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ON THE RATE OF CONVERGENCE OF THE CORE*

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1. INTRODUCTION

The classical theorem of Debreu and Scarf [1963] asserts that the core of a replicated pure exchange economy tends to the set of its competitive allocations as the index k of the replication (the number of traders of each type) tends to infinity. In separate contributions that complement each other nicely, Shapley [1975] and Debreu [1975] have investigated the rate of this convergence. Shapley showed that with C^1 utility functions, the convergence can be as slow as one wishes; whereas Debreu showed that when the utility functions are C^2 and the indifference surfaces have positive Gaussian curvature, the rate of convergence is generically $O(1/k)$. Here "generically" means that for fixed utilities, the rate $O(1/k)$ will obtain except possibly when the initial allocation lies in a closed set of Lebesgue measure 0.

These developments, though they went far toward settling the question of the rate of convergence, still left a gap. Shapley's example depends critically on an essential discontinuity in the second derivative: it becomes unbounded in the neighborhood of the unique competitive allocation. In the introduction to his paper, he speculates that "a condition that bounds the second derivatives... might suffice" to assure $O(1/K)$. On the other hand, the generic nature of Debreu's theorem leaves open the possibility of slow convergence even when the second derivative is bounded and in fact continuous.

Here we will show that the condition of genericity in Debreu's theorem cannot be removed; i.e., there are C^2 (in fact C^∞) utility functions, with indifference surfaces that have positive Gaussian curvature, where the core converges as slowly as we like.

2. THE EXAMPLE

First, some definitions. A *null sequence* is a sequence $\{\delta_k\}$ of positive numbers tending to 0. A *market* is a pure exchange economy with a finite number n of goods, a finite number m of agents, and consumption sets all equal to R^n_+; it is *smooth* if the utility functions are[2] C^2 and strictly quasiconcave, and have positive first derivatives and indifference surfaces with positive Gaussian curvature.

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[2] A function is called C^2 on R^n_+ if it is C^2 on Int R^n_+, continuous on R^n_+, and the second partial derivatives can be extended to continuous functions on R^n_+.

Given a smooth market M, its *k-replication* M_k is the smooth market with km agents and n goods, in which there are k agents corresponding to each agent of M and having the same utility and endowment as that agent; these k agents are said to be of the same *type*. By the "equal treatment principle" (Debreu and Scarf [1963]), each allocation in the core of M_k assigns the same commodity vector to agents of the same type; therefore each core allocation in M_k corresponds to an allocation in M. The set of all allocations in M corresponding to core-allocations in M_k will be denoted C_k. Let W be the set of Walras (i.e., competitive) allocations of M; the Debreu-Scarf Theorem says that $\max\{d(x, W): x \in C_k\} \to 0$ as $k \to \infty$, where $d(x, W)$ denotes the Euclidean distance from the point x to the compact set W.

THEOREM. *Let $\{\delta_k\}$ be a null sequence. Then there is a smooth market M such that*

(1) $$\max\{d(x, W):\ x \in C_k\} > \delta_k$$

for all k.

PROOF. It is sufficient to prove (1) for all sufficiently large k, since by increasing the scale of the market we can multiply the left side of (1) by as large a constant as we want (simultaneously for all k); this will overcome any deficiency in (1) for any finite number of k's.

Before writing down formulas, let us describe the construction geometrically. The markets we consider will have two goods and two agents (before replication). We start out by recalling the condition for an allocation x in such a market to be in C_k. Letting e be the initial allocation, set

(2) $$h^k(x) = x + \frac{1}{k-1}(x - e),$$

$$h_k(x) = x - \frac{1}{k}(x - e).$$

Then

(3) *$x \in C_k$ if and only if $x \in C_1$ and both agents weakly prefer x to both $h^k(x)$ and $h_k(x)$.*

Essentially, this is the condition of Edgeworth [1881, p. 37]. The proof is also indicated in Shapley [1975], and moreover it is a straightforward matter to verify it directly. The condition is illustrated in the Edgeworth box of Figure 1. The point x is in C_k because both $h^k(x)$ and $h_k(x)$ are "beneath" the indifference curve for x from the point of view of both agents; whereas y is *not* in C_k, as $h_k(y)$ is above the indifference curve of y for agent 1; i.e., agent 1 prefers $h_k(y)$ to y.

Rather than proceeding immediately with the construction of M, we construct first an auxiliary market M^*, illustrated in the Edgeworth box of Figure 2. Note the contract curve, which for definiteness we can think of as the diagonal of the

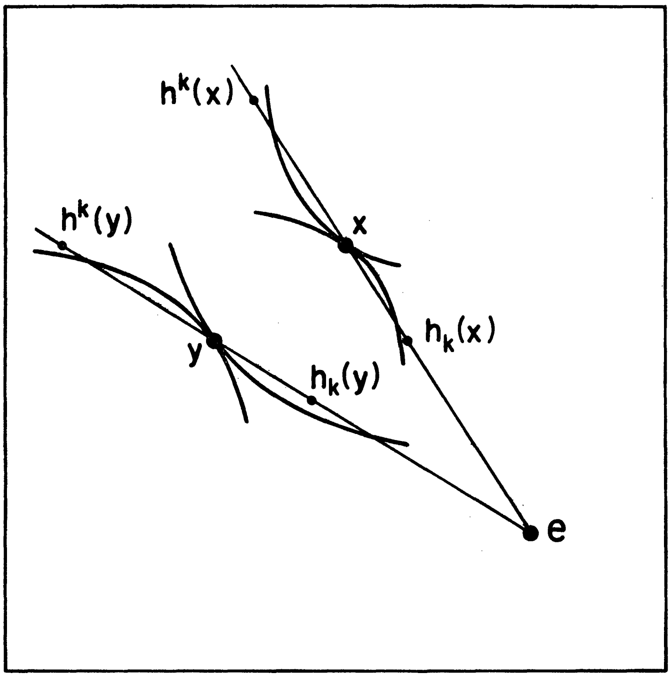

FIGURE 1

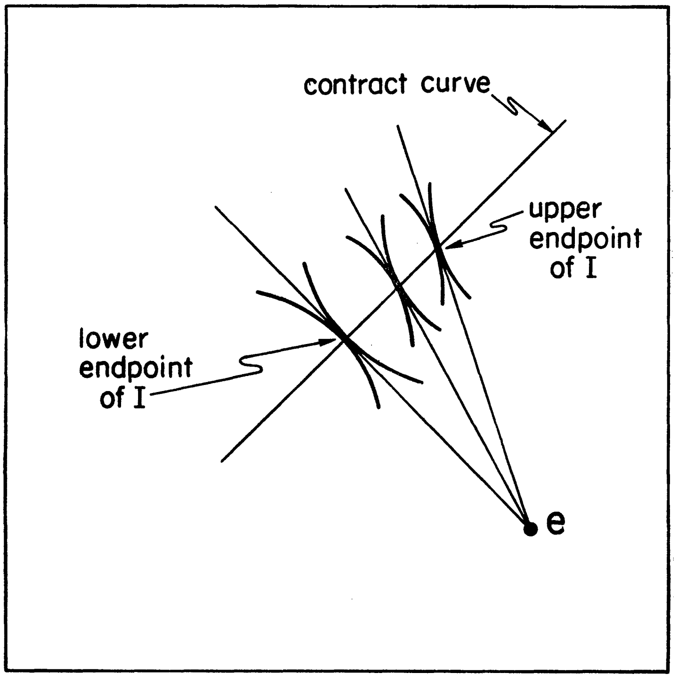

FIGURE 2

box, and the initial allocation e, which is assumed *not* to be on the contract curve. The only special feature we require of M^* is that *every* point in some non-degènerate interval I along the contract curve be a Walras allocation. Obviously these conditions can be satisfied in a smooth market.

Let us denote the lower (from agent 1's standpoint) endpoint of I by w (see Figure 3). To contruct M from M^* we choose a sequence of points a_k on I whose distance from w is, for sufficiently large k, precisely δ_k. Since a_k is a Walras allocation in M^*, the point a_k is *strictly preferred* in M^* by both agents to both $h^k(a_k)$ and $h_k(a_k)$. Let us now modify the utilities so that the slopes of the two indifference curves at a_k are very slightly changed, but remain equal to each other. Then a_k remains in the contract curve but is no longer a Walras allocation; moreover, if the change in the utilities is sufficiently small, a_k will still be preferred to both $h^k(a_k)$ and $h_k(a_k)$, and so will be in C_k (see Figure 3). If we make sure that the slopes of the indifference curves at *all* points of I except w are changed (but remain equal to each other), then I remains part of the contract curve C_1, but there will be no Walras allocations in I other than w. Thus for sufficiently large k,

$$\max\{d(x, W):\ x \in C_k\} \geqq \|a_k - w\| = \delta_k,$$

and so our construction is complete.

The only question that remains is whether the desired modification of the

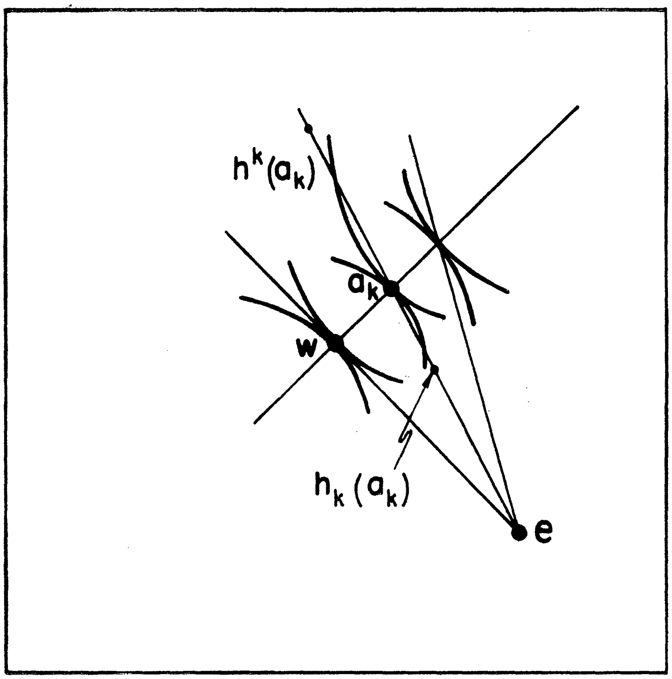

FIGURE 3

utility functions can really be carried out, simultaneously for all k, so that the resulting market is smooth. This we will now show by carrying out the construction more or less explicitly.

For definiteness, let the total endowment of the economy (the sum of 1's and 2's endowments) be (1, 1). Then an allocation — i.e., a "point" x in the Edgeworth box — is of the form $(x^1, x^2, 1-x^1, 1-x^2)$, where the superscripts index the commodities. In particular, each a_k, since it is on the diagonal, has the form

$$a_k = (\alpha_k, \alpha_k, 1 - \alpha_k, 1 - \alpha_k),$$

and similarly w has the form

$$w = (\omega, \omega, 1 - \omega, 1 - \omega).$$

Denote the utility functions of agents 1 and 2 in M^* by u^* and v^* respectively. Let f be a C^2 function from the real line onto itself, with $f' > 0$ everywhere, $f'(\alpha) < \alpha$ when $\omega < \alpha < 1$, and $f(\alpha) = \alpha$ otherwise; such a function will be called *admissible*. Define functions u and v by

$$\text{(4)} \qquad \begin{cases} u(x^1, x^2) = u^*(f(x^1), f(x^2)) \\ v(1 - x^1, 1 - x^2) = v^*(1 - f(x^1), 1 - f(x^2)) \end{cases}$$

when $(x^1, x^2, 1-x^1, 1-x^2)$ is in the box, i.e., when $0 \leqq x^1 \leqq 1$, $0 \leqq x^2 \leqq 1$; for other values of x^1 and x^2, we will define u and v later. For appropriate choice of f, we will take the utilities in M to be u and v.

First we show that the contract curve in M remains the diagonal. To this end, note that for points $(\alpha, \alpha, 1-\alpha, 1-\alpha)$ on the diagonal, we have[3]

$$\text{(5)} \qquad \begin{cases} \nabla u(\alpha, \alpha) = f'(\alpha) \nabla u^*(f(\alpha), f(\alpha)), \\ \nabla v(1 - \alpha, 1 - \alpha) = f'(\alpha) \nabla v^*(1 - f(\alpha), 1 - f(\alpha)). \end{cases}$$

Letting g_u denote the normalized gradient, i.e.,

$$g_u(x_1, x_2) = \frac{\nabla u(x_1, x_2)}{\|\nabla u(x_1, x_2)\|},$$

and defining g_{u^*}, g_v, and g_{v^*} similarly, we deduce from (5) that

$$\text{(6)} \qquad \begin{cases} g_u(\alpha, \alpha) = g_{u^*}(f(\alpha), f(\alpha)) \\ g_v(1 - \alpha, 1 - \alpha) = g_{v^*}(1 - f(\alpha), 1 - f(\alpha)). \end{cases}$$

On the other hand, since the diagonal is the contract curve in M^*, we have

$$g_u(\alpha, \alpha) = g_v(1 - \alpha, 1 - \alpha),$$

which means that the diagonal remains the contract curve in M, as asserted.

We next wish to show that none of the points $(\alpha, \alpha, 1-\alpha, 1-\alpha)$ in I other than

[3] By $\nabla v(1-\alpha, 1-\alpha)$ we mean ∇v evaluated at $(1-\alpha, 1-\alpha)$.

w are Walras allocations in M. Since by construction these points *are* Walras allocations in M^*, it is sufficient for this to show that

(7) $$g_u(\alpha, \alpha) \neq g_{u^*}(\alpha, \alpha).$$

Indeed, when $(\alpha, \alpha, 1-\alpha, 1-\alpha) \in I-w$, then $\omega < f(\alpha) < \alpha$; hence $(f(\alpha), f(\alpha), 1-f(\alpha), 1-f(\alpha))$ is in $I-w$ and is different from $(\alpha, \alpha, 1-\alpha, 1-\alpha)$, and so both are Walras allocations in M^*. Hence

(8) $$g_{u^*}(f(\alpha), f(\alpha)) \neq g_{u^*}(\alpha, \alpha),$$

and combining this with (6), we get (7).

Before proceeding, it will be convenient to establish the following lemma:

LEMMA. *Let $\{\gamma_l\}$ be a sequence of numbers tending to ω, and let $\{\varepsilon_l\}$ be an arbitrary sequence of positive numbers. Then there is an admissible function f such that*

(9) $$|f(\gamma_l) - \gamma_l| \leqq \varepsilon_l$$

for all l. Furthermore, if ε is an arbitrary positive number, then an admissible f can be found satisfying (9) *and, in addition,*

(10) $$|f(\alpha) - \alpha| < \varepsilon, \quad |f'(\alpha) - 1| < \varepsilon, \quad |f''(\alpha)| < \varepsilon$$

for all real α.

PROOF. Since we will define $f(\alpha)=\alpha$ whenever α is not in the open interval $(\omega, 1)$, we may assume w.l.o.g. (without loss of generality) that $\omega < \gamma_l < 1$ for all l. Since ω is the only limit point of the γ_l, we may then also assume w.l.o.g. that γ_l is decreasing in l (no more than a finite number of γ_l can coincide, since $\gamma_l \to \omega$; if, then, there are several equal γ_l, we can simply eliminate all but one and use the smallest of the corresponding ε_l). Finally, we may assume w.l.o.g. that $\varepsilon_l \leqq 1$ for all l and that $\varepsilon_l \to 0$; for if not, we can substitute $\min(\varepsilon_l, 1/l)$ for ε_l.

Now let f_1 be a continuous function on $\boldsymbol{R}$ such that $1 \geqq f_1(\alpha) > 0$ on $(\omega, 1)$,

(11) $$f_1(\alpha) = 0 \qquad \text{outside of} \quad (\omega, 1),$$

and

(12) $$f_1(\gamma_l) = \varepsilon_l \qquad \text{for all} \quad l.$$

For example, such a function can be constructed by defining it by (11), (12), and the requirement that it be linear on each of the segments $(\gamma_1, 1)$ and (γ_{i+1}, γ_i), $i=1, 2, \ldots$. Next, define

$$f_2(\alpha) = \int_0^\alpha f_1(t)dt$$

and

$$f_3(\alpha) = \int_0^\alpha f_2(t)dt.$$

Let f_4 be a C^2 function such that $1 \geqq f_4(\alpha) > 0$ for $\alpha \in (\omega, 1)$ and $f_4(\alpha) = 0$ for $\alpha \geqq 1$ (for example, $f_4(\alpha) = (\alpha - 1)^4$ when $\alpha \leqq 1$, and $f_4(\alpha) = 0$ when $\alpha \geqq 1$). Define $f_5 = f_4 f_3$; then $f_5 \in C^2$, $f_5(\alpha) > 0$ on $(\omega, 1)$, $f_5(\alpha) = 0$ outside of $(\omega, 1)$, and

$$f_5(\gamma_l) \leqq f_3(\gamma_l) < f_1(\gamma_l) \leqq \varepsilon_l$$

for all l.

Let $K_1 = \max f_5$, $K_2 = \max f_5'$, $K_3 = \max f_5''$, $K = \max(K_1, K_2, K_3)$, and define

$$f(\alpha) = \alpha - \left(\frac{\varepsilon}{2K}\right) f_5(\alpha).$$

Then f obeys all the requirements of the lemma, and so the proof of the lemma is complete.

We are now ready to show that when f is appropriately chosen, $a_k \in C_k$ for k sufficiently large. Since a_k is a Walras allocation in M^*, it follows from (3) and (4) that $a_k \in C_k$ if the mapping

$$(x^1, x^2, 1 - x^1, 1 - x^2) \longrightarrow (f(x^1), f(x^2), 1 - f(x^1), 1 - f(x^2))$$

moves the three points a_k, $h^k(a_k)$, and $h_k(a_k)$ sufficiently little. To state this formally, set

$$\begin{aligned} h^k(a_k) &= (\beta_k^1, \beta_k^2, 1 - \beta_k^1, 1 - \beta_k^2) \\ h_k(a_k) &= (\beta_k^3, \beta_k^4, 1 - \beta_k^3, 1 - \beta_k^4). \end{aligned} \tag{13}$$

Since a_k is a Walras allocation in M^*, we have

$$\begin{aligned} &u^*(\beta_k^1, \beta_k^2) < u^*(\alpha_k, \alpha_k) > u^*(\beta_k^3, \beta_k^4) \\ &v^*(1 - \beta_k^1, 1 - \beta_k^2) < v^*(1 - \alpha_k, 1 - \alpha_k) > v^*(1 - \beta_k^3, 1 - \beta_k^4). \end{aligned} \tag{14}$$

From the continuity of u^* and v^* it follows that for each k there is a positive number η_k such that if we change each of the five numbers α_k, β_k^1, β_k^2, β_k^3, β_k^4 by at most η_k, then the strict inequalities in (14) will continue to hold. From this it follows that if f is constructed such that for all k,

$$|f(\alpha_k) - \alpha_k| < \eta_k \quad \text{and} \quad |f(\beta_k^i) - \beta_k^i| < \eta_k \qquad \text{for} \quad i = 1, 2, 3, 4, \tag{15}$$

then we will have $a_k \in C_k$ for all k.

From (2) and (13) we deduce

$$\beta_k^i \longrightarrow \omega \quad \text{as} \quad k \longrightarrow \infty \tag{16}$$

for $i = 1, 2, 3, 4$. We can therefore combine the α_k and the β_k^i into a single sequence $\{\gamma_l\}$ defined by $\gamma_{5k+i} = \beta_k^i$ for $i = 1, 2, 3, 4$ and $\gamma_{5k+5} = \alpha_k$. Setting $\varepsilon_{5k+i} = \eta_k$ $(i = 1, 2, 3, 4, 5)$ and applying the Lemma (in particular (10)), we conclude that it is possible to construct f so that (15) is satisfied, as desired.

Finally, we must show that u and v can be constructed so that they are everywhere C^2 and have positive first derivatives and positive Gaussian curvature.

Let q be a C^2 function from $\boldsymbol{R}$ to itself such that $q(\alpha)=1$ for $\alpha \leqq 1$ and $q(\alpha)=0$ for $\alpha \geqq 2$. For arbitrary $(x^1, x^2) \in R_+^2$, define

$$u(x^1, x^2) = u^*(q(x^2)f(x^1) + (1 - q(x^2))x^1, q(x^1)f(x^2) + (1 - q(x^1))x^2).$$

When $(x^1, x^2, 1-x^1, 1-x^2)$ is in the box, i.e., when $0 \leqq x^1 \leqq 1$ and $0 \leqq x^2 \leqq 1$, this coincides with the definition at (4). When $x^1 > 2$ *or* $x^2 > 2$, then $u(x^1, x^2) = u^*(x^1, x^2)$, in which case the required smoothness properties are inherited from M^*. We must therefore only concern ourselves with the compact set $\{(x^1, x^2): 0 \leqq x^1 \leqq 2 \text{ and } 0 \leqq x^2 \leqq 2\}$. In this set, u^* and its first and second derivatives are bounded, and the first derivatives of u^* and the Gaussian curvatures are bounded away from zero. Hence if we change the first and second derivatives by sufficiently little, the smoothness properties will continue to hold.[4] A straightforward calculation of the derivatives, together with an application of the Lemma (in particular (10)), then yields the desired result.

Similar considerations apply to v. Define $\hat{f}(\alpha) = 1 - f(1-\alpha)$, and for $(y^1, y^2) \in R_+^2$, set

$$v(y^1, y^2) = v^*(q(y^2)\hat{f}(y^1) + (1 - q(y^2))y^1, q(y^1)\hat{f}(y^2) + (1 - q(y^1))y^2).$$

Again, when $0 \leqq y^1 \leqq 1$ and $0 \leqq y^2 \leqq 1$, this coincides with the definition at (4). If we note that the inequalities (10) for $\hat{f}$ follow from the same inequalities for f, then the argument for v may be completed like that for u.

3. REMARKS AND CONCLUSION

The reader who has followed the construction through will realize that it can be carried out at no additional cost with C^∞ utility functions. Furthermore, the example can be constructed so that it has a unique Walras equilibrium. Thus unlike in the Shapley example, there is absolutely nothing pathological here, nothing overt that one might have guessed would lead to "trouble." The conclusion is that though Debreu's theorem shows that a slow-converging core is a rarity, it nevertheless can and does occur in the smoothest of environments, and can apparently not be ruled out by any kind of economically reasonable a priori requirement.

The measure with which we chose to gauge the speed of convergence was simply the Euclidean distance in the "equal treatment allocation space" E^{lm}. Of course, any other norm on E^{lm} would have done as well. But one might ask whether the example would go away if one used some economically more meaningful measure, such as difference in value at, say, the equilibrium prices. Again, the answer is no; the value at *any* price will yield the same slow rate of convergence, since the core is laid out along the diagonal, which is transversal to any budget line.

Finally, it is useful to examine our example in the light of the result of Dierker

[4] The Gaussian curvature is a continuous function of the first and second derivatives of u.

[1975],[5] which assumes no differentiability or even continuity. Roughly speaking, this result says that no matter how large an economy may be, the total deviation of a member of its core from "competitiveness", summed over all agents in the economy, will not exceed some constant (which is independent of the size of the economy). Translated to the replication context, it implies that for each k and each allocation $x=(x_1,\ldots, x_m)$ in C_k, there is a normalized price vector p such that for each agent i, $|p(e_i-x_i)|=O(1/k)$ and $|p(y_i-x_i)|=O(1/k)$, where e_i is i's endowment and y_i is the bundle of minimal value, at prices p, on the indifference surface of x_i. If in these two equations, $O(1/k)$ would be replaced by 0, then x would be a Walras allocation; this, therefore, strongly suggests the $O(1/k)$ convergence. The reason that such a conclusion would be erroneous is that p depends on x and k and need not itself be competitive (i.e., Walras); indeed, our example (and Shapley's) shows that convergence of p to the competitive price may be arbitrarily slow.

Intuitively, though, Dierker's theorem makes it clear that $O(1/k)$ is the "right" rate, even when the actual convergence is slower. Though p is not within $O(1/k)$ of a competitive price, it is a price that is within $O(1/k)$ of being competitive. The distinction here is similar to that between being near a fixed point of a mapping and being at a point that is nearly fixed. It is, of course, the second kind of approximation that has been the object of the approximation algorithms pioneered by Scarf [1973], and that seems, intuitively, to be the more significant concept.

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[5] See Arrow and Hahn [1971, p. 188ff] for a weaker version of this result, and Anderson [1978] for a simpler proof.