62b Value, Symmetry, and Equal Treatment: A Comment on Scafuri and Yannelis[1]

Two participants i and j in a game or economic model are called *substitutes* if they enter the model in the same way; i.e., if interchanging them, while keeping all other elements of the model fixed, constitutes a symmetry of the model. In a market, this means that they have the same endowments and utility functions. In a TU (transferable utility) coalitional game v, it means that $v(S \cup i) = v(S \cup j)$ whenever S is a coalition containing neither i nor j. In an NTU (nontransferable utility) game V, it means that $V(S \cup i)$ transforms into $V(S \cup j)$ if we interchange the x^i and x^j axes,[2] whenever S contains either both i or j or neither one (Wooders, 1983).

Scafuri and Yannelis (1984) construct an example of a market in which the value does not provide *equal treatment*: i.e., there is a value that assigns different utilities to substitutes. They call their example "counterintuitive" and say that it "reinforces" allegedly negative results on the NTU value obtained by others.

It is difficult to understand this view. Equal treatment is provided by almost no multi-valued game-theoretic solution concept, including the most well known and widely applied. Neither the core, nor the solution of von Neumann and Morgenstern (1944), nor the bargaining set[3] provide equal treatment. If indeed this is a valid criticism, why pick just the NTU value as its target?

Markets with nonequal treatment cores abound; they are the rule, not the exception. An explicit example is a 3-agent market in complementary goods, e.g., perfectly divisible right and left "gloves." Agent 1 initially holds two right gloves, whereas 2 and 3 hold one left glove each; all agents have linear utilities in pairs of gloves. This corresponds to the TU game $v(123) = 2$, $v(12) = v(13) = 1$, $v(S) = 0$ otherwise. The core contains the point (1, 0, 1), and so certainly does not provide equal treatment.

There is even a 3-person NTU game whose core, while nonempty, contains *no* equal treatment outcomes.[4] All three players acting together

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2. More precisely, that $\pi^{ij} V(S \cup i) = V(S \cup j)$, where π^{ij} is the reflection of payoff space in the hyperplane $x^i = x^j$.

3. Davis and Maschler (1967), Peleg (1967).

4. We are grateful to B. Peleg for bringing this example to our attention.

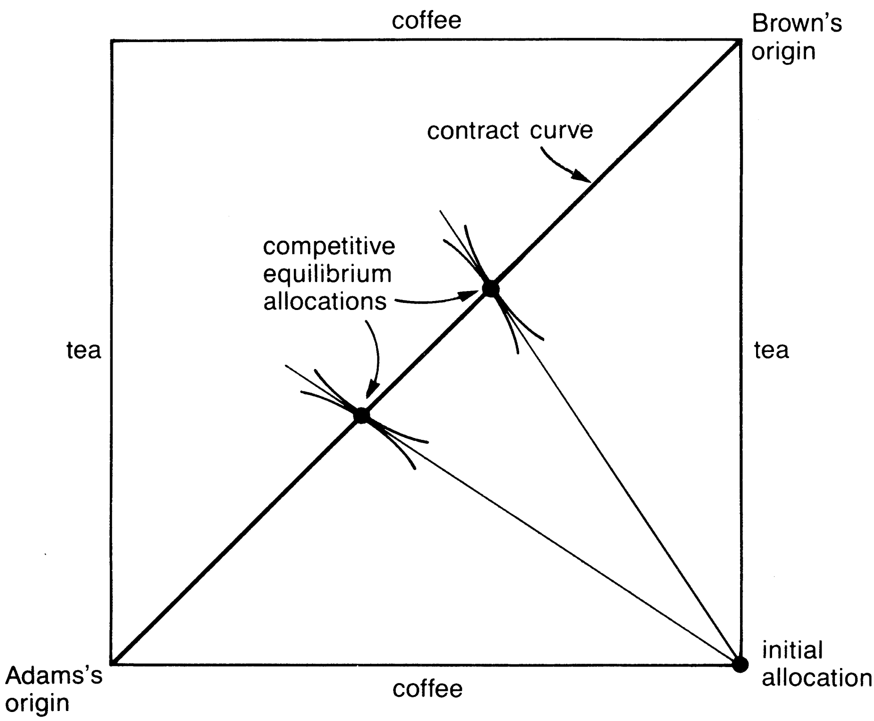

Figure 1

may divide a dollar in any way they please; any two may divide it fifty-fifty, but in no other way; acting alone, each player gets nothing. The game is totally symmetric—all players are substitutes for each other—but the only outcomes in the core are the permutations of (1/2, 1/2, 0), none of which is equal treatment.[5]

Technically, the competitive equilibrium does provide equal treatment; substitutes do always get the same utility. But even there, symmetric[6] agents need not. The Edgeworth box in Figure 1 illustrates a two-commodity, two-agent market with multiple equilibria, that is symmetric under a simultaneous interchange both of the agents and of the commodities. Adams relates to coffee and tea in the same way that Brown relates to tea and coffee, respectively. Adams is endowed with a kilo of tea; symmetrically, Brown is endowed with a kilo of coffee. The agents

5. This cannot happen with the NTU value; when the feasible set $V(N)$ is convex, the methods of Shapley (1969) always yield at least one equal treatment value.

6. We call Agents i and j *symmetric* if there is a symmetry of the game or economy that takes i to j (and hence also one that takes j to i; in Figure 1, the same symmetry does both).

are formally indistinguishable; yet there is a competitive equilibrium that assigns to Adams more tea *and* more coffee than it assigns to Brown. If one likes drama, one can fix things so that Adams gets 999 grams of tea and 999 grams of coffee, and Brown gets only one gram of each.

One must distinguish between symmetry and equal treatment. Symmetry of a multi-valued solution concept refers to the set of *all* outcomes assigned by it to a given game or economy; equal treatment refers separately to *each single* outcome. That the core is symmetric means that as a whole, it is invariant under any symmetry of the game or economy. Thus interchanging substitutes i and j does not change the core as a whole; in utility space, the core is its own reflection in the hyperplane $x^i = x^j$. But equal treatment would mean that substitutes get the same utility at each point of the core, that the hyperplane $x^i = x^j$ actually includes the core.

Symmetry is an eminently reasonable requirement. Indeed, almost all solution concepts in the literature, including the NTU value and the core, do satisfy it. So does the competitive equilibrium: In the coffee-tea example, while one equilibrium gives almost all the goods to Adams, another one does so for Brown.

But equal treatment is a horse of a different color. We have already noted that almost no multi-valued game-theoretic solution satisfies it.[7] Indeed, it seems much too strong for a reasonable general requirement. In the above "divide the dollar" game, the core calls for equal division between two of the three players, but does not say *which* two. This seems perfectly reasonable, but it is not equal treatment. Equal treatment would imply that all three players share the dollar equally. While that also makes sense, it cannot be considered the *only* reasonable outcome.[8]

The reader might object that this is all very well for concepts like the core, which represent some notion of stability; but that the value, which represents an index of power or an arbitrated outcome, should provide equal treatment. But closer examination reveals that this argument is unfounded. Any multi-valued solution concept, no matter what its intuitive content, associates a set of outcomes with each game or economy to which it applies. Choosing a single outcome from this set necessarily calls for more information about the underlying situation, information that is not provided by the game or economy as described. It is this additional information that may well distinguish between substitutes.

7. The one notable exception is the kernel (Davis and Maschler, 1965). But even it may contain outcomes at which symmetric players get unequal utility (Maschler and Peleg, 1967, 598–599).

8. For single-valued solution concepts, of course, symmetry implies equal treatment; this accounts for the fact, noted by Scafuri and Yannelis, that the TU value and the Nash bargaining solution, both of which are by definition single valued, do provide equal treatment.

In an NTU game, two players can show up as formal substitutes even though in fact they are vastly different. For example, this could happen if the units of payoff are dollars and cents respectively, and the utilities of all players are linear in money. Equal treatment would require that one player gets exactly 100 times the payoff of the other; and while this may be quite reasonable in some situations, one certainly would not want to insist on it. The point is that the description of an NTU game does not enable any exogenous comparison between the utilities of different players; from the mere fact that players are substitutes in a given model, one cannot conclude that they are truly "identical." They may be, and then again they may not; the value allows for both possibilities.

Scafuri and Yannelis write that their example "casts doubt on any interpretation of the weights as a meaningful 'endogenous utility comparison' as has been suggested in Shapley (1969)." But "endogenous" does not mean "unique." Scafuri and Yannelis will agree, I hope, that prices are endogenously determined by market forces; yet commodities that appear entirely symmetrically in a market can easily have different prices (as in the economy of Figure 1). The unknowns x and y appear symmetrically in the system $xy = 2$, $x + y = 3$, yet $x \neq y$ in both solutions. Endogeneity has nothing to do with equal treatment.

Turning to the example itself, we find it rather pathological. The allocation in question is indeed associated with nonzero weights; but as the authors themselves point out, the same allocation is also associated with the weight vector (1, 1, 1, 0). Zero weights symptomize degeneracy in the game itself, not in the value; they imply that there are some players who cannot, under any circumstances, contribute to the others. In our case, agents 0 and 3 are endowed with no goods whatsoever, which implies that they cannot contribute anything to other individuals or coalitions. We have not stressed these points because they are secondary; for the reasons stated above, it would not be at all surprising or disconcerting to find a perfectly "healthy" NTU game with a nonequal treatment value. But this particular example must be considered weak.

For the record, we note that with a continuum of agents, the value equivalence principle (e.g., Hart (1977)) shows that value allocations are competitive; this assures equal treatment.

References

Davis, M., and M. Maschler (1965): "The Kernel of a Cooperative Game," *Naval Research Logistics Quarterly*, 12, 223–259.

——— (1967): "Existence of Stable Payoff Configurations for Cooperative Games," in *Essays in Mathematical Economics in Honor of Oskar Morgenstern*, ed. by M. Shubik. Princeton: Princeton University Press, pp. 39–62; also, *Bulletin of the American Mathematical Society*, 69 (1963), 106–108.

Hart, S. (1977): "Values of Non-Differentiable Markets with a Continuum of Traders," *Journal of Mathematical Economics*, 4, 103–116.

Maschler, M., and B. Peleg (1967): "The Structure of the Kernel of a Cooperative Game," *SIAM Journal of Applied Mathematics*, 15, 569–604.

Peleg, B. (1967): "Existence Theorem for the Bargaining set $M_1^{(i)}$," in *Essays in Mathematical Economics in Honor of Oskar Morgenstern*, ed. by M. Shubik. Princeton: Princeton University Press, pp. 53–56; also, *Bulletin of the American Mathematical Society*, 69 (1963), 109–110.

Scafuri, A. J., and N. Yannelis (1984): "Non-Symmetric Cardinal Value Allocations," *Econometrica*, 52, 1365–1368 [Chapter 62a].

Shapley, L. S. (1969): "Utility Comparison and the Theory of Games," in *La Décision: Agrégation et dynamique des ordres de préférences*, ed. by G. Th. Guilbaud. Paris: Editions du Centre National de la Recherche Scientifique, pp. 251–263.

Von Neumann, J., and O. Morgenstern (1944): *Theory of Games and Economic Behavior*. Princeton: Princeton University Press (Third Edition, 1953).

Wooders, M. H. (1983): "The ε-core of a Large Replica Game," *Journal of Mathematical Economics*, 11, 277–300.