boldsymbol{\lambda}})(t) = \lambda_t u_t(x_t), \tag{4.2}$$

where $\phi v_{\boldsymbol{\lambda}}$ denotes the Shapley value of the game $v_{\boldsymbol{\lambda}}$; we say that $\boldsymbol{\lambda}$ and $\boldsymbol{x}$ are *associated* with each other. (Recall that

$$(\phi v_{\boldsymbol{\lambda}})(t) = E(v_{\boldsymbol{\lambda}}(S\cup t) - v_{\boldsymbol{\lambda}}(S)), \tag{4.3}$$

where E denotes "expectation", and S is the set of traders preceding t in a random order on all traders.[15])

At a value allocation, the weighted utility each player receives is equal to his Shapley value in the game $v_{\boldsymbol{\lambda}}$. In other words, if transfers of utility are permitted

[13] If free disposal were permitted, the example would go away; but this is not a reasonable assumption in the absence of monotonicity. Moreover, "disposal" is not really possible in the fixed-price application (Section 10).

[14] A *coalition* is a subset of T.

[15] Shapley's definition (1953) of ϕ is in terms of a set of axioms, from which (4.3) is derived. See Roth (1977) for an interesting discussion of Shapley's axioms.

with exchange rates λ_t, then the Shapley value of the resulting game is achieved, *without transfers of utility*, at the allocation x. Compare Shapley (1969) and Aumann (1975).

5. THE MAIN THEOREM

Let M^1 be a market with satiation; denote the traders by $1, \ldots, k$, the utility functions by $u_1, \ldots, u_k$, the net trade sets by $X_1, \ldots, X_k$, and the satiation sets by $B_1, \ldots, B_k$. The *n-fold replication* M^n of M^1 is the market with satiation in which there are nk traders, n of each of the k "types" in M^1; i.e. the trader space T^n in M^n is the union of k disjoint sets $T_1^n, \ldots, T_k^n$ (the *types*), such that $u_t = u_i$ and $X_t = X_i$ whenever $t \in T_i^n$. We assume as follows:

ELBOW ROOM ASSUMPTION: *For each* $J \subset \{1, \ldots, k\}$,

$$0 \notin \text{bd}\left[\sum_{i \in J} B_i + \sum_{i \notin J} X_i\right]. \tag{5.1}$$

In words, for any coalition J in M^1, if it is at all possible simultaneously to satiate all agents in J, then this can also be done "with room to spare," i.e. when all other agents are restricted to the interiors of their net trade sets.[16]

This assumption holds generically in a certain very natural sense. The left side of (5.1) represents the total endowment of the market, whereas its right side is the boundary of a convex set in R^d, hence at most $(d-1)$-dimensional. Since there are only finitely many J, the assumption holds for all but a $(d-1)$-dimensional set of total endowments. A formal genericity statement can be made in terms of translates of the X_i; translating the net trade set is equivalent to varying the endowment. For details, see Section 11.3.

Note also that since the J's are sets of types, the number of conditions (5.1) is fixed at 2^k, and does not vary with n.

Call an allocation $\hat{x}$ in M^n *equal treatment* if it assigns the same net trade to traders of the same type. Such an allocation $\hat{x}$ defines a k-tuple x of net trades, one for each type; x is an allocation in M^1, which is said to *correspond* to $\hat{x}$.

MAIN THEOREM: *For each n, let* x^n *be an allocation in the unreplicated market* M^1*, corresponding to an equal treatment value allocation in the n-fold replication* M^n*. Let* x^∞ *be a limit point*[17] *of* $\{x^n\}$*. Then there is a nonnegative dividend c that is monotonic in the net trade sets, and a price vector q, such that* (q, c, x^∞) *is a dividend equilibrium in* M^1.

From the monotonicity it follows that the dividends are determined by the net trade sets, i.e., $X_i = X_j$ implies $c_i = c_j$. Thus, the theorem says that all value allocations in large markets with satiation approximate dividend equilibrium

[16] See Section 10 for an interpretation in the fixed price case.

[17] Limit of a subsequence.

allocations, where the dividends depend only on the net trade sets, and monotonically so.[18] In particular, call a dividend equilibrium (q, c, x) *uniform*[19] if all the c_t are the same. Then we have the following:

COROLLARY 5.3: *Under the conditions of the theorem, assume that all traders have the same net trade set. Then (q, c, x^∞) is a uniform dividend equilibrium.*

The following existence result, for which (5.1) is not needed, gives substance to the main theorem:

PROPOSITION 5.4: *For every n, there is an equal treatment value allocation in the n-fold replication M^n.*

Further results will be stated in Section 12.

6. AN INFORMAL DEMONSTRATION OF THE MAIN THEOREM

Let $\boldsymbol{\lambda}^n = (\lambda_1^n, \ldots, \lambda_k^n)$ be a comparison vector associated[20] with x^n. Normalize $\boldsymbol{\lambda}^n$ so that the sum of its coordinates is 1. For simplicity,[21] assume for each i that x_i^n and λ_i^n converge as $n \to \infty$; that each of the x_i^n as well as their limit x_i^∞ is interior to the net trade set X_i; and that the utility function u_i is strictly convex and continuously differentiable on X_i.

In the classical context of monotonicity and nonsatiation, λ_i tends to a positive limit for all types i (Champsaur, 1975). But as we shall see below, in our context some of the λ_i^n may tend to 0. This is the crucial difference between the two contexts, and it is this that leads to the positive dividends.

The situation may in fact be quite complicated; there may be differences in order of magnitude even between those λ_i^n that tend to 0. In this section, though, we will assume for simplicity that those λ_i^n that do tend to 0 all have the same order of magnitude.[22] If there are such types, they are called *lightweight*, the others heavyweight.[23]

By definition, the allocation x^n is optimal[24] for the all-trader set T^n. This implies that all the (weighted) utility gradients $\lambda_i^n u_i'(x_i^n)$ are equal; otherwise

[18] Of course, the dividends are endogenously determined by all the data of the market, including all the utilities. Yet, in any given market, traders with the same net trade set have the same dividend.

[19] This concept is due to Drèze and Müller (1980), who showed, using fixed point methods, that uniform dividend equilibria always exist. They worked in a fixed price context, using "uniform coupons equilibrium" for what we call "uniform dividend equilibrium."

[20] More precisely, λ^n is a k-dimensional vector corresponding to an nk-dimensional equal treatment comparison vector $\hat{\boldsymbol{\lambda}}^n$ associated with the equal treatment allocation $\hat{x}^n$ corresponding to x^n. There must always be such a $\hat{\boldsymbol{\lambda}}^n$; cf. footnote 33.

[21] Use of the phrase "for simplicity" means that the restriction involved is for purposes of the informal demonstration in this section only, and is not needed for the rigorous treatment in the ensuing sections.

[22] More precisely, $\lim_{n\to\infty} \lambda_i^n/\lambda_j^n$ exists and is positive whenever both λ_i^n and λ_j^n tend to 0.

[23] The formal definition of "lightweight" in Section 9 is slightly different, but yields the same result.

[24] An allocation is called *optimal* for a coalition S if it achieves the maximum total utility for S when the utilities u_i are multiplied by the weights λ_i^n.

transfers could lead to gains in total utility. Denote the common value of all these gradients by q^n; thus

(6.1) $$\lambda_i^n u_i'(x_i^n) = \lambda_j^n u_j'(x_j^n) = q^n$$

for all i and j. If we let $n \to \infty$ and set $\lambda_i^\infty := \lim_{n\to\infty} \lambda_i^n$, $q^\infty := \lim_{n\to\infty} q^n$, we get

(6.2) $$\lambda_i^\infty u_i'(x_i^\infty) = \lambda_j^\infty u_j'(x_j^\infty) = q^\infty.$$

Our demonstration is based on (4.3), which says that the Shapley value of a given trader t is his expected contribution to a randomly chosen coalition; more precisely, to the coalition S of traders before t in a randomly chosen order on all traders.

If S is large, it is very likely to be a good sample of all the traders, i.e. to have approximately equal numbers of traders of each type. The allocation that is optimal for S is then approximately the same as the allocation $\mathbf{x}^n$ that is optimal for the all-trader coalition T^n.

Adding t to S will not change this optimal allocation by much; each trader will be allocated approximately the same net trade as before. In particular, if t is of type i, he will be allocated approximately x_i^n. Since all the net trades must sum to 0, the net trade of t must somehow be divided among all the traders, with each trader subtracting a small part of x_i^n from his net trade. Since all the utility gradients are q^n, the utility of each trader is decreased by q^n times that small part. Altogether, this causes a change in total utility of $-q^n \cdot x_i^n$. To this must be added the utility $\lambda_i^n u_i(x_i^n)$ that t himself now enjoys. Thus t's contribution to S (the worth of $S \cup t$ less the worth of S) is given approximately by

(6.3) $$\Delta := \lambda_i^n u_i(x_i^n) - q^n \cdot x_i^n.$$

All this is valid only when S is reasonably large. Otherwise—e.g. when S has no more than a fixed finite number of traders (such as one or a hundred or a thousand)—the reasoning breaks down. Denote by P^n the probability that S is "small," so that t's contribution is not measured by Δ. This event is perhaps not very well defined, but in this section we are making no attempt at precision. It is in any case clear that $P^n \to 0$ as $n \to \infty$, and that the order of P^n is at least $1/n$ (obtained already when S has only one trader).

Letting δ denote the conditional expectation of t's contribution given that S is small,[25] we conclude that

(6.4) $$(\phi v_{\boldsymbol{\lambda}^n})(t) \approx (1 - P^n)\Delta + P^n \delta.$$

Note that we are ignoring the probability that S is large but nevertheless not a good sample of the entire population; this probability is very small indeed, much smaller than P^n, and in fact has no influence on the value. Note that δ is uniformly bounded; this follows, e.g., from the continuous differentiability of the utilities u_i on the compact sets X_i.

The definition (4.2) of the value stipulates that

$$(\phi v_{\boldsymbol{\lambda}^n})(t) = \lambda_i^n u_i(x_i^n).$$

[25] Both Δ and δ depend on n; we suppress the corresponding superscript to keep our notation as uncluttered as possible.

Together with (6.4) and the definition of Δ, this yields

(6.5) $\quad q^n \cdot x_i^n \approx \varepsilon^n(\delta - \lambda_i^n u_i(x_i^n)),$

where $\varepsilon^n := P^n/(1-P^n) \to 0$.

The case in which none of the λ_i^n tend to 0 is the simplest, and we dispose of it first. With strict convexity, the elbow room assumption (5.1) rules out the possibility of simultaneously satiating all traders, a situation that is in any case rather uninteresting. Hence $u_i'(x_i^\infty) \neq 0$ for at least one i; and since $\lambda_i^\infty > 0$ for all i, it follows from (6.2) that $\mathbf{x}^\infty$ satiates no one, that $q^\infty \neq 0$, and that for all i, the gradient of u_i at x_i^∞ is in the direction of q^∞. Letting $n \to \infty$ in (6.5), we obtain $q^\infty \cdot x_i^\infty = 0$; hence the net trade x_i^∞ maximizes u_i over the budget set $\{x \in X_i: q \cdot x \leqslant 0\}$. Thus $\mathbf{x}^\infty$ is an ordinary competitive allocation, and hence trivially part of a dividend equilibrium that satisfies the appropriate conditions.

Up to now the analysis has been as in the classical context of monotonicity and nonsatiation, where limits of value allocations are always competitive (Champsaur, 1975). But as we saw in Section 3, in our context there are situations in which competitive allocations need not even exist. By the above analysis, then, there must be some lightweights; of course there must also be heavyweights, since the sum of the k weights λ_i^n is normalized to be 1. This is the case of central interest in this paper, to which we now turn.

If we take i lightweight in (6.2), we find $q^\infty = 0$; hence $u_j'(x_j^\infty) = 0$ for heavyweight j, and hence $\mathbf{x}^\infty$ satiates all heavyweights. Suppose now that t is a lightweight trader of type i. Since $q^\infty = 0$, letting $n \to \infty$ in (6.5) as before would simply yield $0 = 0$. For something more informative, we must look at the fine structure, at the second order effects. This is done by dividing (6.5) by $\|q^n\|$, which, like q^n, tends to 0 as $n \to \infty$.

Assume for simplicity that $q^n/\|q^n\|$ actually tends to a limit q; note that $\|q\| = 1$. We shall see below that t's expected contribution δ to small coalitions is of larger order of magnitude than the term $\lambda_i^n u_i(x_i^n)$ on the right side of (6.5). Hence dividing (6.5) by $\|q^n\|$ and letting $n \to \infty$, we find

(6.6) $\quad q \cdot x_i = \lim_{n\to\infty}(\varepsilon^n/\|q^n\|)\delta =: c_i;$

the limit exists on the right because it exists on the left.

By definition, q^n is proportional to the unweighted utility gradient $u_i'(x_i^n)$; therefore its direction $q^n/\|q^n\|$ is equal to the direction of $u_i'(x_i^n)$. Letting $n \to \infty$, we deduce that q is the direction of $u_i'(x_i^\infty)$ whenever x_i^∞ does not satiate t. In that case, therefore, (6.6) says that x_i^∞ maximizes t's utility over the dividend budget set defined by prices q and dividend c_i. Of course, when x_i^∞ *does* satiate t, his utility is maximized globally, and a fortiori over his dividend budget set.

For lightweight i, it remains therefore only to show that c_i depends monotonically on the net trade set X_i, and in particular is independent of the utility u_i. To see this, let us examine the contribution of t when joining a fixed small coalition S. This may be divided into three components:

(i) t's own utility after joining;

(ii) the change in the total utility of the lightweight traders in S due to t's joining; and

(iii) the change in the total utility of the heavyweight traders in S due to t's joining.

In the first two of these three components, the utilities have weights tending to 0; in the third they do not. Thus for large n, the contribution of t to himself and to other lightweights is negligible; the importance of his contribution to S comes from what he can do to improve the lot of the heavyweights in S. Therefore he should distribute his resources so as to maximize the heavyweights' gain in utility, paying no attention to his own. His ability to do this is limited only by his net trade set, and has nothing to do with his utility. Moreover, the larger his net trade set, the more he can do, and this yields the monotonicity.

The reasoning works only when S is a fixed coalition of relatively small size. In that case the heavyweights in S cannot, in general, all be simultaneously satiated; since S is small, they will then be a significant distance from satiation.[26] When t joins, he brings in resources (not utility!) that could be used significantly to improve the lot of at least one heavyweight trader, perhaps even to bring that one all the way to satiation; that would be a good deal more worthwhile than using the resources for himself or for other lightweight traders, whose utilities have weights tending to 0. A more even handed distribution of the resources among the heavyweights would yield still more, but giving it all to one gives us a lower bound on t's contribution, and indicates that it is of larger order than λ_i^n.

If, however, S is large, it is probably a good sample of all agents, and then all types j will be close to x_j^∞; in particular, the heavyweights will already be satiated, even before t joins. Thus by joining, t cannot improve the heavyweights by much. The upshot is that no matter how t uses his resources—whether for himself, for his lightweight colleagues, or for the heavyweights—the increment in total utility will be the same; in the first two cases the utilities are weighted by small weights of the order λ_i^n, and in the last, the increase in the utility u_j is small.

We come finally to the case in which the type i of the additional trader t is heavyweight. In calculating the contribution δ to small coalitions, the significant components are now (i) and (iii), rather than just (iii); component (ii) remains negligible. Note, though, that on the right side of (6.5) we now have not δ, but something close to $\delta-\lambda_i^\infty u_i(x_i^\infty)$. Since $\lambda_i^\infty u_i(x_i^\infty)$ is the absolute maximum that t can get, component (i) of δ is certainly at least cancelled out, and very likely more than cancelled out. Thus what is left is at most component (iii). The rest of the argument is as before, with (6.6) modified to read:[27]

$$(6.7) \qquad q\cdot x_i^\infty \leq \lim_{n\to\infty} (\varepsilon^n/\|q^n\|)[\text{component (iii) of } \delta] =: c_i.$$

[26] In principle, the equality of marginal utilities expressed by (6.1) should still hold when x_i^n and x_j^n are replaced by y_i^n and y_j^n, where y^n is optimal for an arbitrary (fixed) S. In fact, when S is small and n large, y_i^n is very likely to be on the boundary of X_i, so that we have a corner situation, in which marginal utilities need not be equal. We therefore cannot deduce that y_j^n is close to satiation for heavyweight j, and indeed it will usually not be.

[27] For technical reasons, the definition of the dividend c_i for heavyweight types i that we use in Section 9 is a little different from (6.7). Since x_i^∞ is generally in the interior of the dividend budget set when i is heavyweight, there is sometimes a little leeway in defining the dividend. Of course, if $X_i=X_j$ for some lightweight j, then we must have $c_i=c_j$, so the leeway disappears.

Since i is heavyweight, x_i^∞ satiates; hence we must only show that it satisfies the budget inequality, which (6.7) indeed shows.

We end with a word of caution. The argument in this section is meant only to be indicative, and cannot easily be made rigorous. The difficulties we will encounter in the rigorous treatment below are intrinsic; they are not due to the generality of the treatment. Assuming differentiability, strict concavity, and so on enabled a simplified presentation in this section, but it would not help appreciably in the treatment below.

7. THE EXISTENCE PROOF

In this section we prove Proposition 5.4. Define a *generalized comparison vector* on the trader space T of a market with satiation to be a vector $\boldsymbol{\lambda} = (\lambda_t)_{t\in T}$ of nonnegative real numbers not all of which vanish. A *generalized value allocation* is defined like a value allocation, except that the comparison vector is replaced by a generalized comparison vector.

In 1969, Shapley proved that every game has a nontransferable utility value, if generalized comparison vectors are admitted. In this proof, the only properties of the Shapley value that are used are continuity in the comparison weights, Pareto optimality, and individual rationality (see the theorem on p. 261 of Shapley (1969)); any function of the comparison vector enjoying these properties will be called a *pseudo-value.*

Fix the replication index n. To each generalized comparison vector $\boldsymbol{\lambda}$ on $\{1, \dots, k\}$, there corresponds a generalized comparison vector $\hat{\boldsymbol{\lambda}}$ on T^n, which assigns weight $\hat{\lambda}_t = \lambda_i$ to each of the n traders t of type i. Define $v_{\hat{\lambda}}$ on the subsets of T^n as in (4.1). By the symmetry of the Shapley value ϕ, traders of the same type in T^n are assigned equal values by $\phi v_{\hat{\lambda}}$; thus $\phi v_{\hat{\lambda}}$ defines a k-dimensional vector $\psi_{\boldsymbol{\lambda}}$, whose ith coordinate is $(\phi v_{\hat{\lambda}})(t)$ for any t of type i in T^n. Then the function $\boldsymbol{\lambda} \to \psi_{\boldsymbol{\lambda}}$ is a pseudo-value, and so by the theorem of Shapley quoted above, there is an allocation $\boldsymbol{x}$ in M^1 and a generalized comparison vector $\boldsymbol{\lambda}$ such that $(\psi_{\boldsymbol{\lambda}})_i = \lambda_i u_i(x_i)$ for all $i = 1, \dots, k$. Now define an allocation $\hat{\boldsymbol{x}}$ in M^n by $\hat{x}_t = x_i$ whenever t in T^n is of type i; then

$$(7.1) \qquad (\phi v_{\boldsymbol{\lambda}})(t) = (\psi_{\boldsymbol{\lambda}})_i = \lambda_i u_i(x_i) = \hat{\lambda}_t u_t(\hat{x}_t),$$

and so $\hat{\boldsymbol{x}}$ is an equal treatment generalized value allocation in M^n.

It remains only to show that $\boldsymbol{\lambda}$ is in fact a comparison vector, i.e., that $\lambda_i > 0$ for all i. W.l.o.g. $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$; hence $\lambda_1 > 0$. Suppose $\lambda_k = 0$. Let $b \in B_1$; since 0 does not satiate, $u_1(b) > u_1(0)$. Since $0 \in \operatorname{int} X_k$, we have $-\theta b \in X_k$ for $\theta > 0$ sufficiently small. Hence if t is a type k trader and S a coalition in M^n consisting of a single type 1 trader, then

$$(7.2) \qquad v_{\boldsymbol{\lambda}}(S \cup t) - v_{\boldsymbol{\lambda}}(S) \geqslant \lambda_1 u_1(\theta b) + \lambda_k u_k(-\theta b) - \lambda_1 u_1(0)$$
$$= \lambda_1(u_1(\theta b) - u_1(0)) \geqslant \lambda_1 \theta(u_1(b) - u_1(0)) > 0,$$

by the concavity of u_1. On the other hand, for any $S \subset T^n$ not containing t, the superadditivity of v_λ yields

(7.3) $$v_\lambda(S \cup t) - v_\lambda(S) \geq v_\lambda(t) = \lambda_k u_k(0) = 0.$$

Combining (7.2) and (7.3) with (4.3), and noting that S is as in (7.2) with positive probability, we deduce $(\phi v_{\hat{\lambda}})(t) > 0$. Since $\hat{\lambda}_t = \lambda_k$, (7.1) then yields $\lambda_k > 0$ after all. Hence $\lambda_i > 0$ for all i. *Q.E.D.*

8. SOME TOOLS

If f is a concave function on a convex set X in a Euclidean space R^d, define the *superdifferential*[28] $\partial f(x)$ of f at a point x in X by,[29]

(8.1) $$\partial f(x) = \{p \in R^d : f(z) - f(x) \leq p \cdot (z - x) \text{ for all } z \in X\}.$$

Note that $\partial f(x)$ is always nonempty, closed, and convex, and that when x is interior and f is differentiable, it consists of the gradient only.

As in the foregoing, for $i = 1, \ldots, k$, let u_i be a concave continuous function on a compact convex subset[30] X_i of R^d. Throughout the rest of the paper, write $\sum$ for $\sum_{i=1}^k$ or $\sum_{j=1}^k$. Define a closed convex cone U in R_+^k by

(8.2) $$U = \{y \in R_+^k : 0 \in \sum y_i X_i\};$$

note that U coincides with R_+^k whenever each X_i contains 0. Define a real-valued function w on U by

(8.3) $$w(y) = \operatorname{Max}\{\sum y_i u_i(x_i) : x_i \in X_i \text{ for all } i, \text{ and } \sum y_i x_i = 0\};$$

the maximum is attained because of the compactness of the X_i. Note that

(8.4) w is 1-homogeneous,[31] concave, and superadditive (i.e. $w(y + y') \geq w(y) + w(y')$).

The function w plays a vital role in the sequel. The following proposition, related to the core equivalence theorem, characterizes the superdifferentials of w in terms of the superdifferentials of u_i (which play the role of prices). It is proved in Appendix A.

PROPOSITION 8.5: *Let y in R_{++}^k be such that $0 \in \operatorname{Int} \sum y_i X_i$, and let $w(y)$ be attained at $(x_1, \ldots, x_k)$. Then $p \in \partial w(y)$ if and only if there is an element q of $\bigcap_{i=1}^k \partial u_i(x_i)$ such that for each i,*

(8.6) $$p_i = u_i(x_i) - q \cdot x_i.$$

[28] Cf. Rockafellar (1970, p. 215), where this notation is used for *sub*differentials. The superdifferential $\partial f(x)$ is denoted $P(x; f)$ by Aumann and Shapley (1974, p. 216 ff.).

[29] The symbol p in this and the next section bears no relation whatever to the fixed price vector $\bar{p}$ in Sections 1 and 10.

[30] In this section, the X_i need not have interiors, nor contain 0, nor satisfy $0 \notin B_i$, nor satisfy (5.1).

[31] Homogeneous of degree 1.

The second tool presented in this section is of an explicitly game-theoretic nature. As in the foregoing, let T^n be a set with nk members (the *players*), divided into disjoint subsets $T_1^n, \ldots, T_k^n$ (the *types*) with n members each. For each $S \subset T^n$, let $\eta(S) \in R_+^k$ denote the *profile* of S, i.e.,

$$(8.7) \qquad \eta_i(S) = |S \cap T_i^n|,$$

where $|\cdot|$ denotes cardinality; in words, $\eta_i(S)$ is the number of type i players in S. Let ϕ denote the Shapley value.

PROPOSITION 8.8: *Let $w_1, w_2, \ldots$ be 1-homogeneous continuous functions on R_+^k that are uniformly bounded on $[0, 1]^k$. Assume that on a convex neighborhood of $(1, \ldots, 1)$, the w_n are concave and pointwise approach a concave function*[32] w_∞. *Define a game v_n on the subsets of T^n by $v_n = w_n \circ \eta$, and define p^n in R^k by*

$$(8.9) \qquad p_i^n = (\phi v_n)(T_i^n)/n \qquad (i = 1, \ldots, k).$$

Then p^n is bounded, and every limit point of $\{p^n\}$ as $n \to \infty$ is in $\partial w_\infty(1, \ldots, 1)$.

Propositions 8.5 and 8.8 constitute the technical foundation on which the proof of the main theorem rests. Proposition 8.8, proved in Appendix B, tells us that as the number of traders increases, the value approaches the superdifferential of the appropriate w-function; and Proposition 8.5 tells us that this superdifferential can be interpreted in terms of prices in the original market.

9. PROOF OF THE MAIN THEOREM

9.1. *Preliminaries*

We return now to the situation described in Sections 2 through 5. Assume w.l.o.g. that $u_i(0) = 0$ for all i. Let $\hat{\boldsymbol{x}}^n$ be an equal treatment value allocation in the n-fold replication M^n, let $\hat{\boldsymbol{\lambda}}^n$ be an equal treatment comparison vector[33] associated with $\hat{\boldsymbol{x}}^n$, let $\boldsymbol{x}^n$ be the allocation in M^1 corresponding to $\hat{\boldsymbol{x}}^n$, and let $\boldsymbol{\lambda}^n$ be the comparison vector on $T^1 = \{1, \ldots, k\}$ corresponding to $\hat{\boldsymbol{\lambda}}^n$. Assume w.l.o.g. that $\sum \lambda_i^n = 1$. Then there is a sequence of positive integers, called the *convergence indicator*,[34] such that if $n \to \infty$ over members of this sequence only, then $\boldsymbol{x}^n \to \boldsymbol{x}^\infty$, and also $\{\boldsymbol{\lambda}^n\}$ converges. From now on, all finite values of n will be in the convergence indicator; in particular, $n \to \infty$ means that n tends to infinity over members of the convergence indicator only.

By possibly taking a smaller convergence indicator, we may assume w.l.o.g. that λ_i^n/λ_j^n approaches a (finite or infinite) limit as $n \to \infty$, for all i and j. We will assume the types arranged in the limiting order of size of the weights λ_i^n; that

[32] The w_n and w_∞ are not necessarily related to the w of (8.3).

[33] One that assigns equal weight to traders of the same type. That there is one such associated with $\hat{\boldsymbol{x}}^n$ follows from the fact that $\hat{\boldsymbol{x}}^n$ is itself equal treatment. Indeed, if $\hat{\boldsymbol{\mu}}^n$ is any comparison vector associated with $\hat{\boldsymbol{x}}^n$, then we may define an equal treamtent $\hat{\boldsymbol{\lambda}}^n$ associated with $\hat{\boldsymbol{x}}^n$ by taking $\hat{\boldsymbol{\lambda}}_t^n$, for each trader t, to be the average of the weights $\hat{\mu}_s^n$ over traders s of t's type.

[34] We are grateful to Lloyd Shapley for suggesting this name.

is, $\lim \lambda_{i+1}^n/\lambda_i^n \leq 1$ for all $i<k$. W.l.o.g. x^∞ does not satiate all types;[35] let ℓ be the "heaviest" type not satiated by x^∞ (i.e., $\ell=\min\{i: x_i^\infty \notin B_i\}$). Call type i *lightweight* ($i \in L$) if $\lambda_i^n = 0(\lambda_\ell^n)$, *heavyweight* ($i\in H$) otherwise.

Define w^n on R_+^k as in (8.3), except that u_i is replaced by $\lambda_i^n u_i$. Define an open convex cone U_L in R_{++}^k by

$$U_L=\left\{y\in R_{++}^k\colon 0\in \operatorname{Int}\left(\sum_{i\in H} y_iB_i+\sum_{i\in L} y_iX_i\right)\right\}.$$

For each lightweight trader i, define

$$\xi_i=\lim_{n\to\infty}\lambda_i^n/\lambda_\ell^n.$$

For $y\in U_L$, define $w_L(y)$ as in (8.3), except that for heavyweight i, X_i is replaced by B_i and u_i by 0; and for lightweight i, u_i is replaced by $\xi_i u_i$ (that the constraint set is nonempty follows from $y\in U_L$). For $n<\infty$, define w_H^n on R_+^k as in (8.3), except that u_i is replaced by $\lambda_i^n u_i$ for heavyweight i, and by 0 for lightweight i. Also for $n<\infty$, define w_L^n on R_+^k by

$$(9.1)\qquad w^n=w_H^n+\lambda_\ell^n w_L^n.$$

Recall that $\eta(S)$ denotes the profile of the coalition S (see (8.7)). On T^n, define games v^n, v_L^n, and v_H^n by

$$v^n=w^n\circ\eta,\quad v_L^n=w_L^n\circ\eta,\quad v_H^n=w_H^n\circ\eta.$$

From the concavity of the u_i it follows that v^n is the same as the $v_{\hat\lambda_n}$ defined on $S\subset T^n$ as in (4.1). Hence by (4.2),

$$(9.2)\qquad (\phi v^n)(T_i^n)=n\lambda_i^n u_i(x_i^n)\qquad (i=1,\dots,k).$$

9.2. *Outline*

Before proceeding, we briefly outline the proof; the reader may also wish to consult the flow charts in Appendix C. The thought that first comes to mind is to apply the value convergence theorem (Proposition 8.8) directly to the v^n. But this would wipe out all information about all types i for which $\lambda_i^n\to 0$; i.e., about all but the heaviest of the heavyweights. We therefore decompose w^n as in (9.1), and deduce

$$(9.3)\qquad \phi v^n=\phi v_H^n+\lambda_\ell^n\phi v_L^n.$$

After showing $w_L^n\to w_L$ (Lemma 9.8), we apply Proposition 8.8 to deduce $\phi v_L^n\to p\in \partial w_L(1,\dots,1)$. We then use Proposition 8.5 to express p in terms of prices q in the commodity space. By plugging all this back into (9.3), and applying (9.2), we obtain our result.

[35] Otherwise the theorem is immediate.

Intuitively, $v_H^n(S)$ represents what the coalition S can do for the heavyweight traders in it; i.e., the result of ignoring completely the utilities of the lightweights in S. What is left over for the lightweights (possibly negative!) is of order[36] λ_ℓ^n, since that is the order of all their utilities. Dividing by λ_ℓ^n to get $v_L^n(S)$ is a way of looking at the situation through a microscope, so to speak on the scale of the lightweight traders themselves; it yields something of order 1. If $v_L := w_L \circ \eta$, then $v_L(S)$ represents what the lightweights in S, when assigned their limiting "relative" weights ξ_i, can do for themselves (again, on their own "scale"), after first satiating all the heavyweights in S; it is defined only if the heavyweights can indeed be satiated.

If S is large, then its profile is likely to be close to the "diagonal" (i.e., to a multiple of $(1,\ldots,1)$); hence the heavyweights can be satiated (Lemma 9.11). Then $v_L^n(S)$ will for large n be close to $v_L(S)$ (Lemma 9.8), and hence an additional trader of type i contributes approximately p_i to v_L^n by joining S, where $p = \partial w_L(1,\ldots,1) \approx \partial w_L(\eta(S))$. To v_H^n he contributes nothing if he is lightweight, since the heavyweights are already satiated. Any contribution to v_H^n by lightweights therefore comes from the coalitions with few traders, which stand a reasonable chance of not being able to satiate all their heavyweights.

Figure 2 depicts the path of the profile of a coalition $\tilde{Q}$ that grows as traders are added to it in a random order (cf. (4.3)). When $\tilde{Q}$ is small, there is a good chance that $\eta(\tilde{Q})$ is outside of the cone U_L. During this initial period, lightweight traders of type i make heavyweight contributions, which sum to $(\phi v_H^n)(T_i^n)$; but it is a short period, and relative to n it tends to zero. (There is also a lightweight contribution during this period, but it is negligible.) Afterwards, $\eta(\tilde{Q})$ is in the cone U_L and in fact close to the diagonal; lightweight traders of type i are no longer making heavyweight contributions, but altogether their contributions are over a much longer period, and the total, which is $\approx n\lambda_\ell^n p_i$, has the same order of magnitude as $(\phi v_H^n)(T_i^n)$. By the definition of the NTU value, all the contributions together must add up to the total (weighted) utility of type i; this yields

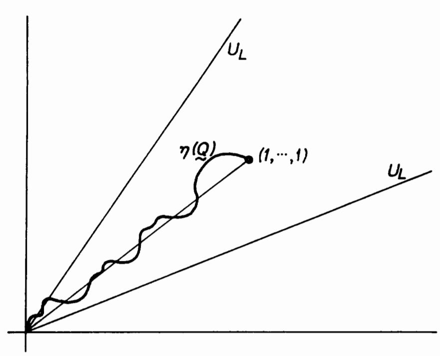

FIGURE 2

[36] By Corollary 9.26, $\xi_i > 0$ for all $i \in L$; hence the lightweight i are those with λ_i^n of the lowest order of magnitude (that of λ_k^n).

(9.25). The v_L^n term gives us the prices q, as in ordinary markets without satiation (Cf. Champsaur (1975)); the v_H^n term gives us the dividends. The analysis is similar when i is heavyweight.

We now proceed with the formal proof, which we divide into three parts.

9.3. w_L^n *Satisfies the Hypotheses of Proposition* 8.8

If $y \in R_+^k$, a *y-allocation* is a k-tuple $\boldsymbol{x} = (x_1, \ldots, x_k)$, with $x_i \in X_i$ for all i, and $\sum y_i x_i = 0$. An *allocation* is simply a $(1, \ldots, 1)$-allocation, i.e., an allocation in M^1. A y-allocation *satiates* type i if $x_i \in B_i$. Denote

$$\mu_i = \text{Max}\,\{u_i(x) \colon x \in X_i\}.$$

LEMMA 9.4. *Let $y \in U_L$, and for $n < \infty$, let $w^n(y)$ be attained at $\boldsymbol{x}^n(y)$. Let $\boldsymbol{x}^\infty(y)$ be a limit point of $\{\boldsymbol{x}^n(y)\}$. Then w_L and w_H^n are attained at $\boldsymbol{x}^\infty(y)$; that is, $\boldsymbol{x}^\infty(y)$ is a y-allocation that satiates all heavyweight types,*

$$w_L(y) = \sum_{i \in L} y_i \xi_i u_i(x_i^\infty(y)), \tag{9.5}$$

and

$$w_H^n(y) = \sum_{i \in H} y_i \lambda_i^n u_i(x_i^\infty(y)) = \sum_{i \in H} y_i \lambda_i^n \mu_i. \tag{9.6}$$

Finally, if $\boldsymbol{x}^n(y) \to \boldsymbol{x}^\infty(y)$ for some sequence of n, then for n in that sequence and for all i, we have, as $n \to \infty$,

$$\lambda_i^n u_i(x_i^n(y)) = \lambda_i^n u_i(x_i^\infty(y)) + o(\lambda_\ell^n). \tag{9.7}$$

PROOF: We restrict attention to a subsequence of the convergence indicator for which $\boldsymbol{x}^n(y) \to \boldsymbol{x}^\infty(y)$. Since the $\boldsymbol{x}^n(y)$ are y-allocations, so is $\boldsymbol{x}^\infty(y)$. Suppose that there is a heavyweight i with $x_i(y) \notin B_i$. Since $y \in U_L$, there is a y-allocation z that satiates all j in H. Now

$$\left(\sum y_j \lambda_j^n u_j(z_j)\right) - w^n(y) = \sum y_j \lambda_j^n (u_j(z_j) - u_j(x_j^n(y))).$$

Since $z_j \in B_j$ for all j in H, each term on the right with $j \in H$ is nonnegative. Moreover, for n in the subsequence we have chosen, the ith term is of the order λ_i^n. But the terms with $j \in L$ are of smaller order of magnitude; so for n sufficiently large, the right side is positive. Thus z satisfies the constraints in the definition of $w^n(y)$ and yields a larger value than $\boldsymbol{x}^n(y)$, a contradiction. Hence indeed $\boldsymbol{x}^\infty(y)$ is a y-allocation that satiates all heavyweight types.

Now let z be any other such y-allocation. Then

$$\begin{aligned}
&\sum_{i \in H} y_i \lambda_i^n \mu_i + \sum_{i \in L} y_i \lambda_i^n u_i(x_i^n(y)) \\
&\quad \geq \sum y_i \lambda_i^n u_i(x_i^n(y)) = w^n(y) \\
&\quad \geq \sum y_i \lambda_i^n u_i(z_i) = \sum_{i \in H} y_i \lambda_i^n \mu_i + \sum_{i \in L} y_i \lambda_i^n u_i(z_i).
\end{aligned}$$

Cancelling $\sum_{i\in H} y_i\lambda_i^n\mu_i$, dividing by λ_ℓ^n, and allowing $n\to\infty$, we obtain

$$\sum_{i\in L} y_i\xi_i u_i(x_i^\infty(y)) \geqslant \sum_{i\in L} y_i\xi_i u_i(z_i),$$

which yields (9.5).

As for (9.6), this follows from the fact that $x^\infty(y)$ satisfies the constraints in the definition of $w_H^n(y)$ and yields the maximum value for each of the u_i appearing in this definition.

Finally, to prove (9.7), note that

$$\sum y_i\lambda_i^n u_i(x_i^\infty(y)) \leqslant w^n(y) = \sum y_i\lambda_i^n u_i(x_i^n(y)).$$

Hence

$$\sum_{i\in H} y_i\lambda_i^n(u_i(x_i^\infty(y)) - u_i(x_i^n(y)))$$
$$\leqslant \sum_{i\in L} y_i\lambda_i^n(u_i(x_i^n(y)) - u_i(x_i^\infty(y))) = o(\lambda_\ell^n).$$

Since each summand on the left is nonnegative, and since each y_i is positive, (9.7) follows. *Q.E.D.*

LEMMA 9.8: *w_L^n is concave and* 1*-homogeneous on U_L, and for each y in U_L,*

$$w_L^n(y) \to w_L(y) \quad \text{as} \quad n\to\infty.$$

PROOF: The concavity and 1-homogeneity of w_L^n follow from its definition (9.1), from the concavity and 1-homogeneity (8.4) of w^n, and from the linearity (9.6) of w_H^n on U_L. The convergence follows from

$$(9.9)\qquad w^n(y) = w_H^n(y) + \lambda_\ell^n w_L(y) + o(\lambda_\ell^n)$$

for each y in U_L. To demonstrate (9.9) for a sequence of n for which $x^n(y)$ converges, write $w^n(y) = \sum y_i\lambda_i^n u_i(x_i^n(y))$ and apply (9.7), (9.6), (9.5), and $\lambda_i^n = \lambda_\ell^n\xi_i + o(\lambda_\ell^n)$ for $i\in L$; the truth of (9.9) in general follows from its truth for those special sequences. *Q.E.D.*

LEMMA 9.10: *$w_L^n(y) = O(1)$ as $n\to\infty$, uniformly for y in $[0,1]^k$.*

PROOF: Let $w^n(y)$ and $w_H^n(y)$ be attained at the y-allocations $x^n(y)$ and z^n respectively. Then

$$w^n(y) = \sum_{i\in H} y_i\lambda_i^n u_i(x_i^n(y)) + \sum_{i\in L} y_i\lambda_i^n u_i(x_i^n(y))$$
$$\leqslant w_H^n(y) + O(\lambda_\ell^n),$$

where the O is uniform because the u_i are uniformly bounded. Next,

$$w_H^n(y) = \sum y_i\lambda_i^n u_i(z_i^n) - \sum_{i\in L} y_i\lambda_i^n u_i(z_i^n)$$
$$\leqslant w^n(y) + O(\lambda_\ell^n),$$

where again the O is uniform. *Q.E.D.*

LEMMA 9.11: *U_L is a convex neighborhood of* $(1, \ldots, 1)$.

PROOF: U_L is open and convex by definition. By the definition of H, the allocation x^∞ satiates all heavyweight types; hence $0 \in \sum_{i\in H} B_i + \sum_{i\in L} X_i$, and hence by[37] the elbow room assumption (5.1), $0 \in \operatorname{Int}(\sum_{i\in H} B_i + \sum_{i\in L} X_i)$; i.e., $(1, \ldots, 1) \in U_L$. *Q.E.D.*

9.4. *Monotonicity Properties of v_H^n*

In a finite coalitional game v, a player s is called *at least as desirable as*[38] a player t, written $s \succcurlyeq t$, if $v(S \cup s) \geq v(S \cup t)$ for each[39] coalition S that contains neither s nor t.

PROPOSITION 9.12: *If $s \succcurlyeq t$, then $(\phi v)(s) \geq (\phi v)(t)$.*

PROOF: Suppose there are h players in all. The orders $\mathcal{R}$ on the players can be divided into $h!/2$ pairs, in each of which the two orders are the same, except that the positions of s and t are interchanged. Let $\mathcal{R}$ and $\mathcal{R}^*$ constitute such a pair; suppose s precedes t in $\mathcal{R}$. Let S_1 be the set of players preceding s in $\mathcal{R}$, and let S_2 be the set of players strictly between s and t. Setting $S_3 = S_1 \cup S_2$, we find that

$$\begin{cases} \phi^{\mathcal{R}}(s) = v(S_1 \cup s) - v(S_1), \\ \phi^{\mathcal{R}}(t) = v(S_3 \cup s \cup t) - v(S_3 \cup s), \end{cases}$$

where $\phi^{\mathcal{R}}(s)$ and $\phi^{\mathcal{R}}(t)$ are the contributions of s and t respectively in the order $\mathcal{R}$. Similarly,

$$\begin{cases} \phi^{\mathcal{R}^*}(s) = v(S_3 \cup s \cup t) - v(S_3 \cup t), \\ \phi^{\mathcal{R}^*}(t) = v(S_1 \cup t) - v(S_1). \end{cases}$$

Using $s \succcurlyeq t$, we conclude that

$$\phi^{\mathcal{R}}(s) + \phi^{\mathcal{R}^*}(s) \geq \phi^{\mathcal{R}}(t) + \phi^{\mathcal{R}^*}(t),$$

and the proposition follows. *Q.E.D.*

COROLLARY 9.13: *Suppose $i, j \in L$, and $X_i \subset X_j$. Then $(\phi v_H^n)(T_i^n) \leq (\phi v_H^n)(T_j^n)$.*

If S is a coalition in T^n, an *S-allocation* is a vector $x = (x_t)_{t\in S}$ such that $x_t \in X_i$ whenever t is of type i, and $\sum_{t\in S} x_t = 0$. Whereas every $\eta(S)$-allocation may be viewed as an S-allocation, the converse is false; an S-allocation allows traders of the same type to have different net trades, which an $\eta(S)$-allocation does not.

LEMMA 9.14: *Suppose $i \in H$, $j \in L$, and $X_i \subset X_j$. Then*

$$(\phi v_H^n)(T_i^n) \leq (\phi v_H^n)(T_j^n) + n\lambda_i^n \mu_i.$$

[37] (5.1) is used only here.
[38] Cf. Maschler and Peleg (1966).
[39] $\{s\}$ and $\{t\}$ are abbreviated by s and t.

PROOF: Let t and s be type i and j traders respectively. First we show that

$$v_H^n(S \cup t) \leq v_H^n(S \cup s) + \lambda_i^n \mu_i \tag{9.15}$$

for all coalitions S containing neither t nor s. Indeed, suppose $v_H^n(S \cup t)$ is attained at the $(S \cup t)$-allocation $\boldsymbol{x}$. Now transfer the amount x_t from t to s. Since $X_i \subset X_j$, this yields an $(S \cup s)$-allocation z; z coincides with $\boldsymbol{x}$ on S, and obeys $z_s = x_t$. Since lightweight traders have utilities 0 in the definitions of w_H^n and v_H^n, it follows that

$$v_H^n(S \cup s) \geq \sum_{r \in S} \hat{\lambda}_r^n u_r(x_r),$$

where the comparison function $\hat{\boldsymbol{\lambda}}^n$ on T^n corresponds to $\boldsymbol{\lambda}^n$ on T. Hence

$$v_H^n(S \cup t) = \sum_{r \in S \cup t} \hat{\lambda}_r^n u_r(x_r) \leq v_H^n(S \cup s) + \hat{\lambda}_t^n u_t(x_t),$$

and (9.15) follows. Proceeding from (9.15) as in the proof of Proposition 9.12, we find

$$\phi^{\mathscr{R}}(t) + \phi^{\mathscr{R}^*}(t) \leq \phi^{\mathscr{R}}(s) + \phi^{\mathscr{R}^*}(s) + 2\lambda_i^n \mu_i.$$

Summing over all $(nk)!/2$ pairs $(\mathscr{R}, \mathscr{R}^*)$ and dividing by $(nk)!$, we obtain

$$(\phi v_H^n)(t) \leq (\phi v_H^n)(s) + \lambda_i^n \mu_i,$$

and the lemma follows. *Q.E.D.*

9.5. *Derivation of Prices and Dividends*

By restricting the convergence indicator, we may assume that $(\phi v_L^n)(T_i^n)/n$ has a (finite or infinite) limit for each i; denote

$$p_i = \lim_{n \to \infty} (\phi v_L^n)(T_i^n)/n. \tag{9.16}$$

By Lemma 9.11, U_L is a convex neighborhood of $(1, \ldots, 1)$; by Lemma 9.8, on U_L the w_L^n are concave and 1-homogeneous, and converge to the concave function w_L; the w_L^n are 1-homogeneous throughout R_+^k by definition, and by Lemma 9.10 are uniformly bounded on $[0, 1]^k$. Therefore by Proposition 8.8, all p_i are finite, and setting $p = (p_1, \ldots, p_k)$, we have

$$p \in \partial w_L(1, \ldots, 1). \tag{9.17}$$

Now construct a vector q corresponding to p in accordance with Proposition 8.5. Here w is replaced by w_L, i.e. u_i is replaced by $\xi_i u_i$ for lightweight i, and by 0 for heavyweight i; and X_i is replaced by B_i for heavyweight i. Applying Lemma 9.4 to $y = (1, \ldots, 1)$, we see that $w_L(1, \ldots, 1)$ is achieved at $\boldsymbol{x}^\infty$; and $0 \in \operatorname{Int} (\sum_{i \in H} B_i + \sum_{i \in L} X_i)$ by Lemma 9.11. Hence (8.6) yields

$$p_i = \begin{cases} \xi_i u_i(x_i^\infty) - q \cdot x_i^\infty, & i \in L, \\ -q \cdot x_i^\infty, & i \in H. \end{cases} \tag{9.18}$$

As for q, the condition in Proposition 8.5 implies that

$$q \in \partial \xi_i u_i(x_i^\infty), \quad i \in L. \tag{9.19}$$

Since $\ell \in L$, $\xi_\ell = 1$, and $u_\ell(x_\ell^\infty) < \mu_\ell$, it follows from (9.19) that

$$q \neq 0 \tag{9.20}$$

(choose $z \in B_\ell$ in the definition (8.1) of superdifferential).

Note that v_H^n is monotonic,[40] since $u_i(0) = 0$ for all i; hence by (4.3),

$$(\phi v_H^n)(T_i^n) \geq 0, \quad \text{all } i. \tag{9.21}$$

LEMMA 9.22: *If i is lightweight, then*

$$q \cdot x_i^\infty = \frac{(\phi v_H^n)(T_i^n)}{n\lambda_\ell^n} + o(1). \tag{9.23}$$

If i is heavyweight, j lightweight, and $X_i \subset X_j$, *then*

$$q \cdot x_i^\infty = \frac{(\phi v_H^n)(T_i^n) - n\lambda_i^n \mu_i}{n\lambda_\ell^n} + o(1). \tag{9.24}$$

PROOF: By (9.2) (which is part of the definition of the λ_i^n), (9.1), and (9.16),

$$\begin{aligned} n\lambda_i^n u_i(x_i^n) &= (\phi v^n)(T_i^n) = (\phi v_H^n)(T_i^n) + \lambda_\ell^n (\phi v_L^n)(T_i^n) \\ &= (\phi v_H^n)(T_i^n) + n\lambda_\ell^n p_i + o(n\lambda_\ell^n) \end{aligned} \tag{9.25}$$

for all i. Moreover, $x^n \to x^\infty$ and the continuity of u_i yield

$$u_i(x_i^n) = u_i(x_i^\infty) + o(1).$$

Hence if i is lightweight, then $\lambda_i^n = \lambda_\ell^n \xi_i + o(\lambda_\ell^n)$, together with (9.25) and (9.18), yield

$$\begin{aligned} n\lambda_\ell^n \xi_i u_i(x_i^\infty) + o(n\lambda_\ell^n) &= n\lambda_i^n u_i(x_i^n) \\ &= (\phi v_H^n)(T_i^n) + n\lambda_\ell^n \xi_i u_i(x_i^\infty) - n\lambda_\ell^n q \cdot x_i^\infty + o(n\lambda_\ell^n). \end{aligned}$$

Cancelling, transposing, and dividing by $n\lambda_\ell^n$, we obtain (9.23).

To prove the second part of the lemma, note that $\mu_i = u_i(x_i^\infty)$, since i is heavyweight. Hence (9.7), (9.25), and (9.18) yield

$$\begin{aligned} n\lambda_i^n u_i(x_i^\infty) + o(n\lambda_\ell^n) = n\lambda_i^n u_i(x_i^n) &= (\phi v_H^n)(T_i^n) + n\lambda_\ell^n p_i + o(n\lambda_\ell^n) \\ &= (\phi v_H^n)(T_i^n) - n\lambda_\ell^n q \cdot x_i^\infty + o(n\lambda_\ell^n); \end{aligned}$$

transposing and dividing by $n\lambda_\ell^n$, we obtain (9.24). *Q.E.D.*

[40] $v^n(S \cup t) \geq v^n(S)$.

COROLLARY 9.26: *$\xi_i > 0$ for all lightweight i.*

PROOF: If $\xi_i = 0$, then by (9.19), the definition (8.1) of superdifferential, $0 \in \text{Int } X_i$ (see (2.3)), and $q \neq 0$ (see (9.20)), we have

$$q \cdot x_i^\infty = \min \{q \cdot x \colon x \in X_i\} < 0;$$

by (9.21), this contradicts (9.23). *Q.E.D.*

We now define the dividend *c*. First, define

$$\text{(9.27)} \qquad c_i := \lim_{n \to \infty} \frac{(\phi v_H^n)(T_i^n)}{n \lambda_\ell^n}, \quad \text{when} \quad i \in L;$$

the limit exists by (9.23). When $i \in H$, define $J(i) := \{j \in L \colon X_i \subset X_j\}$, and

$$\text{(9.28)} \qquad c_i := \min \{c_j \colon j \in J(i)\}, \quad \text{when} \quad i \in H \quad \text{and} \quad J(i) \neq \varnothing.$$

If $J(i)$ is empty, then, as usual for the minimum over the empty set, c_i should be taken as $+\infty$. Since, however, we want c_i finite for all i, we define it as a sufficiently large finite number; specifically,

$$\text{(9.29)} \qquad c_i := \max \{q \cdot x \colon x \in \textstyle\sum X_j\}, \quad \text{when} \quad i \in H \quad \text{and} \quad J(i) = \varnothing.$$

With these definitions it follows from (9.21) that the dividends are nonnegative, and from Corollary 9.13 that they are monotonic in the net trade sets. By (9.23),

$$\text{(9.30)} \qquad q \cdot x_i^\infty = c_i \quad \text{when} \quad i \in L;$$

and by (9.23), (9.24), and Lemma 9.14,

$$q \cdot x_i^\infty \leqslant c_i \quad \text{for all } i,$$

i.e., x_i^∞ is in the dividend budget set $\{x \in X_i \colon q \cdot x \leqslant c_i\}$ (note that q is a price vector by (9.20)). If i is heavyweight, then x_i^∞ is a global maximum of u_i, and a fortiori maximizes u_i over the dividend budget set. If i is lightweight, then from (9.19), (9.30), and $\xi_i > 0$ (Corollary 9.26), it follows that x_i^∞ maximizes u_i over the dividend budget set. This completes the proof of the main theorem.

10. FIXED PRICE MARKETS

A (finite) fixed price market is defined by:

(10.1) a finite set T (the *trader space*);

(10.2) an integer $d+1$ that is at least 2 (the *number of commodities*);

(10.3) for each trader t, a point e_t in R_{++}^{d+1} (t's *endowment*);

(10.4) for each trader t, a concave continuous function u_t^* on R_+^{d+1} (t's *utility function*); and

(10.5) a point $\bar{p}$ in R_{++}^{d+1} (the *fixed price vector*).

A fixed price market is just like an ordinary market (without satiation), except that all trade is constrained to take place at the exogenously given prices $\bar{p}$. In effect, this means that each trader t can only consume bundles in his *fixed price hyperplane*

$$H_t := \{y \in R_+^{d+1}: \bar{p} \cdot y = \bar{p} \cdot e_t\}.$$

The utility u_t^* is defined on the entire orthant R_+^{d+1} only for convenience; all that is actually used is its restriction to H_t.

An *allocation* in a fixed price market is a vector $\boldsymbol{v} = (y_t)_{t \in T}$, where $y_t \in H_t$ for each t (trading takes place at prices $\bar{p}$ and all consumptions are nonnegative), and $\sum_{t \in T} y_t = \sum_{t \in T} e_t$ (trading does not affect the total quantity of each good). A *coupons price vector* is a member q^* of R^{d+1} not proportional to $\bar{p}$ (i.e., unequal to $\alpha\bar{p}$ for any real α). A *coupons endowment* for agent t is a real number c_t. A *coupons equilibrium* consists of a coupons price vector q^*, a vector $\boldsymbol{c} = (c_t)_{t \in T}$ of coupons endowments, and an allocation $\boldsymbol{y}$, such that for all traders t, y_t maximizes u_t^* over the *coupons budget set*

$$\{y \in R_+^{d+1}: \bar{p} \cdot y = \bar{p} \cdot e_t \text{ and } q^* \cdot y \leq q^* \cdot e_t + c_t\}.$$

The notion of coupons equilibrium is due to Drèze and Müller (1980). If the traders maximize their utilities subject only to the fixed prices $\bar{p}$, then in general the market will not clear. To obtain market clearing, one introduces an auxiliary currency, *in addition* to the ordinary currency in which the fixed prices $\bar{p}$ are stated. This auxiliary currency may be thought of as rationing "coupons"; each transfer of commodities must be paid for both in ordinary money, at prices $\bar{p}$, and in coupons, at prices q^*. Coupons may not be exchanged for ordinary money.

The coupons endowments c_t are called *monotonic in the commodity endowments* if $e_t \geq e_s$ (coordinatewise) implies $c_t \geq c_s$; and *uniform* if all the c_t are the same.

In a $(d+1)$-commodity fixed price market M^*, the spaces $H_t - e_t =: X_t$ of net trades are compact convex subsets of the d-dimensional subspace Q of R^{m+1} that is othogonal to $\bar{p}$; M^* may be viewed as a d-commodity market M with satiation. If $(q, c, \boldsymbol{x})$ is a dividend equilibrium in this market, then q is a linear functional on Q, and $\boldsymbol{x}$ is in Q^k; extending q in an arbitrary way to a linear functional q^* on all of R^{d+1} yields a coupons equilibrium $(q^*, c, \boldsymbol{x}+\boldsymbol{e})$ in M^*, where $\boldsymbol{e}$ is the initial allocation in M^*. In brief, dividend equilibria in M correspond naturally to coupons equilibria in M^* (cf. Drèze and Müller, 1980, p. 133). Hence our main theorem implies that in fixed price markets with k types, limiting value allocations are associated with coupons equilibria enjoying the appropriate monotonicity properties.

Clearly, monotonicity in the net trade sets is equivalent to monotonicity in the endowments:

(10.6) $\quad X_t \supset X_s$ if and only if $e_t \geq e_s$ (coordinatewise).

Thus, *all value allocations in large fixed price markets approximate coupons equilibrium allocations, where the coupons endowments depend only on the commodity endowments, and monotonically so.* In particular, if all commodity endowments are the same, we are led to uniform coupons equilibria.

Note that in the context of fixed prices, the elbow room assumption is satisfied if there is no set J of types whose aggregate demand for some good h precisely exhausts the total supply of that good.[41]

So much for the technical treatment. We end this section with some remarks of a more conceptual nature, which relate this work to other work on fixed prices.

As mentioned in the introduction, our interest in markets with satiation arose from the desire to discover what kind of allocations the Shapley value would generate, in markets with fixed prices. The study of these markets in economic theory has been mostly oriented towards equilibria with one-sided, market-by-market rationing. The specific features of the equilibrium concept are either imposed directly, as in Drèze (1975), or derived from more basic assumptions (no involuntary trading, efficient recourse to a set of admissible trades, . . .), as in Malinvaud and Younès (1977). By contrast, the analysis presented here imposes no conditions on the problem or its solution, beyond the constraint that all trading should take place at exogenously given prices.[42] Rather, we apply a general solution concept (the Shapley value) to the problem. The equilibrium concept (coupons equilibrium) and its specific features (nonnegative coupons endowments monotonically geared to initial resources) are an output of the analysis, not an input.

Whatever further properties Shapley value allocations may be found to possess, these properties will also emerge from the problem formulation, kept here to essentials. By this we mean in particular keeping out of the problem formulation elements like "market-by-market rationing," which make the solution set depend upon inessentials like the definition of commodities.[43]

To clarify this point, note that the formal description of a market specifies for each trader a set (the consumption set), a real function on it (the utility function), and a point in it (the endowment). The set must also be endowed with an additive structure (to enable us to describe transfers between traders). Nothing more is required to describe a market from the economic viewpoint; the above structure completely specifies the opportunities as well as the incentives.

This suggests that one might want an economic "solution" (such as an equilibrium concept) to depend only on this structure, to be invariant under "inessential" changes, changes in the specification of the situation that leave this basic structure invariant. Familiar examples of such "inessential" changes are changing the units of the commodities, or using different commodities that are utility substitutes, or permuting the commodities. But the principle of invariance under inessential changes applies equally well in less familiar cases, e.g. for rotations or other affine transformations.[44]

[41] That is, $\sum_{i\in J} y_i^h = \sum e_i^h$, where y_i maximizes u_i^* over H_i.

[42] This is not the place to discuss the rationale for studying markets with fixed prices.

[43] Indeed, in applied work, identifying specific "commodities" is often quite difficult.

[44] For example, suppose that each of two mutual funds is composed of stock in the same two companies, but in different proportions. Suppose that the companies themselves are not public, but that the funds are: in essence, therefore, one can buy into the companies in any proportion between those offered by the two funds. Then it should make no difference whether the "commodities" are defined to be company stock or fund stock.

The Walras equilibrium is invariant in this sense; so are the kinds of dividend equilibrium and coupons equilibrium defined in this paper.[45] But the "market-by-market" rationing equilibria mentioned above are not; they depend on identifying specific "commodities," which are not present in the opportunities or the incentives.

All this is reflected in the game-theoretic treatment. Game theory gets at the essence; it separates the intrinsic from the conventional. Thus it is not surprising that the game-theoretic analysis leads to a "commodity-free" solution.

When institutional aspects (like market-by-market rationing) are deemed important, they should perhaps be introduced exogenously into the problem formulation. This brings us to the basic methodological dichotomy, whether economic theory should be concerned with "explaining" the genesis of institutions, or with "predicting" their consequences. In the end, each of these activities has its own validity.

11. ALTERNATIVE APPROACHES

11.1 *The Core*

The core of a market is the set of all allocations that cannot be improved upon by any coalition S. "Improved upon" has two possible meanings:

(a) Some members of S are better off and none worse off.

(b) All members of S are better off.

In classical markets these two meanings lead to the same core, but here they do not. Neither core is very interesting; the first is too small, the second is too large.

Let M^1 be a market with satiation, M^n its n-fold replication. For simplicity we assume that the utilities are strictly convex and that not all traders can be simultaneously satiated; the elbow room assumption (5.1) is, however, unnecessary here.

Under (a), the Debreu-Scarf Theorem (1963) applies; the proof goes through without difficulty. Specifically, the a-core of M^n enjoys the equal treatment property. Hence it may be represented by a set C_a^n of allocations in the unreplicated market M^1. Then $C_a^{n+1} \subset C_a^n$, and the limiting a-core, $C_a^\infty := \bigcap_{n=1}^\infty C_a^n$, coincides with the set of competitive allocations.

As we saw in Section 3, markets with satiation often have no competitive allocations; the limiting a-core is then empty. This is what we meant by "too small".

Under (b), the core of M^n may be very large, and does not even enjoy the equal treatment property. If we nevertheless confine ourselves to equal treatment allocations, we are, as above, led to a set C_b^n of allocations in M^1, where again $C_b^{n+1} \subset C_b^n$. We are interested in $C_b^\infty := \bigcap_{n=1}^\infty C_b^n$.

For each allocation $\mathbf{x}$ in M^1, let $G_i(\mathbf{x}) := \{x \in X_i \colon u_i(x) > u_i(\mathbf{x}_i)\}$ be the set of net trades preferred by i to $\mathbf{x}$. Then C_b^∞ consists precisely of those allocations $\mathbf{x}$

[45] The monotonicity condition for coupons equilibria ostensibly involves the commodities, but (10.6) shows that it is merely a restatement of an "invariant" condition.

for which 0 is not in the convex hull of the union of the preferred sets $G_i(x)$; since the preferred set is empty for satiated traders, we may take the union over unsatiated i only. For example, any dividend equilibrium allocation with nonnegative dividends is in the limiting b-core, even if the dividends are not in any sense monotonic; they may even be different for identical[46] types. Any individually rational allocation x at which only one type is unsatiated will also be in the limiting b-core. What is happening is that the satiated traders are useless as partners in an improving scheme; the unsatiated must fend for themselves, and they may well lack the resources for this. This makes the b-core very large.

We note for the record that the limiting value allocations are in the limiting b-core; but in general they constitute only a small subset. This fits in well with our experience in other market contexts with cores. For example, in large transferable utility markets with nondifferentiable utilities the core may be quite big, but if it has a center of symmetry, then the value is that center of symmetry (Hart, 1977a). More generally (asymmetric core, nontransferable utility), the value allocations in a large nondifferentiable market often constitute a small, "central" subset of a relatively big core (Hart, 1977b, 1980; Mertens, 1986; Tauman, 1981).

11.2. *The Continuum Approach*

There is no difficulty in defining nonatomic markets with satiation. One simply replaces the trader space T by a nonatomic measure space $(T, \mathscr{C}, \mu)$ with $\mu(T)=1$, and requires that the net trade sets X_t and the utility functions u_t be measurable in an appropriate sense, and the u_t uniformly bounded. An *allocation* is now a measurable function x from T to R^d with $x(t)\in X_t$ for all t and $\int_T x=0$. As before, we assume that no allocation satiates all traders, but do not require anything like the "elbow room" assumption (5.1). The definition of competitive equilibrium remains literally unchanged.

A *generalized comparison function* is an integrable function from T to R^1_+ with a positive integral; if it is to R^1_{++}, it is a *comparison function.* A *coalition* is a measurable subset of T (i.e. a member of $\mathscr{C}$). Given a generalized comparison function $\boldsymbol{\lambda}$, define a nonatomic game $v_{\boldsymbol{\lambda}}$ by

$$v_{\boldsymbol{\lambda}}(S) := \max\left(\int_S \lambda(t)u_t(x(t))\mu(dt)\colon \int_S x=0 \quad \text{and } x(t)\in X_t \text{ for all } t\in S\right)$$

for each coalition S. A (*generalized*) *value allocation* is an allocation x for which there exists a (generalized) comparison function $\boldsymbol{\lambda}$ such that

(11.1) $\quad (\phi v_{\boldsymbol{\lambda}})(dt) = \boldsymbol{\lambda}(t)u_t(x(t))\mu(dt)$

for all "infinitesimal" agents dt, where ϕ is an appropriate[47] value operator. (A more formal statement of (11.1) is that $(\phi v_{\boldsymbol{\lambda}})(S)=\int_S \boldsymbol{\lambda}(t)u_t(x(t))\mu(dt)$ for all coalitions S.)

[46] Having the same utilities and net trade sets. Confining ourselves to equal treatment allocations in M^n does not mean that identical types get the same net trade; it only means that in M^n, different replicas of the same trader in M^1 get the same net trade.

[47] See, e.g., Kannai (1966), Aumann and Shapley (1974), Hart (1980), Mertens (1980, 1986), or Neyman and Tauman (1979). What is needed here is a value with the "diagonal" property.

So much for the definitions. Unfortunately, the results are rather disappointing. All we can say is:

(11.2) *Every value allocation is competitive.*

(11.3) *An allocation is a generalized value allocation if and only if it is competitive or satiates some agents.*

We have already noted that markets with satiation often have no competitive equilibria; in that case there are *no* value allocations in the continuum approach. The *generalized* value allocations, on the other hand, constitute a very large set even then, consisting of all allocations satiating at least one agent. There is no restriction at all on what nonsatiated agents get; they may even be assigned individually irrational net trades.

To demonstrate these results, assume for simplicity that the u_t are continuously differentiable and strictly convex, and that the allocations in question are interior (i.e., $\boldsymbol{x}(t) \in \operatorname{int} X_t$ for all t). Let $\boldsymbol{x}$ be a generalized value allocation. As at (6.1), there is a vector q such that

(11.4) $\boldsymbol{\lambda}(t)u_t'(\boldsymbol{x}(t)) = q,$

for all t. Moreover, the value $(\phi v_{\boldsymbol{\lambda}})(dt)$ is the average contribution of dt to a coalition S. In the continuum case, "almost all" coalitions are large; hence as at (6.3),

$$(\phi v_{\boldsymbol{\lambda}})(dt) = \boldsymbol{\lambda}(t)u_t(\boldsymbol{x}(t))\mu(dt) - q \cdot \boldsymbol{x}(t)\mu(dt),$$

and together with (11.1), we obtain that for all t,

(11.5) $q \cdot \boldsymbol{x}(t) = 0.$

If $\boldsymbol{x}$ is a value allocation, i.e., $\boldsymbol{\lambda}(t) > 0$ for all t, then $u_t'(\boldsymbol{x}(t))$ is either equal to 0 for all t or is unequal to 0 for all t. If it is equal to 0 for all t, then all traders are simultaneously satiated, which we have ruled out. Hence it is unequal to 0 for all t, whence $q \neq 0$; together, (11.4) and (11.5) then assert precisely that $(q, \boldsymbol{x})$ is a competitive equilibrium, whence $\boldsymbol{x}$ is a competitive allocation.

If $\boldsymbol{x}$ is not a value allocation, i.e., $\boldsymbol{\lambda}(t) = 0$ for some t, then $q = 0$. Hence for those t for which $\boldsymbol{\lambda}(t) \neq 0$, we must have $u_t'(\boldsymbol{x}(t)) = 0$, and hence these t are satiated. Conversely, if $\boldsymbol{x}$ is any allocation that satiates some traders, let $\boldsymbol{\lambda}$ be a generalized comparison function that assigns weight 0 to all traders unsatiated at $\boldsymbol{x}$. The value $(\phi v_{\boldsymbol{\lambda}})(dt)$ is the average contribution of dt to a diagonal[48] coalition θT, where θ ranges from 0 to 1. The case $\theta = 0$ has no effect on the average and may be ignored. As soon as $\theta > 0$, there are enough unsatiated traders in θT to supply all the resources desired by the satiated traders in θT. Thus dt contributes nothing if he is unsatiated, and only his own utility $\boldsymbol{\lambda}(t)u_t(\boldsymbol{x}(t))\mu(dt)$ if he is satiated, which means that (11.1) is satisfied. Thus any such $\boldsymbol{x}$ is a generalized value allocation.

[48] See Aumann and Shapley (1974, Chapter III).

The reader will have realized that what prevents the continuum approach from achieving a more satisfactory result is that "small" coalitions play no role. Coalitions either have positive measure, in which case they behave like "large" coalitions, or have measure 0, in which case they are ignored. The crucial coalitions in the limit approach[49] (the one used in this paper) are those whose size is positive but of smaller order of magnitude than that of the all-trader set; the continuum approach is not equipped to take account of such coalitions.

11.3. *The Elbow Room Assumption in a Model with Explicit Endowments*

To show that (5.1) holds generically, we must reformulate the definition of a market with satiation so that the endowments appear explicitly. Accordingly, define a *market with satiation and explicit endowments* to consist of a finite set T (the *trader space*), a positive integer m (the number of *commodities*), and for each trader t,

a compact convex subset X_t^0 of R^d (t's *consumption set*);

a concave continuous function u_t^0 on X_t^0 (t's *utility function*); and

a point e_t in the interior of X_t^0 (t's *endowment*).

To regain from this a market with satiation as in Section 2, simply define $X_t := X_t^0 - e_t$ (algebraic subtraction!) and $u_t(x) := u_t^0(x + e_t)$. The remainder of the treatment is then exactly as before.

In this formulation, the elbow room assumption is equivalent to the following: for each $J \subset \{1, \dots, k\}$,

$$\sum_{i=1}^{k} e_i \notin \operatorname{bd}\left[\sum_{i \in J} B_i^0 + \sum_{i \notin J} X_i^0\right], \tag{11.6}$$

where e_i is the endowment of a type i trader, and B_i^0 is the set of points in X_i^0 at which u_i^0 is maximized. Note that for each J, the right side of (11.6) represents the boundary of a compact convex set in R^d that is independent of the endowments; such sets are closed and of measure 0. The left side is simply the total endowment. Since there are only finitely many different possible choices of J, it follows that the elbow room assumption holds for all total endowment vectors except for a closed set of measure 0 in R^d; hence also for all k-tuples $(e_1, \dots, e_k)$ of endowments except for a closed set of measure 0 in R^{dk}.

Conceptually, the situation here is perhaps a little different from that of other generic theorems in the literature. The exceptional set is entirely explicit and has transparent geometric and economic meanings; in any given market one can, so to speak, "see at a glance" whether or not the elbow room assumption is satisfied.

[49] Of course, the value operator ϕ may itself be defined by a limit approach, even when it is applied to nonatomic games. Nevertheless, the kind of second-order effect that is crucial in the proof of the main theorem does not obtain then.

12. FURTHER RESULTS

The main theorem (Section 5) shows that the dividends depend only on the net trade sets, and monotonically so. In fact, we know much more about them, both qualitatively and quantitatively. This section discusses additional properties of the dividends, as well as the order of magnitude of the weights λ_i^n, and some results concerning the differentiable case and that of a single commodity. The next section applies these results to computing some examples.

We will maintain here the terminology, notations, and conventions introduced previously, especially in Sections 5, 8, and 9. The allocations $\boldsymbol{x}^n$, the comparison vectors $\boldsymbol{\lambda}^n$, and the limiting allocation $\boldsymbol{x}^\infty$ will be fixed throughout; so will the price vector q and the dividend $\boldsymbol{c}$, which are taken to be as in the proof (Section 9) of the main theorem. We will call a type i *satiated* (*unsatiated*) if it is satiated (unsatiated) at x_i^∞.

The competitive case is from our point of view less interesting, and it is convenient to exclude it; thus throughout this section, we assume

(12.1) $(q, \boldsymbol{x}^\infty)$ *is not a competitive equilibrium.*

A summary of the results is as follows. The dividends c_i are all strictly positive (rather than just nonnegative); in addition to monotonicity, they satisfy a concavity condition; and there is an "explicit" formula for them. The order of magnitude of the weights λ_i^n of each lightweight type i is exactly $1/n$. Lightweight types whose utilities are differentiable, and whose maxima are interior, are unsatiated; hence if all utilities are differentiable and all maxima interior, then the heavyweights are precisely the satiated types, the lightweights precisely the unsatiated. In the case of one commodity ($d=1$), all traders on the "short" side are satiated.

We start the formal treatment by discussing concavity of the dividends. Let J be a set of types. A dividend $\boldsymbol{c}$ in M^1 is called *monotonically concave in the net trade sets of the types in* J, if

(12.2) $$c_j \geq \sum_{i\in J} \alpha_i c_i$$

for all j in J and constants $\alpha_i \geq 0$ such that $\sum_{i\in J} \alpha_i \leq 1$ and

(12.3) $$X_j \supset \sum_{i\in J} \alpha_i X_i.$$

This implies that if a convex combination of net trade sets X_i is itself a net trade set X_j, then the dividend c_j is at least as great as the corresponding combination of the dividends c_i. As a function of the net trade sets, so to speak, the dividends are concave.

Actually, the condition involves somewhat more, in two directions. First, note that in (12.3) we have inclusion, not only equality. This expresses a kind of amalgamation of concavity and monotonicity. We have already seen that "ordinary" concavity is implied; note that by taking $\alpha_i = 1$ for one of the i in (12.3), we also get ordinary monotonicity.

Second, note that we demand only that $\sum_{i \in J} \alpha_i \leqslant 1$, not $\sum_{i \in J} \alpha_i = 1$. Intuitively, one can think of this as if we were talking about ordinary convex combinations ($\sum_{i \in J} \alpha_i = 1$), but adding the additional "virtual" element $X_0 := \{0\}$, with $c_0 := 0$.

Denote by $\mathscr{C}_J$ the set of all convex compact subsets of linear combinations of the X_i (for $i \in J$) that contain 0; note that $\mathscr{C}_J$ itself is a convex set. M. Perles has shown (private communication) that c is concave over J if and only if there is a concave monotonic function on $\mathscr{C}_J$ that vanishes at $\{0\}$ and coincides with c_i at each X_i with $i \in J$.

We now state our first results.

THEOREM 12.4: *$c_i > 0$ for all i.*

THEOREM 12.5: ***c** is monotonically concave in the net trade sets of the lightweight types.*

In particular, since by definition, all unsatiated types are lightweight, it follows that $\boldsymbol{c}$ is monotonically concave in the net trade sets of the unsatiated types.

THEOREM 12.6: *If i is lightweight, then λ_i^n has order of magnitude $1/n$ exactly.*

By possibly restricting the convergence indicator, we may assume that $\lim_{n \to \infty} n\lambda_\ell^n$ exists; denote it by λ_*. The above theorem asserts that

(12.7) $\quad 0 < \lambda_* < \infty.$

We come next to the explicit formula. Define

(12.8) $\quad \lambda_i^\infty := \lim_{n \to \infty} \lambda_i^n.$

Define w_H on R_+^k as in (8.3), except that u_i is replaced by $\lambda_i^\infty u_i$ for all i. Consider a pool of traders containing infinitely many of each of the k types. If S is a finite coalition chosen from this pool, let $v_H(S)$ be the maximum total utility that S can achieve by trading within itself, with utilities weighted by the λ_i^∞; formally, $v_H := w_H \circ \eta$, where $\eta(S)$ is S's profile ($\eta_i(S)$ is the number of type i traders in S). Consider next an infinite sequence of independent random choices from this pool, which pick traders from the k types with probability $1/k$ each. Denote by $\underset{\sim}{Q}_m$ the coalition of traders chosen up to stage m, and let $\underset{\sim}{\rho}_i^m = 1$ or 0 according as to whether or not i is chosen at stage m. Let t_i denote a separate trader of type i, not included in the pool.

THEOREM 12.9: *For lightweight i,*

$$c_i = \frac{1}{\lambda_*} \sum_{m=1}^{\infty} E((v_H(\underset{\sim}{Q}_m) - v_H(\underset{\sim}{Q}_{m-1}))\underset{\sim}{\rho}_i^m)$$

$$= \frac{1}{k\lambda_*} \sum_{m=0}^{\infty} E(v_H(\underset{\sim}{Q}_m \cup t_i) - v_H(\underset{\sim}{Q}_m)).$$

In words, the first line says that the dividend is the expected total contribution of all type i traders in the infinite sequence, divided by λ_*. The second expression is the sum (over m) of the expected contributions made by a type i trader to the coalition chosen up to stage m, divided by $k\lambda_*$. The two expressions are equal because at each stage, a type i trader is picked with probability $1/k$ exactly.

Another alternative expression, directly in terms of w_H, will be given in Appendix C (Proposition C.15). There we will also discuss analogous formulas for heavyweight types (Corollary C.19).

A noteworthy feature of this formula is that it involves *independent* choices (like in sampling with replacement), which are much easier to work with than the more complicated permutations (sampling without replacement) that are usually associated with values. For example, it is the independence that enables the proof of concavity of the dividends (Theorem 12.5). Another noteworthy feature is that the limiting order represented here—a discrete infinite sequence—is quite different both from the continuous interval associated with the familiar diagonal formula,[50] and from the denumerable dense order type (the order type of the rationals) introduced into value theory by N. Z. Shapiro.[51]

The last two results are shallower than the others, but are useful to keep in mind, particularly when computing examples.

PROPOSITION 12.10: *Let i be a lightweight type with $B_i \subset \text{Int } X_i$, and u_i differentiable at each point of B_i. Then i is unsatiated.*

The final result deals with the case of one commodity ($d = 1$), which is of considerable special interest, both theoretically—because of its relative simplicity—and in the applications, e.g. to unemployment.

In this case, each net trade set X_i is a compact interval on the real line, that contains 0 in its interior. For each i, the bliss set B_i is also a compact interval; we have assumed (Section 2) that it does *not* contain 0. If $0 \in \sum B_j$, then all agents can be simultaneously satiated. Since value allocations are Pareto optimal, it follows that $\boldsymbol{x}^\infty$ satiates all agents, a trivial case that was excluded in Section 9. Thus, $0 \notin \sum B_j$. Since $\sum B_j$ is a compact interval, it is included in either the strictly positive or the strictly negative half-line. We shall say that Type i is on the *short* (*long*) *side of the market* if B_i is on the opposite (same) side of 0 from $\sum B_j$.

Intuitively, $\sum B_j$ represents total demand. If $\sum B_j$ is, say, in the positive half-line, then there is excess demand for the single good;[52] the good is scarce. A type i with B_i in the negative half-line wants to supply this scarce good, which is what we mean by "being on the short side."

PROPOSITION 12.11: *If $d = 1$, then $\lambda_i^\infty > 0$ for all types i on the short side of the market; in particular, they are satiated.*

[50] See Footnote 47.

[51] See Shapley (1962).

[52] This is the convention in the examples below; it is convenient because then $q > 0$. In applications, the case of interest is often that of excess supply. Practically, there is no difference; demand for positive amounts is the same as supply of negative amounts.

All results stated in this section will be proved in Appendix C. Also, some useful generalizations and strengthenings of these results may be found there.

13. EXAMPLES

EXAMPLE 13.1: Unless $\boldsymbol{x}^\infty$ is competitive, there are at least two orders of magnitude for the weights: $1/n$ for the lightweights, and 1 for at least some of the heavyweights. Here we show that there may be more than two orders of magnitude, by bringing an example with three. We will not compute the third one explicitly, but will use theoretical arguments to show that it is different from both 1 and $1/n$. Indeed, we do not know what it is.

$$\text{Let} \quad d=1, \quad k=3, \quad X_1=X_2=X_3=[-3,3],$$

$$u_1(x)=4-(x+2)^2, \quad u_2(x)=1-(x-1)^2, \quad u_3(x)=4-(x-2)^2.$$

The maxima are at -2, 1, and 2 respectively. The short side consists of Type 1 only, and so by Proposition 12.11, Type 1 is satiated, and indeed $\lambda_1^\infty>0$. Since all net trade sets are equal, all dividends are equal, which means that the maxima of 2 and 3 over their dividend budget sets are achieved at 1. At this point, 3 is unsatiated and so must be lightweight. Type 2 is satiated and so, by Proposition 12.10, must be heavyweight; that is, λ_2^n has order of magnitude greater than $1/n$. If $\lambda_2^\infty>0$, then by Lemma C.28, $1=x_2^\infty<x_3^\infty=1$. Hence $\lambda_2^\infty=0$; that is, λ_2^n has order of magnitude less than 1. *Q.E.D.*

EXAMPLE 13.2: In this example, the net trade sets are not all the same, and we use the "explicit formula" (Theorem 12.9) to calculate the dividends. Let $d=1$, $k=3$,

$$X_2=[-1,1], \quad X_1=X_3=[-2,2],$$

$$u_1(x)=1-|x+1|, \quad u_2(x)=u_3(x)=1-(x-1)^2.$$

The short side consists of Type 1 only, so $\lambda_1^\infty>0$ (Proposition 12.11), and $q>0$. Because $c_i>0$ for all i (Theorem 12.4), both long types, 2 and 3, have a positive consumption. Therefore if either one is satiated, the market does not clear; there is excess demand. Hence they are both unsatiated, hence lightweight, and so Theorem 12.9 applies to them. Together, they consume the supply of Type 1, and hence

$$1=x_2^\infty+x_3^\infty=\frac{c_2}{q}+\frac{c_3}{q}.$$

Hence it suffices to calculate c_2/c_3, which we can do from Theorem 12.9 without knowing λ_*. The normalization $\sum \lambda_i^\infty=1$ yields $\lambda_1^\infty=1$.

Denote $\underset{\sim}{Q}_m$'s profile by $\underset{\sim}{y}^m:=(\underset{\sim}{y}_1^m, \underset{\sim}{y}_2^m, \underset{\sim}{y}_3^m)$. Set

$$f(y):=y_1-y_2-2y_3.$$

If $f(\underset{\sim}{y}^m) \leq 0$, then the long types together can satiate Type 1, and so an additional long trader t_i contributes nothing to v_H. If $f(\underset{\sim}{y}^m) = 1$, then the whole long side together falls short by just one unit of trading capacity to be able to satiate Type 1. This unit of capacity is supplied by t_i whether $i = 2$ or 3, and so in either case, t_i contributes exactly 1. If $f(\underset{\sim}{y}^m) \geq 2$, then t_i contributes 1 if $i = 2$, and 2 if $i = 3$. The expected contribution of t_i is therefore

$$\text{(13.2.1)} \quad \text{prob}\,\{f(\underset{\sim}{y}^m) = 1\} + (i-1)\,\text{prob}\,(f(\underset{\sim}{y}^m) \geq 2\}.$$

Since $f(\underset{\sim}{y}^m)$ is the sum of m i.i.d. r.v.'s distributed like $f(\underset{\sim}{y}^1)$, the generating function of $f(\underset{\sim}{y}^m)$ is $g(x) := ((x + x^{-1} + x^{-2})/3)^m$. The first probability in (13.2.1) is the coefficient of x in $g(x)$, and the second is the sum of all coefficients starting with that of x^2. Using this we find

$$\text{(13.2.2)} \quad x_2^\infty = c_2/q = .414,$$
$$x_3^\infty = c_3/q = .586,$$

to within .002. Of course, $x_1^\infty = -1$.

Several aspects of this example are worthy of special note. First, the limiting value allocation $\boldsymbol{x}^\infty$ is unique, though there are no symmetries that would lead to this conclusion in any obvious way. Second, our method does *not* involve calculating $\boldsymbol{x}^n$ for large n, say by some fixed point method. On the contrary, we use a formula that is valid only "in the limit." This formula enables a precise calculation of $\boldsymbol{x}^\infty$, to within errors with definite, theoretically proven bounds; but it does *not* enable even an approximate calculation of $\boldsymbol{x}^n$, for any finite n.

Third, the outcome is fairly insensitive to the utility functions on the long side (Types 2 and 3). For simplicity, let each bliss set B_i consist of a unique point b_i. The above reasoning certainly continues to apply when u_2 and u_3 are changed only cardinally,[53] e.g., if $u_2(x) = x$, $u_3(x) = 1 - (x-1)^4$. But even if there are ordinal changes, i.e., the bliss points are changed, $\boldsymbol{x}^\infty$ will not change if b_2 and b_3 remain sufficiently large; e.g., if $b_2 > \frac{1}{2}$ and $b_3 > \frac{2}{3}$ (see C.19 and C.20). Of course, if they are sufficiently small—e.g., $b_2 \leq .412$ or $b_3 \leq .584$—then either 2 or 3 must be satiated, and the result necessarily changes. (We are assuming that u_1 and the X_i do not change.) Again, all this applies to $\boldsymbol{x}^\infty$ only; each $\boldsymbol{x}^n$ may, and in general will, change considerably, even if u_2 or u_3 are only changed cardinally. Indeed, the $\boldsymbol{x}^n$ may perhaps be nonunique, even though $\boldsymbol{x}^\infty$ is unique.

On the other hand, the outcome *is* quite sensitive to changes in u_1, even if they are only cardinal. If u_1 is strictly concave, then the fact that Type 3 has more capacity than Type 2 matters less than when u_1 is linear. Specifically, if $u_1 = 1 - (x+1)^\alpha$, then $c_3/c_2 \to 1$ when $\alpha \to \infty$, and hence $x_3^\infty \to \frac{1}{2}$, $x_2^\infty \to \frac{1}{2}$. Indeed, when α is large, the lion's share of the contribution to Type 1 comes on the very first occasion when a trader on the long side joins the market, and it comes with the first unit that that trader contributes. When α is moderate, additional units remain more significant.

[53] This is as in ordinary markets, without satiation. In such markets, of course, *all* agents are unsatiated; and indeed, under appropriate smoothness conditions, the limiting value allocations coincide with the competitive ones (Mas-Colell, 1977), and so are ordinally invariant.

Thus on one end of the scale, when the satiated traders are very risk averse, the unsatiated traders get dividends that are almost independent of their trading capacity. One might have thought that on the other end of the scale, when the satiated traders are risk neutral, the dividends of the satiated will be proportional to their capacities. Our example shows that this is not so. Type 3 traders have 100% more capacity than type 2, but their dividends are only about 40 per cent larger. This is because of the possibility that the shortfall, $f(\tilde{y}^m)$, in the capacity to satiate Type 1 is exactly 1, in which case the extra capacity of Type 3 is useless.[54]

Intuitively, it may seem strange that what the unsatiated traders get depends on the cardinal utility of the satiated traders, but not on their own cardinal utility. The reason is that because the unsatiated traders are on the "long" side, they contribute very little to Society. When there is unemployment, an additional employer is a lot more welcome than an additional worker. This is reflected in the weights, which are much higher for the employers; the workers are judged not by what they can do for themselves, but by what they can do for the employers.

It should not puzzle the reader that the dividends may depend on cardinal utilities, in spite of our statement that they depend only on the net trade sets. The latter statement refers to different types within the same market; such types do get the same dividend as long as they have the same net trade sets, even if their utilities are different. But if one changes the utility of one type, everybody's dividend may change. The situation is similar in an ordinary market, without satiation. Incomes depend only on endowments, not on utilities, in the sense that agents with the same endowment have the same incomes, even if their utilities are different. But prices, and therefore incomes, are determined by everybody's utilities, and may change when a utility changes.

EXAMPLE 13.3: In this example, there are two types on the long side with identical utilities (but different net trade sets), one of whom is satiated and the other not. The satiated one of the two has $\lambda_i^\infty > 0$.

Let $k=3$, $d=1$, the X_i and u_1 as in the previous example, and

$$u_2(x) = u_3(x) = (.55)^2 - (x - .55)^2.$$

As above, the short side consists of Type 1 only; therefore Type 1 is satiated ($x_1^\infty = -1$), and indeed $\lambda_1^\infty > 0$. Type 2 cannot be satiated, because then by monotonicity, Type 3 is also satiated, and then the market does not clear (there is excess demand). Hence $\lambda_2^\infty = 0$. If also $\lambda_3^\infty = 0$, then v_H is as in the previous example, and hence $x_2^\infty < .416 < .45$. Hence $c_3/q \geqslant x_3^\infty > .55$. But then Type 3 is not maximizing over its budget set, since the maximum is at .55. Hence $\lambda_3^\infty > 0$, and hence $x_3^\infty = .55$, $x_2^\infty = .45$.

[54] It is worthy of note that this happens in a significant proportion of the cases, not only when m is small, but also when it is large. When $m = 40$, the probability is about .0003 that Type 1 cannot be satiated. In 37 per cent of these cases, the shortfall is exactly 1, though it ranges up to 40. This phenomenon is closely related to the exponential decay in Lemma B.9; it lies at the root of our results.

14. OPEN PROBLEMS

Foremost among the open problems is that of the converse. To what extent are the necessary conditions that we have found for limiting value allocations also sufficient? In the case of ordinary markets, without satiation, this is related to smoothness: see Mas-Colell (1977) and Hart (1977b). It is quite likely that smoothness is relevant here too.

Another interesting task is to dispense with the finite type assumption. As we have seen, one cannot simply use a continuum; what is called for is a limiting approach, in which the limit is a continuum of different types. There is a large literature on this type of model in connection with the core equivalence principle; cf. Hildenbrand's book (1974) and survey article (1982). Another approach that could conceivably be helpful for this purpose is that of nonstandard analysis (cf. Brown and Robinson, 1972).

One might also like to explore the consequences of dispensing with the equal treatment restriction, the elbow room assumption, or the assumption that 0 is in the interior of each net trade set.

Perhaps most interesting at this stage would be to derive additional qualitative properties of the solution in particular contexts. In the case $d=1$ (one commodity), for example, what happens to the dividends when the capacity of the long side is much larger than the supply of the short side? When they are almost equal? Can this kind of result, once obtained, be generalized to $d>1$? The "explicit formula" (12.9) gives us a powerful tool for investigating these and other questions arising in particular contexts.

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Manuscript received June, 1984; final revision received February, 1986.

APPENDIX A

PRICE CHARACTERIZATION OF $\partial w(y)$

In this Appendix we prove Proposition 8.5.

LEMMA A.1: *Let g be concave and 1-homogeneous on a convex cone V, and let $y \in V$. Then $p \in \partial g(y)$ if and only if*

(A.2) $$p \cdot y = g(y),$$

and

(A.3) $$p \cdot y' \geqslant g(y') \quad \textit{for all } y' \textit{ in } V.$$

PROOF: By definition, $p \in \partial g(y)$ if and only if

(A.4) $$g(y') - g(y) \leqslant p \cdot (y' - y) \quad \text{for all } y'.$$

Obviously (A.2) and (A.3) imply (A.4). Conversely, applying (A.4) to $y' = 2y$ and $y' = 0$ yields $g(y) \leqslant p \cdot y$ and $g(y) \geqslant p \cdot y$ respectively, and therefore $p \cdot y = g(y)$. Again applying (A.4), we then obtain $p \cdot y' \geqslant g(y')$. *Q.E.D.*

PROOF OF PROPOSITION 8.5: Intuitively, Lemma A.1 shows that one can view the superdifferential $\partial w(y)$ as the core of a transferable utility market with a nonatomic continuum of traders of types $1, \ldots, k$, in which type i has measure y_i. Proposition 8.5 is then simply an expression of the core equivalence principle. It needs to be reproved because the published versions use different assumptions (e.g., they do not permit satiation), but the proof follows known ideas.

Suppose $q \in \bigcap_{i=1}^k \partial u_i(x_i)$, and define p by (8.6). Let $y' \in R_+^k$, and let $w(y')$ be attained at $(x'_1, \ldots, x'_k)$. From $\sum y_i x_i = 0 = \sum y'_i x'_i$ it follows that

$$\sum y'_i q \cdot (x'_i - x_i) = \sum (y'_i - y_i)(-q \cdot x_i).$$

Hence

$$\begin{aligned} w(y') - w(y) &= \sum (y'_i u_i(x'_i) - y_i u_i(x_i)) \\ &= \sum [(y'_i - y_i) u_i(x_i) + y'_i(u_i(x'_i) - u_i(x_i))] \\ &\leq \sum [(y'_i - y_i) u_i(x_i) + y'_i q \cdot (x'_i - x_i)] \\ &= \sum (y'_i - y_i)(u_i(x_i) - q \cdot x_i) = p \cdot (y' - y), \end{aligned}$$

i.e., $p \in \partial w(y)$.

Conversely, let $p \in \partial w(y)$. Let G_i denote the "strict subgraph" of $u_i(x) - p_i$, i.e.,

$$G_i = \{(x, \theta) \colon x \in X_i \text{ and } \theta < u_i(x) - p_i\}.$$

We assert that

(A.5) 0 is not in the convex hull of $\bigcup_i G_i$.

Indeed, suppose 0 *is* in this convex hull. Then there are nonnegative $y'_1, \ldots, y'_k$ summing to 1, and (x'_i, θ_i) in G_i, such that $\sum y'_i(x'_i, \theta_i) = 0$, i.e.

(A.6) $0 = \sum y'_i x'_i$

and

(A.7) $0 = \sum y'_i \theta_i$.

From (A.6) it follows that

$$\sum y'_i u_i(x'_i) \leq w(y');$$

hence from $\theta_i < u_i(x'_i) - p_i$ and (A.7) we obtain

$$0 < \sum [y'_i u_i(x'_i) - y'_i p_i] \leq w(y') - p \cdot y',$$

in contradiction to (A.3). Thus (A.5) is established.

From (A.5) it follows that there is a hyperplane through 0 that supports all the G_i; i.e., a nonzero vector $(-q, q_0)$ in R^{d+1} such that

(A.8) $-q \cdot x + q_0 \theta \leq 0$

whenever $\theta < u_i(x) - p_i$, and hence also whenever $\theta < u_i(x) - p_i$. Choosing θ to be a negative number with a large absolute value shows that q_0 cannot be negative. If $q_0 = 0$, then $q \neq 0$ and $q \cdot x \geq 0$ for all x in all X_i; but this contradicts $0 \in \operatorname{Int} \sum y_i X_i$. Thus $q_0 > 0$, so w.l.o.g. $q_0 = 1$. Setting $\theta = u_i(x) - p_i$ in (A.8) we then obtain

(A.9) $-q \cdot x + u_i(x) - p_i \leq 0$ for all $x \in X_i$,

and in particular when $x = x_i$. If strict inequality would hold in (A.9) for one of the x_i, then since $y \in R_{++}^k$,

$$0 > \sum y_i(-q \cdot x_i + u_i(x_i) - p_i) = -q \cdot \sum y_i x_i + w(y) - p \cdot y = 0,$$

where the last equation follows from $\sum y_i x_i = 0$ and (A.2). Therefore

(A.10) $p_i = u_i(x_i) - q \cdot x_i$

for all *i*. From (A.10) and (A.9) we obtain $q \in \partial u_i(x_i)$ for all *i*, and together with (A.10), this is what was to be proved.

APPENDIX B

THE VALUE CONVERGENCE THEOREM

In this Appendix we prove Proposition 8.8. Results that are in many respects similar appear in Champsaur (1975) and Hart (1977a); but they do not contain quite what we need here,[55] and it is simpler and quicker to prove what we need directly than to derive it from those results.

Lemmas B.9 and B.10 constitute the probabilitistic foundations of the paper. The proofs of these lemmas do not depend on what goes before.

Let U be the convex neighborhood of $(1, \ldots, 1)$ mentioned in the proposition;[56] since the w_i are 1-homogeneous, we may take U to be an open cone. Denote by w an arbitrary concave 1-homogeneous function on U, and by $w'(y, z)$ its directional derivative $\lim_{\theta\to 0+} [w(y+\theta z) - w(y)]/\theta$ in the direction z.

LEMMA B.1: *Let $y_n \to y \in U$ and $z_n \to z \in R^k$. Then*

$$\liminf_{n\to\infty} w'_n(y_n; z_n) \geq w'_\infty(y; z).$$

PROOF: Rockafellar (1970, Theorem 2.4.5, p. 233). *Q.E.D.*

For $z \in R^k$, set $\|z\| = \max_i z_i$.

COROLLARY B.2: *Let $y \in U$ and $z \in R^k$. Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$w'_n(\hat{y}; \hat{z}) \geq w'_\infty(y; z) - \varepsilon$$

whenever

$$\|\hat{y} - y\| < \delta, \quad \|\hat{z} - z\| < \delta, \quad \text{and} \quad n > 1/\delta.$$

PROOF: Immediate from Lemma B.1.

LEMMA B.3: *Let $y \in U$ and $z \in R^k$. Then*

(B.4) $\quad w'(y; z) \geq w(y+z) - w(y) \quad \text{if} \quad y+z \in U;$

(B.5) $\quad -w'(y; -z) \geq w'(y; z);$

(B.6) $\quad w(y) - w(y-z) \geq w'(y; z) \quad \text{if} \quad y-z \in U;$

(B.7) $\quad$ *$w'(y; z)$ is 0-homogeneous*[57] *in y and 1-homogeneous in z.*

PROOF: The concavity of w yields (B.4). For (B.5), see Rockafellar (1970, Theorem 2.3.1, p. 214). For (B.6), substitute $-z$ for z in (B.4) and apply (B.5). The definition of $w'(y; z)$ yields (B.7) directly. *Q.E.D.*

LEMMA B.8: *Let $y \in U$, and let p in R^k be such that $p \cdot z \geq w'(y; z)$ for all z in R^k_+, and $p \cdot y = w(y)$. Then $p \in \partial w(y)$.*

PROOF: Let $y' \in U$. Then $y' \in R^k_{++}$, since U is an open cone in R^k_+. Hence $\theta y' = y$ for sufficiently large positive θ, and hence there is a z in R^k_+ such that $y + z = \theta y'$. Hence by (B.4),

$$p \cdot z \geq w'(y; z) \geq w(y+z) - w(y) = w(y+z) - p \cdot y;$$

hence by the homogeneity of w,

$$p'(\theta y') = p \cdot (y+z) \geq w(y+z) = w(\theta y') = \theta w(y');$$

hence $p \cdot y' \geq w(y')$, and hence by Lemma A.1, $p \in \partial w(y)$. *Q.E.D.*

LEMMA B.9: *Let $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots$ be independent identically distributed random variables with mean 0, all bounded in absolute value by 1. Set $\underset{\sim}{\sigma}_m = \underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_m$, and let $0 < \delta < 1$. Then for all m,*

$$\text{prob}\left\{\left|\frac{\underset{\sim}{\sigma}_m}{m}\right| \geq \delta\right\} \leq 2e^{-(\delta^2/4)m}.$$

[55] For example, we do not have concavity on the entire nonnegative orthant, but only on a conical neighborhood of the diagonal.

[56] Not the U of (8.2).

[57] Homogeneous of degree 0.

REMARK: This provides an explicit bound for the residual probability in the weak law of large numbers. The particular expression $\delta^2/4$ is of no importance; for our purposes, it can be replaced by any positive constant depending on δ only (but not on m or on the distribution of the x_i). We are grateful to B. Weiss for providing the following elegant direct proof.

PROOF: Let $0<\lambda<1$. Since $E\underset{\sim}{x}_i=0$, we have

$$\begin{aligned}\text{prob}\,\{\underset{\sim}{\sigma}_m \geqslant m\delta\}\, e^{\lambda m\delta} &\leqslant Ee^{\lambda \underset{\sim}{\sigma}_m} = (E\,e^{\lambda \underset{\sim}{x}_1})^m \\ &= \left(1+\lambda(E\underset{\sim}{x}_1)+\frac{\lambda^2}{2}(E\underset{\sim}{x}_1^2)+\cdots\right)^m \\ &\leqslant \left(1+\frac{\lambda^2}{2}+\frac{\lambda^3}{3!}+\cdots\right)^m \leqslant (1+\lambda^2)^m \leqslant e^{m\lambda^2}.\end{aligned}$$

Setting $\lambda=\delta/2$, we deduce

$$\text{prob}\,\{\underset{\sim}{\sigma}_m \geqslant m\delta\} \leqslant e^{-m\delta^2/4}.$$

By substituting $-\underset{\sim}{x}_i$ for $\underset{\sim}{x}_i$, we deduce that also

$$\text{prob}\,\{\underset{\sim}{\sigma}_m \leqslant -m\delta\} \leqslant e^{-m\delta^2/4}.$$

Q.E.D.

We now proceed to calculate ϕv_n, using (4.3). Pick at random an order $\underset{\sim}{\mathscr{R}}^n$ on T^n, assigning equal probability to all $(nk)!$ orders. For $0\leqslant m \leqslant nk$, denote by $\underset{\sim}{Q}_m^n$ the coalition[58] consisting of the first m agents in the order $\underset{\sim}{\mathscr{R}}^n$.

LEMMA B.10: *For* $1>\delta>0$ *and* $S\subset T^n$, *we have*

$$\text{prob}\left\{\left|\frac{|\underset{\sim}{Q}_m^n\cap S|}{m}-\frac{|S|}{nk}\right|\geqslant\delta\right\}\leqslant 2e^{-(\delta^2/4)m}.$$

REMARK: This is the weak law of large numbers for sampling without replacement, with an explicit bound as in the previous lemma.

PROOF: Fix S, and write $\underset{\sim}{r}_m^n=|S\cap \underset{\sim}{Q}_m^n|$. The sequence $\underset{\sim}{r}_1^n,\ldots,\underset{\sim}{r}_{nk}^n$ may be obtained by choosing traders at random, one at a time from the population of all nk traders, *without* replacement; if choosing an agent in S is considered as "success," then $\underset{\sim}{r}_m^n$ is the number of successes in the first m trials. Now let $\underset{\sim}{s}_m$ be the number of successes in m trials when traders are chosen at random from the same population, but *with* replacement (here m may be as large as we like); i.e., the number of successes in m independent Bernoulli trials, each with success probability $|S|/nk$. Note that both $\underset{\sim}{r}_m^n$ and $\underset{\sim}{s}_m$ have mean $m|S|/nk$. Our proof is based on the principle that "sampling without replacement is uniformly better than sampling with replacement."[59] Here "uniformly better" means that the probability of any given deviation from the mean is smaller; thus if

$$\mathscr{F}=\left\{\left|\frac{\underset{\sim}{r}_m^n}{m}-\frac{|S|}{nk}\right|\geqslant\delta\right\},\quad \mathscr{G}=\left\{\left|\frac{\underset{\sim}{s}_m}{m}-\frac{|S|}{nk}\right|\geqslant\delta\right\},$$

then prob $\mathscr{F}\leqslant$ prob $\mathscr{G}$. Noting that $(\underset{\sim}{s}_m/m)-(|S|/nk)$ is of the form $\underset{\sim}{\sigma}_m/m$ in the previous lemma completes the proof. *Q.E.D.*

PROOF OF PROPOSITION 8.8: Let e^i denote the ith unit vector $(0,\ldots,0,1,0,\ldots,0)$ in R^k, and let $e=(1,\ldots,1)$. Let z be an arbitrary but, for the time being, fixed point in $[0,1]^k/k$. Let $S^1,S^2,\ldots$ be a sequence of coalitions in $T^1,T^2,\ldots$ respectively such that

$$\text{(B.11)}\qquad \lim_{n\to\infty}\frac{\eta(S^n)}{nk}=z;$$

in fact, we choose S^n so that $\eta(S^n)\neq 0$ and in the maximum norm,

$$\text{(B.12)}\qquad \left\|\frac{\eta(S^n)}{nk}-z\right\|\leqslant\frac{1}{nk}.$$

[58] The wiggles are to indicate that $\underset{\sim}{\mathscr{R}}^n$ and $\underset{\sim}{Q}_m^n$ are random variables.

[59] Cf. e.g. Aumann and Shapley (1974, p. 135, Note 1), or Champsaur (1975, p. 415, 6.13).

Let $\pi^n_m = 1$ or 0 according as to whether the mth trader in the order $\mathscr{R}^n$ is or is not in S^n. Set

$$\text{(B.13)} \qquad y^{mn} = \eta(Q^n_m), \quad z^{mn} = \frac{\eta(S^n \cap Q^n_m)}{m}.$$

For all $z \in R^k_+$, set $z_* = z/\sum z_i$ if $z \neq 0$, $z_* = 0$ if $z = 0$.

The value of a type i player in v_n is p^n_i, where p^n is as in (8.9); hence by the efficiency of the value,

$$p^n \cdot ne = p^n \cdot \eta(T^n) = (\phi v_n)(T^n) = v_n(T^n) = w_n(\eta(T^n)) = w_n(ne),$$

and so by the 1-homogeneity of w_n,

$$\text{(B.14)} \qquad p^n \cdot e = w_n(e) \to w_\infty(e) \quad \text{as} \quad n \to \infty.$$

Next, by (4.3) we have

$$\text{(B.15)} \qquad p^n \cdot \eta(S^n) = (\phi v_n)(S^n) = E \sum_{m=1}^{nk} (v_n(Q^n_m) - v_n(Q^n_{m-1}))\pi^n_m$$

$$= \sum_{m=1}^{nk} E[(w_n(\eta(Q^n_m)) - w_n(\eta(Q^n_{m-1})))\pi^n_m] = \sum_{m=1}^{nk} E\Delta^n_m,$$

the Δ^n_m being defined by the expressions in square brackets. Now Δ^n_m is either 0, or it is $w_n(y^{mn}) - w_n(y^{mn} - e^i)$ for some type i. The latter happens if and only if the mth trader in $\mathscr{R}^n$ is of type i *and* is in S^n; given Q^n_m, the probability of this is $\eta_i(S^n \cap Q^n_m)/m = z^{mn}_i$. Setting

$$\text{(B.16)} \qquad \Gamma^n_m = E(\Delta^n_m \mid Q^n_m),$$

we thus conclude that

$$\text{(B.17)} \qquad \Gamma^n_m = \sum_{i=1}^{k} [w_n(y^{mn}) - w_n(y^{mn} - e^i)]z^{mn}_i.$$

Now define the event $\mathscr{E}^n_m$ by

$$\text{(B.18)} \qquad \mathscr{E}^n_m = \{y^{mn} - [0,1]^k \subset U\}.$$

If $\mathscr{E}^n_m$ obtains, then the concavity of w_n on U, (B.6), and (B.7) yield

$$\text{(B.19)} \qquad \Gamma^n_m = \left(\sum_{i=1}^{k} [w_n(y^{mn}) - w_n(y^{mn} - e^i)]z^{mn}_{*i}\right) \sum z^{mn}_i$$

$$\geq (w_n(y^{mn}) - w_n(y^{mn} - z^{mn}_*)) \sum z^{mn}_i$$

$$\geq w'_n(y^{mn}; z^{mn}_*) \sum z^{mn}_i$$

$$= w'_n(y^{mn}; z^{mn}) = w'_n\left(\frac{y^{mn}}{m}; z^{mn}\right).$$

Now let $\varepsilon > 0$ be given. Choose δ in accordance with Corollary B.2, with $y = e/k$ and z as chosen at the beginning of the proof. Define events $\mathscr{D}^n_m$, $\mathscr{C}^n_m$, and $\mathscr{B}^n_m$ by

$$\mathscr{D}^n_m = \left\{\left\|\frac{y^{mn}}{m} - \frac{e}{k}\right\| < \delta\right\}, \quad \mathscr{C}^n_m = \left\{\left\|z^{mn} - \frac{\eta(S^n)}{nk}\right\| < \frac{\delta}{2}\right\}, \quad \mathscr{B}^n_m = \mathscr{D}^n_m \cap \mathscr{C}^n_m.$$

By setting $S = T^n_i$ in Lemma B.10, we deduce

$$\text{Prob (not } \mathscr{D}^n_m) \leq 2k\, e^{-m\delta^2/4}.$$

Similarly, by setting $S = S^n \cap T^n_i$, we deduce

$$\text{prob (not } \mathscr{C}^n_m) \leq 2k\, e^{-m\delta^2/16},$$

and hence

$$\text{(B.20)} \qquad \text{prob (not } \mathscr{B}^n_m) \leq 4k\, e^{-m\delta^2/16}.$$

Now from (B.12) we deduce that

$$\text{(B.21)} \qquad \mathscr{B}^n_m \subset \mathscr{C}^n_m \subset \{\|z^{mn} - z\| < \delta\} \quad \text{whenever} \quad n > 2/\delta.$$

Moreover, we may assume w.l.o.g. that δ is chosen sufficiently small so that

$$\left\| y - \frac{e}{k} \right\| < 2\delta \quad \Rightarrow \quad y \in U;$$

as U is a cone, this yields

$$\left\| \frac{y}{m} - \frac{e}{k} \right\| < 2\delta \quad \Rightarrow \quad \frac{y}{m} \in U \quad \Rightarrow \quad y \in U.$$

Since $x \in [0, 1]^k$ and $m > 1/\delta$ imply $\|x/m\| < \delta$, we deduce

$$\text{(B.22)} \qquad \mathscr{B}^n_m \subset \mathscr{D}^n_m \subset \mathscr{E}^n_m \quad \text{whenever} \quad m > 1/\delta.$$

Combining (B.19), (B.21), (B.22), Corollary B.2 and (B.7), we obtain that for $n > 2/\delta$ and $m > 1/\delta$

$$\text{(B.23)} \qquad E(\Gamma^n_m | \mathscr{B}^n_m) \geq w'_\infty\left(\frac{e}{k}; z\right) - \varepsilon = w'_\infty(e; z) - \varepsilon.$$

Since $n \geq m/k$, (B.23) holds whenever $m > 2k/\delta$.

Let μ be a uniform bound on the $|w_n|$ in $[0, 1]^k$. By (B.17), we always have

$$\text{(B.24)} \qquad |\Gamma^n_m| \leq 2k\mu m.$$

By (B.16), $E(\Delta^n_m) = E(\Gamma^n_m)$. Hence by (B.15), (B.23), and (B.24), for $m > 2k/\delta$ we have

$$\begin{aligned} p^n \cdot \eta(S^n) &= \sum_{m=1}^{nk} E(\Delta^n_m) = \sum_{m=1}^{nk} E(\Gamma^n_m) \\ &= \sum_{m=1}^{nk} [E(\Gamma^n_m | \mathscr{B}^n_m) \operatorname{prob} \mathscr{B}^n_m + E(\Gamma^n_m | \text{not } \mathscr{B}^n_m) \operatorname{prob} (\text{not } \mathscr{B}^n_m)] \\ &\geq nkw'_\infty(e; z) - nk\varepsilon - \sum_{m=1}^{nk} 4k\mu m \operatorname{prob} (\text{not } \mathscr{B}^n_m); \end{aligned}$$

and hence by (B.20)

$$\text{(B.25)} \qquad p^n \cdot \frac{\eta(S^n)}{nk} \geq w'_\infty(e; z) - \varepsilon - \left[\frac{1}{nk} \sum_{m=1}^{nk} m\, e^{-m\delta^2/16}\right] 16k^2\mu.$$

Denote by c_n the expression in square brackets. Then c_n is the Cesaro mean of the sequence $\{m\, e^{-m\delta^2/16}\}$, which tends to 0 as $m \to \infty$; hence also $c_n \to 0$ as $n \to \infty$. In particular, for n sufficiently large, we have $16k^2\mu c_n < \varepsilon$, and then

$$\text{(B.26)} \qquad p^n \cdot \frac{\eta(S^n)}{nk} \geq w'_\infty(e; z) - 2\varepsilon.$$

In the particular case in which $z = e^i$, we may choose $S^n = T^n_i$, and then (B.26) yields

$$p^n_i = p^n \cdot \frac{\eta(S^n)}{n} \geq kw'_\infty(e; e^i) - 2k\varepsilon.$$

Thus the p^n_i are bounded from below, and by (B.14), $\sum p^n_i$ is bounded from above. Hence $\{p^n\}$ is bounded. If p is a limit point of $\{p^n\}$, then (B.12) and (B.26) yield

$$p \cdot z \geq w'_\infty(e; z) - 2\varepsilon.$$

Since this is true for each ε, we deduce

$$\text{(B.27)} \qquad p \cdot z \geq w'_\infty(e; z).$$

Now (B.27) holds for any z in $[0, 1]^k/k$, and so by (B.7), for any z in R^k_+. By (B.14), $p \cdot e = w_\infty(e)$. Hence by Lemma B.8, $p \in \partial w_\infty(e)$, as was to be proved.

APPENDIX C

FURTHER PROPERTIES OF DIVIDENDS AND WEIGHTS

This Appendix is devoted to proving the results stated in Section 12.

W.l.o.g. $\boldsymbol{x}^\infty$ does not satiate all types (see footnote 35), which implies that there are lightweight types. If none of the λ_i^n tend to 0, it follows that all types are lightweight; hence by (9.27), $c_i = 0$ for all i, and hence $(q, \boldsymbol{x}^\infty)$ is a competitive equilibrium, which we have excluded (12.1). Thus

(C.1) *there are both heavyweight and lightweight types.*

Formulas (9.27) through (9.29) indicate that to gain information about the dividends, we should study $(\phi v_H^n)(T_i^n)$ for lightweight i. Define $\underset{\sim}{\mathscr{R}}^n$, $\underset{\sim}{Q}_m^n$ and $\underset{\sim}{y}^{mn}$ as in Appendix B (just before Lemma B.10, and (B.13)). Let $\underset{\sim}{\pi}_i^{mn} = 1$ or 0 according as to whether the mth trader in the order $\underset{\sim}{\mathscr{R}}^n$ is or is not of Type i, and set

(C.2) $$\underset{\sim}{\Delta}_i^{mn} := (v_H^n(\underset{\sim}{Q}_m^n) - v_H^n(\underset{\sim}{Q}_{m-1}^n))\underset{\sim}{\pi}_i^{mn} = (w_H^n(\underset{\sim}{y}^{mn}) - w_H^n(\underset{\sim}{y}^{(m-1)n}))\underset{\sim}{\pi}_i^{mn}.$$

$\underset{\sim}{\Delta}_i^{mn}$ represents the contribution of the mth player to v_H^n *if* he is of Type i; otherwise it is 0. Hence

(C.3) $$(\phi v_H^n)(T_i^n) = E \sum_{m=1}^{nk} \underset{\sim}{\Delta}_i^{mn} = \sum_{m=1}^{nk} E\underset{\sim}{\Delta}_i^{mn}.$$

LEMMA C.4: *If i is lightweight and N a nonnegative integer, then uniformly in n,*

$$(\phi v_H^n)(T_i^n) = \sum_{m=1}^{N} E\underset{\sim}{\Delta}_i^{mn} + o(1).$$

REMARK: The error $o(1)$ is to be understood as a function of N. In words, the lemma says that the total contribution $(\phi v_H^n)(T_i^n)$ of *all* Type i traders to the heavyweights can be approximated arbitrarily closely by the contribution of the Type i traders that are among the first N traders of the order, where N is a fixed finite number that is independent of the number n of times that the market is replicated. We will in fact show that the error decreases exponentially, i.e. that there is a positive α, independent of n, such that the error is $O(e^{-\alpha N})$.

PROOF: By Lemma 9.11, we may choose δ sufficiently small so that $\delta < 1$ and

$$\left\| y - \frac{e}{k} \right\| < \delta \quad \Rightarrow \quad y \in U_L.$$

Define[60]

$$\mathscr{D}_m^n := \left\{ \left\| \frac{\underset{\sim}{y}^{mn}}{m} - \frac{e}{k} \right\| < \delta \right\}.$$

Using Lemma B.10 with $S = T_j^n$ for each j, we find

$$\text{prob (not } \mathscr{D}_m^n) \leq 2k\, e^{-m\delta^2/4},$$

$$\text{prob (not } \mathscr{D}_{m-1}^n) \leq 2k\, e^{-(m-1)\delta^2/4} \leq 3k\, e^{-m\delta^2/4}.$$

Setting

$$\mathscr{F}_m^n := \{\underset{\sim}{y}^{mn} \text{ and } \underset{\sim}{y}^{(m-1)n} \text{ are in } U_L\},$$

we deduce

(C.5) $$\text{prob (not } \mathscr{F}_m^n) \leq 5k\, e^{-m\delta^2/4}.$$

Since i is lightweight, it follows from (9.6) and (C.2) that

(C.6) $$\underset{\sim}{\Delta}_i^{mn} = 0 \quad \text{whenever } \mathscr{F}_m^n \text{ obtains.}$$

[60] This is as in Appendix B, with U_L instead of U.

Letting $\mu_i := \max u_i$ as in Section 9.3, and setting $\mu := \sum_{i\in H}\mu_i$, we find that $0 \le w_H^n(y) \le \mu$ whenever $y\in[0,1]^k$. From (C.2) and the 1-homogeneity of w_H^n (see (8.4)), it follows that always

(C.7) $\quad 0 \le \underset{\sim}{\Delta}_i^{mn} \le m\mu.$

Combining (C.5) through (C.7) with (C.3) and (C.2), we find

(C.8) $$(\phi v_H^n)(T_i^n) = \sum_{m=1}^{N} E\underset{\sim}{\Delta}_m^n + \sum_{m=N+1}^{nk} E(\underset{\sim}{\Delta}_i^{mn} \mid \text{not } \mathscr{F}_m^n)\,\text{prob}\,(\text{not } \mathscr{F}_m^n)$$

$$\le \sum_{m=1}^{N} E\underset{\sim}{\Delta}_i^{mn} + 5k\mu \sum_{m=N+1}^{nk} m\, e^{-m\delta^2/4}$$

The second term in the last expression is part of the tail of a convergent series, and therefore tends to 0 uniformly as $N\to\infty$. This proves the lemma. The explicit bound on the error term in the remark follows from a slightly closer look at the tail. *Q.E.D.*

COROLLARY C.9: *If i is heavyweight and N is a nonnegative integer, then uniformly in n,*

$$(\phi v_H^N)(T_i^n) = \left(n - \frac{N}{k}\right)\lambda_i^n \mu_i + \sum_{m=1}^{N} E\underset{\sim}{\Delta}_i^{mn} + o(1).$$

REMARK: As in Lemma C.4, the error is a function of N, and decreases exponentially. The difference between the two cases is in the first term on the right, which reflects the fact that a heavyweight trader makes a contribution to the heavyweights even when they are all satiated, simply by adding his own consumption to that of the coalition. As $n\to\infty$, this contribution of the heavyweights cannot be approximated by a fixed finite N.

PROOF: Follows the proof of Lemma C.3. Formula (C.5) must be replaced by

(C.10) $\quad \underset{\sim}{\Delta}_i^{mn} = \lambda_i^n \mu_i$ whenever $\mathscr{F}_m^n$ obtains.

In (C.8), this leads to the additional term

$$\sum_{m=N+1}^{nk} E(\underset{\sim}{\Delta}_i^{mn} \mid \mathscr{F}_m^n)\,\text{prob}\,\mathscr{F}_m^n = \left(n - \frac{N}{k}\right)\lambda_i^n\mu_i + o(1),$$

where, as before, the term $o(1)$ decreases exponentially in N, uniformly in n. *Q.E.D.*

LEMMA C.11: *If i is lightweight, then as n varies, $(\phi v_H^n)(T_i^n)$ remains bounded and bounded away*[61] *from 0. If i is heavyweight, then as n varies, $(\phi v_H^n)(T_i^n) - n\lambda_i^n\mu_i$ remains bounded.*

PROOF: For the boundedness part, which holds in both cases, choose $N=0$ in Lemma C.4 (for $i\in L$) and Corollary C.10 (for $i\in H$); the result follows since the error term in those results is uniform in n. To show the boundedness away from 0 for lightweight i, note that $\underset{\sim}{\Delta}_i^{mn}$ is always ≥ 0, so it is enough to show that Δ_i^{2n}, say, is bounded away from 0. Now the probability is $> 1/k^2$ that the first trader in $\mathscr{R}^n$ is of Type 1 (which is heavyweight by (C.1)), and the second trader is of Type i. In that case

$$\underset{\sim}{\Delta}_i^{2n} = \lambda_1^n \max\{u_1(x) : x \in X_1 \cap (-X_i)\}.$$

Since 0 does not satiate 1, and is in the interior of both X_1 and $-X_i$ (see (2.3)), it follows that the max on the right is a fixed positive number, say $\zeta > 0$. The λ_i^n are all positive by definition, and since 1 is the "heaviest" type, they do not tend to 0; therefore they have a positive minimum, say $\beta > 0$. It follows that

$$(\phi v_H^n)(T_i^n) \ge E\underset{\sim}{\Delta}_i^{2n} > \beta\zeta/k^2 > 0. \qquad Q.E.D.$$

[61] Greater than a positive constant.

PROOF OF THEOREM 12.6: W.l.o.g. restrict the convergence indicator so that $n\lambda_\ell^n$ converges or tends to infinity. By (9.23), if i is lightweight, then

$$(C.12)\qquad (\phi v_H^n)(T_i^n)=n\lambda_\ell^n q\cdot x_i+o(n\lambda_\ell^n).$$

By Lemma C.11, the left side is bounded away from 0, so

$$(C.13)\qquad 0<\lim n\lambda_\ell^n.$$

Suppose $\lim n\lambda_\ell^n=\infty$. Since the left side of (C.12) is bounded (Lemma C.11), it follows that $q\cdot x_i^\infty=0$; this holds for all $i\in L$. If $i\in H$, then (9.24) yields

$$(\phi v_H^n)(T_i^n)-n\lambda_i^n\mu_i=n\lambda_\ell^n q\cdot x_i^\infty+o(n\lambda_\ell^n).$$

By Lemma C.11, the left side is bounded in n, and so again, since $\lim n\lambda_\ell^n=\infty$, we must have $q\cdot x_i^\infty=0$. Thus $q\cdot x_i^\infty=0$ both for $i\in L$ and for $i\in H$, i.e., for all i. But this implies that (q, x^∞) is a competitive equilibrium (cf. the end of Section 9), which we have ruled out (C.1). Hence $\lim n\lambda_\ell^n<\infty$, and so by (C.13),

$$0<\lim n\lambda_\ell^n<\infty.$$

Since by Corollary 9.26, the λ_i^n have the same order of magnitude for all lightweight i, the theorem is proved. *Q.E.D.*

PROOF OF THEOREM 12.4: Suppose $i\in L$. By Lemma C.11, $(\phi v_H^n)(T_i^n)$ is bounded away from 0; by Theorem 12.6, proven above, $n\lambda_\ell^n$ is bounded. Hence $c_i>0$ by (9.27). If $i\in H$, the result follows from (9.28), (9.29), and from its truth for $i\in L$. *Q.E.D.*

We come next to the explicit formula for the dividends, Theorem 12.9. Note that the $\underset{\sim}{\rho}^m$ are i.i.d. r.v.'s, which take on each of the values $e^1,\dots,e^k$ in R_+^k with equal probability (recall that $e^i=(0,\dots,0,1,0,\dots,0)$). Set

$$(C.14)\qquad \underset{\sim}{y}^m:=\underset{\sim}{\rho}^1+\cdots+\underset{\sim}{\rho}^m.$$

PROPOSITION C15: *For lightweight i,*

$$0<\lambda_* c_i=\lim_{n\to\infty}(\phi v_H^n)(T_i^n)=\sum_{m=1}^{\infty}E((w_H(\underset{\sim}{y}^m)-w_H(\underset{\sim}{y}^{m-1}))\underset{\sim}{\rho}_i^m)$$
$$=\sum_{m=0}^{\infty}\frac{1}{k}E(w_H(\underset{\sim}{y}^m+e^i)-w_H(\underset{\sim}{y}^m)).$$

REMARK: In particular, it is asserted that the limit exists and is finite, and the series converge to a finite limit.

PROOF: The positivity, $0<\lambda_* c_i$, follows from (12.7) and Theorem 12.4 (proven above). The equality of $\lambda_* c_i$ with the limit is (9.27). The equality between the two sums is straightforward. It remains only to prove that the limit equals the first sum.

First note that if $z^n\to z$, then

$$(C.16)\qquad w_H^n(z)\to w_H(z).$$

Let us now examine the behavior of $E\underset{\sim}{\Delta}_i^{mn}$ for fixed m, as $n\to\infty$. Each of the random variables $\underset{\sim}{y}^{mn}$ and $\underset{\sim}{y}^m$ has precisely k^m possible values; as $n\to\infty$, the distribution of $\underset{\sim}{y}^{mn}$ approaches that of $\underset{\sim}{y}^m$. Noting that $\underset{\sim}{\pi}_i^{mn}=\underset{\sim}{y}_i^{mn}-\underset{\sim}{y}_i^{(m-1)n}$ and $\underset{\sim}{\rho}_i^m=\underset{\sim}{y}_i^m-\underset{\sim}{y}_i^{m-1}$, and using (C.16), we deduce that as $m\to\infty$,

$$(C.17)\qquad E\underset{\sim}{\Delta}_i^{mn}\to E((w_H(\underset{\sim}{y}^m)-w_H(\underset{\sim}{y}^{m-1}))\underset{\sim}{\rho}_i^m).$$

Now let $\varepsilon>0$ be given. By Lemma C.4, there is an N_0 such that for all $N>N_0$, and for all n,

$$\left|(\phi v_H^n)(T_i^n)-\sum_{m=1}^{N}E\underset{\sim}{\Delta}_i^{mn}\right|\leq\varepsilon.$$

Letting $n\to\infty$ on the left while keeping N fixed, and using (C.17), yields

$$\left|\lim_{n\to\infty}(\phi v_H^n)(T_i^n)-\sum_{m=1}^{N}E((w_H(\underset{\sim}{y}^m)-w_H(\underset{\sim}{y}^{m-1}))\underset{\sim}{\rho}_i^m)\right|\leq\varepsilon$$

for all $N>N_0$. Since ε was chosen arbitrarily, the proof is complete. *Q.E.D.*

PROOF OF THEOREM 12.9: Follows from Proposition C.15, by noting that

$$(C.18)\qquad \underset{\sim}{y}^m=\eta(\underset{\sim}{Q}_m),\quad w_H(\underset{\sim}{y}^m)=v_H(\underset{\sim}{Q}^m),\quad w_H(\underset{\sim}{y}^m+e^i)=v_H(\underset{\sim}{Q}^m\cup t_i).\qquad Q.E.D.$$

COROLLARY C.19: *For all heavyweight i,*

$$\lambda_* q \cdot x_i^\infty = \lim_{n\to\infty} ((\phi v_H^n)(T_i^n) - n\lambda_i^n \mu_i) = \frac{1}{k}\sum_{m=0}^{\infty} E(w_H(\underset{\sim}{y}^m + e^i) - w_H(\underset{\sim}{y}^m) - \lambda_i^\infty \mu_i)$$
$$= \frac{1}{k}\sum_{m=0}^{\infty} E(v_H(\underset{\sim}{Q}_m \cup t_i) - v_H(\underset{\sim}{Q}_m) - \lambda_i^\infty \mu_i).$$

PROOF: The first equality is (9.24). The second equality follows as in the proof of Proposition C.15, using Corollary C.9 instead of Lemma C.4. The third equality follows from (C.18). *Q.E.D.*

Before proceeding, we should point out that whereas v_H is formally defined as $w_H \circ \eta$, the two represent slightly different concepts. In $v_H(S)$, the maximum is over S-allocations that need not be equal treatment; that may give different bundles to traders of the same type (cf. (4.1)). This was used already in the proof of Lemma 9.14, and will be used again in the proof of Lemma C.20 below. Because the X_i are convex and the u_i concave, the maximum is in fact achieved at equal treatment allocations, and this enables us to express v_H by $w_H \circ \eta$.

Denote by T_i^∞ the infinite pool of Type i traders, and set $T^\infty = T_1^\infty \cup \cdots \cup T_k^\infty$. Recall that t_i denotes an additional Type i trader, outside of the pool. Like before, $u_t = u_i$, $X_t = X_i$, and $\lambda_t^\infty = \lambda_i^\infty$ whenever t is of Type i.

PROPOSITION C.20: *Let $\alpha_1, \ldots, \alpha_k$ be nonnegative numbers whose sum α is $\leqslant 1$, let $S \subset T^\infty$, and let j be such that $\lambda_j^\infty = 0$ and*

(C.21) $\quad X_j \supset \sum \alpha_i X_i.$

Then

(C.22) $\quad v_H(S \cup t_j) - v_H(S) \geqslant \sum \alpha_i (v_H(S \cup t_i) - v_H(S) - \lambda_i^\infty \mu_i).$

PROOF: Assume first that $\alpha = 1$. Suppose that $v_H(S \cup t_i)$ is achieved at an $(S \cup t_i)$-allocation that assigns to each t in $S \cup t_i$ the bundle x_t^i; that is, $x_t^i \in X_t$,

(C.23) $\quad \lambda_i^\infty u_i(x_{t_i}^i) + \sum_{t\in S} \lambda_t^\infty u_t^\infty(x_t^i) = v_H(S \cup t_i),\quad$ and

(C.24) $\quad x_{t_i}^i + \sum_{t\in S} x_t^i = 0.$

Now for $t \in S$, assign to t the bundle $x_t := \sum \alpha_i x_t^i$, and to t_j, assign the bundle $x_{t_j} := \sum \alpha_i x_{t_i}^i$. The concavity of X_t yields $x_t \in X_t$ when $t \in S$, (C.21) yields $x_{t_j} \in X_j$, and (C.24) yields $\sum_{t\in S\cup t_j} x_t = 0$. Thus x is an $(S \cup t_j)$-allocation, and hence using $\lambda_j^\infty = 0$, the concavity of the u_t, and (C.23), we get

(C.25)
$$v_H(S \cup t_j) \geqslant \sum_{t\in S} \lambda_t^\infty u_t(x_t) + \lambda_j^\infty u_j(x_j)$$
$$= \sum_{t\in S} \lambda_t^\infty u_t(x_t) \geqslant \sum_{t\in S} \lambda_t^\infty \sum \alpha_i u_t(x_t^i)$$
$$= \sum \alpha_i \sum_{t\in S} \lambda_t^\infty u_t(x_t^i) = \sum \alpha_i (v_H(S \cup t_i) - \lambda_i^\infty u_i(x_{t_i}^i))$$
$$\geqslant \sum \alpha_i (v_H(S \cup t_i) - \lambda_i^\infty \mu_i);$$

this yields the result for $\alpha = 1$.

When $\alpha < 1$, we add an additional type, with index 0, net trade set $X_0 = \{0\}$, utility defined by $u_0(0) = 0$, and weight $\lambda_0^\infty = 0$. This violates the conditions $0 \in \operatorname{Int} X_i$ and $0 \notin B_i$, but in the above proof for the case $\alpha = 1$, no use was made of these conditions. Setting $\alpha_0 := 1 - \alpha$, and applying the previous case to this situation, yields (C.22) in this case as well. *Q.E.D.*

PROOF OF THEOREM 12.5: Since $\lambda_i^\infty \mu_i = 0$ for lightweight i, the theorem follows from Theorem 12.9 (proven above), (12.7), and Proposition C.20. *Q.E.D.*

LEMMA C.26: *If $q \cdot x_i^\infty \leqslant 0$, then $\lambda_i^\infty > 0$.*

PROOF: If $\lambda_i^\infty = 0$, then by (9.30), Proposition C.15, and Corollary C.19,

$$\lambda_* q \cdot x_i^\infty = \frac{1}{k}\sum_{m=0}^{\infty} E(w_H(\underset{\sim}{y}^m + e^i) - w_H(\underset{\sim}{y}^m)).$$

By the superadditivity (8.4) of w_H, each term on the right is nonnegative, and as in the proof of Lemma C.11, at least one must be positive. Hence $q \cdot x_i^\infty > 0$.

REMARK C.27: Let L' be the set of i with $\lambda_i^\infty = 0$. Theorem 12.5 says that the dividends are monotonically concave over L. We can replace L by L', thus strengthening the theorem, but only at the cost of revising the formulas for the dividends. Set $H' := H \backslash L'$. For $i \in L'$, define $c_i' := q \cdot x_i^\infty$. For $i \in H'$, define c_i' analogously to the definition of c_i for $i \in H$ ((9.28) and (9.29)); in words, c_i' is the smallest c_j' among those with $j \in L'$ and $X_j \supset X_i$, when there is such a j; otherwise, it is a sufficiently large finite number. By Lemma C.26, $x_i' > 0$ for i in L', and hence for i in H' as well. From Proposition C.20 it follows that c' is monotonically concave in the net trade sets of L'. It may, incidentally, be seen that c', like c, is monotonic over all the net trade sets. *Q.E.D.*

LEMMA C.28: *If* $\lambda_i^\infty > 0$, $\lambda_j^\infty = 0$, *and* $X_i \subset X_j$, *then* $q \cdot x_i^\infty < q \cdot x_j^\infty$.

PROOF: By (9.30), Proposition C.15, and Corollary C.19,

$$q \cdot x_i^\infty = \frac{1}{\lambda_* k} \sum_{m=0}^{\infty} E(v_H(\underset{\sim}{Q}_m \cup t_i) - v_H(\underset{\sim}{Q}_m) - \lambda_i^\infty \mu_i), \tag{C.29}$$

$$q \cdot x_j^\infty = \frac{1}{\lambda_* k} \sum_{m=0}^{\infty} E(v_H(\underset{\sim}{Q}_m \cup t_j) - v_H(\underset{\sim}{Q}_m)). \tag{C.30}$$

By Proposition (C.20) with $\alpha_i = 1$, each term on the right of (C.29) is less than or equal to the corresponding term in (C.30). In fact it is strictly less than, because there is a positive probability that $\underset{\sim}{Q}_m$ consists of Type i traders only, in which case

$$v_H(\underset{\sim}{Q}_m \cup t_i) - v_H(\underset{\sim}{Q}_m) - \lambda_i^\infty \mu_i = -\lambda_i^\infty \mu_i < 0 \leq v_H(\underset{\sim}{Q}_m \cup t_j) - v_H(\underset{\sim}{Q}_m).$$

Hence $q \cdot x_i^\infty < q \cdot x_j^\infty$.

PROOF OF PROPOSITION 12.10: If i is satiated at x_i^∞, then $x_i^\infty \in B_i$, so u_i is differentiable at x_i^∞, so $\partial u_i(x_i^\infty) = \{0\}$, so $\partial \xi_i u_i(x_i^\infty) = \{0\}$, so $q = 0$ (by (9.19)), contradicting (9.20). *Q.E.D.*

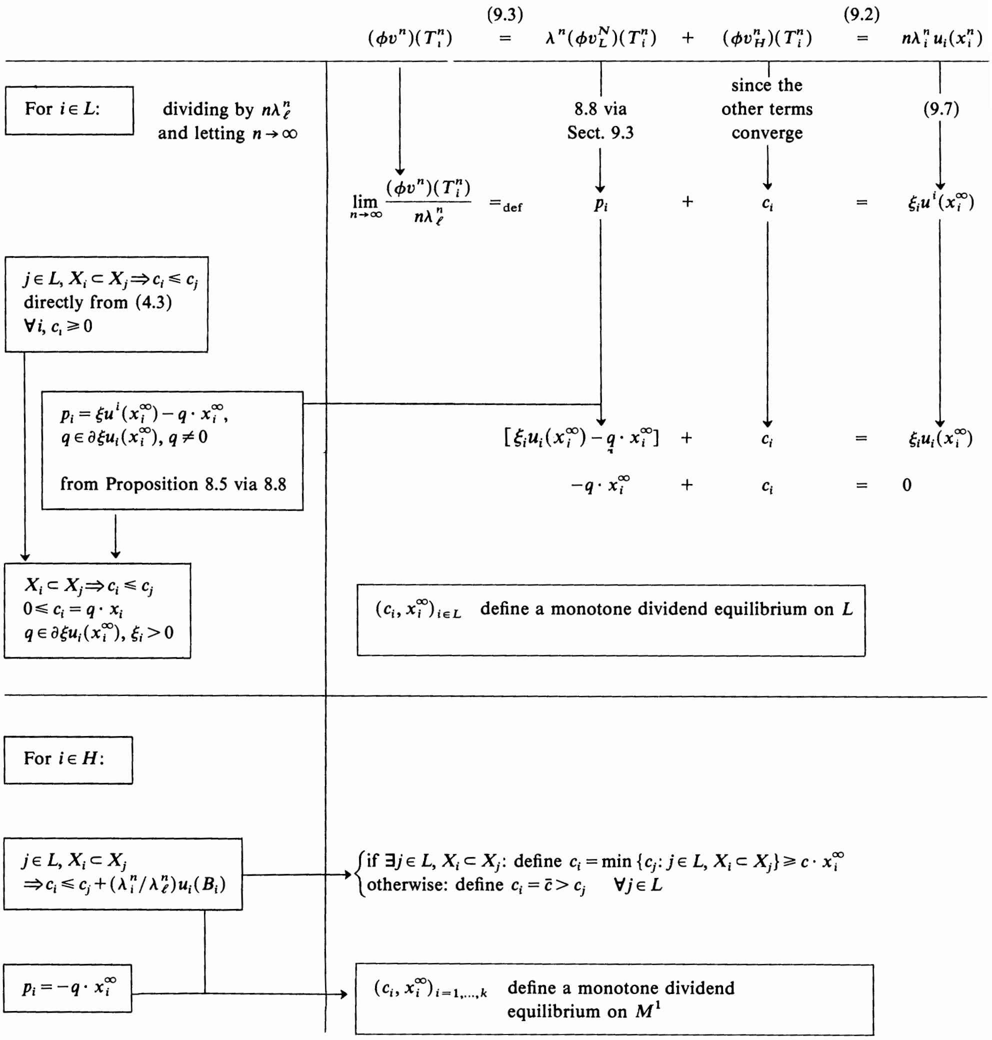
PROOF OF PROPOSITION 12.11: Since $d = 1$, the price vector q is a real number $\neq 0$, w.l.o.g. > 0 (otherwise reflect around 0). We say that i's demand is *positive* (*negative*) if $B_i \subset R_{++}^1 (B_i \subset -R_{++}^1)$; since all dividends are positive, the bliss sets B_i of types i with negative demand are included in i's dividend budget set, and hence $x_i^\infty \in B_i \subset -R_{++}^1$. Hence all unsatiated i have positive demand. The corresponding B_i cannot intersect i's dividend budget set, and so must be to the right of (greater than) x_i^∞. Thus for each unsatiated i, there is a point y_i in B_i with $y_i > x_i^\infty$. Setting $y_i := x_i^\infty \in B_i$ for unsatiated i, we find

$$0 = \sum x_i^\infty < \sum y_i \in \sum B_i,$$

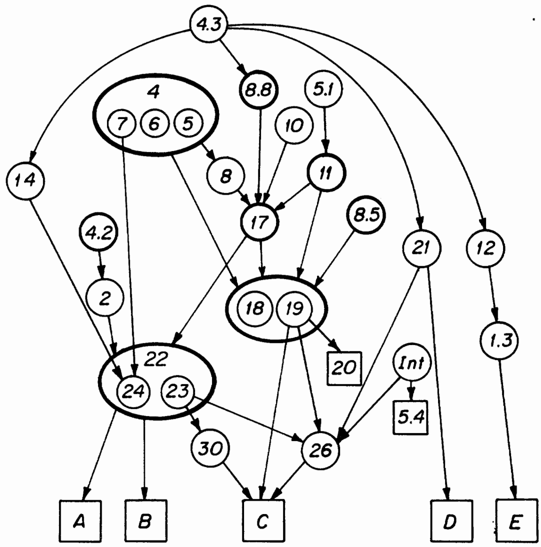
the first equality being the feasibility of x^∞. Thus the short side of the market consists precisely of types with negative demand. But for such types i, we have already seen that $x_i^\infty \in -R_{++}^1$, hence $q \cdot x_i^\infty < 0$, and hence $\lambda_i^\infty > 0$ by Lemma C.26. *Q.E.D.*

APPENDIX D

OVERVIEW OF THE PROOF OF THE MAIN THEOREM



LOGICAL STRUCTURE OF THE PROOF OF THE MAIN THEOREM



Key to flow chart

Results that form part of the conclusion are identified in a square.

Results that are grouped naturally together (e.g. different parts of the same lemma) are encircled together.

Results that are more "central" are enclosed in a heavy circle. Since everything is used, this is necessarily somewhat subjective.

Numbers without decimals refer to Section 9 (e.g., 20 means 9.20).

"Int" is the assumption that 0 is in the interior of each net trade set (see (2.3)).

Lettered results are *not* identified by letter in the text:

A. The dividends c_i are well defined.

B. x_i^∞ is in i's dividend budget set ($q \cdot x_i^\infty \leqslant c_i$).

C. i weakly prefers x_i^∞ to everything in his dividend budget set ($q \cdot x \leqslant c_i \Rightarrow u_i(x_i^\infty) \geqslant u_i(x)$).

D. The dividends are nonnegative ($c_i \geqslant 0$).

E. The dividends are monotonic in the net-trade sets ($X_i \subset X_j \Rightarrow c_i \leqslant c_j$).

REFERENCES

AUMANN, R. J. (1975): "Values of Markets with a Continuum of Traders," *Econometrica*, 43, 611-646.

AUMANN, R. J., AND L. SHAPLEY (1974): *Values of Non-Atomic Games.* Princeton, N.J.: Princeton University Press.

BROWN, D. J., AND A. ROBINSON (1972): "A Limit Theorem on the Cores of Large Standard Exchange Economies," *Proceedings of the National Academy of Sciences of the U.S.A.*, 69, 1258-1260.

CHAMPSAUR, P. (1975): "Cooperation versus Competition," *Journal of Economic Theory*, 11, 394-417.

DEBREU, G., AND H. SCARF (1963): "A Limit Theorem on the Core of an Economy," *International Economic Review*, 4, 225-246.

DRAZEN, A. (1980): "Recent Developments in Macroeconomic Disequilibrium Theory," *Econometrica*, 48, 283-306.

DRÈZE, J. H. (1975): "Existence of an Exchange Equilibrium under Price Rigidities," International Economic Review, 16, 301-320.

DRÈZE, J. H., AND H. MÜLLER (1980): "Optimality Properties of Rationing Schemes," *Journal of Economic Theory*, 23, 150-159.

HART, S. (1977a): "Asymptotic Value of Games with a Continuum of Players," *Journal of Mathematical Economics*, 4, 57–80.

——— (1977b): "Values of Non-Differentiable Markets with a Continuum of Traders," *Journal of Mathematical Economics*, 4, 103–116.

——— (1980): "Measure-Based Values of Market Games," *Mathematics of Operations Research*, 5, 197–228.

HILDENBRAND, W. (1974): *Core and Equilibria of a Large Economy*. Princeton, N.J.: Princeton University Press.

——— (1982): "Core of an Economy" in *Handbook of Mathematical Economics*, Volume II, ed. by K. J. Arrow and M. D. Intriligator. Amsterdam: North-Holland Publishing Company, Ch. 18, pp. 831–877.

KANNAI, Y. (1966): "Values of Games with a Continuum of Players," *Israel Journal of Mathematics*, 4, 54–58.

MALINVAUD, E., AND Y. YOUNÈS (1977): "Une nouvelle formulation générale pour l'étude de certains fondements microéconomiques de la macroéconomie," *Cahiers du Séminaire d'Econométrie*, 18, 63–112.

MAS-COLELL, A. (1977): "Competitive and Value Allocations of Large Exchange Economies," *Journal of Economic Theory*, 14, 419–438.

MASCHLER, M., AND B. PELEG (1966): "A Characterization, Existence Proof and Dimension Bounds for the Kernel of a Game," *Pacific Journal of Mathematics*, 18, 289–328.

MERTENS, J.-F. (1980): "Values and Derivatives," *Mathematics of Operations Research*, 5, 523–552.

——— (1986): "The Shapley Value in the Non-Differentiable Case," *International Journal of Game Theory*, forthcoming.

NEYMAN, A., AND Y. TAUMAN (1979): "The Partition Value," *Mathematics of Operations Research*, 4, 236–264.

ROCKAFELLAR, R. T. (1970): *Convex Analysis*. Princeton, N.J.: Princeton University Press.

ROTH, A. E. (1977): "The Shapley Value as a von Neumann-Morgenstern Utility," *Econometrica*, 45, 657–664.

SHAPLEY, L. (1953): "A Value for *n*-Person Games" in *Contributions to the Theory of Games*, Volume II, ed. by H. W. Kuhn and A. W. Tucker. Princeton, N.J.: Princeton University Press, pp. 307–317.

——— (1962): "Values of Games with Infinitely Many Players" in *Recent Advances in Game Theory* (Proceedings of a Princeton University Conference, October 4–6, 1961), ed. by M. Maschler. Philadelphia: Ivy Curtis Press, pp. 113–118.

—— (1969): "Utility Comparisons and the Theory of Games," in *La Décision*, ed. by G. Th. Guilbaud. Paris: Editions du CNRS, pp. 251–263.

SHAPLEY, L., AND M. SHUBIK (1969): "Pure Competition, Coalitional Power and Fair Division," *International Economic Review*, 10, 337–362.

TAUMAN, Y. (1981): "Value on a Class of Non-Differentiable Market Games," *International Journal of Game Theory*, 10, 155–162.