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MARKETS WITH A CONTINUUM OF TRADERS

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It is suggested that the most natural mathematical model for a market with "perfect competition" is one in which there is a continuum of traders (like the continuum of points on a line). It is shown that the core of such a market coincides with the set of its "equilibrium allocations," i.e., allocations which constitute a competitive equilibrium when combined with an appropriate price structure.

1. INTRODUCTION

THE NOTION of *perfect competition* is fundamental in the treatment of economic equilibrium. The essential idea of this notion is that the economy under consideration has a "very large" number of participants, and that the influence of each individual participant is "negligible." Of course, in real life no competition is perfect; but, in economics, as in the physical sciences,[1] the study of the ideal state has proved very fruitful, though in practice it is, at best, only approximately achieved.

Though writers on economic equilibrium have traditionally assumed perfect competition,[2] they have, paradoxically, adopted a mathematical model that does not fit this assumption. Indeed, the influence of an individual participant on the economy cannot be mathematically negligible, as long as there are only finitely many participants. Thus *a mathematical model appropriate to the intuitive notion of perfect competition must contain infinitely many participants.* We submit that the most natural model for this purpose contains a *continuum* of participants, similar to the continuum of points on a line or the continuum of particles in a fluid. Very succinctly, the reason for this is that one can integrate over a continuum, and changing the integrand at a single point does not affect the value of the integral, that is, the actions of a single individual are negligible.

This paper is confined to "pure exchange economies," i.e., markets. Presumably, the results could be extended to economies with production, but we have not done this. We investigate and compare two known concepts of market equilibrium—the *competitive equilibrium* and the *core*. By means of the continuous model, it is possible to express the relation between these two concepts in a particularly simple and transparent manner. In fact, in the continuous model—but in no finite model—the two concepts are essentially equivalent.

The market model that we consider consists of a set of traders, each of whom starts out with an initial commodity bundle to be used for trading, and each of whom has a well-defined preference order on the set of all commodity bundles. A *trade* (or *allocation*) is a redistribution of the commodities in the initial bundles

[1] Think of a "freely falling body" (no air resistance), an "ideal gas" (the molecules do not collide), an "ideal fluid" (incompressible and nonviscous), and so on.

[2] A discussion of the literature will be found in Section 5.

among the traders. This is a perfectly standard model in economic theory with the sole exception that the set of traders has heretofore been assumed finite.

A *competitive equilibrium* is a state of the market arrived at via "the law of supply and demand"; it consists of a price structure p (one price for each commodity) at which the total supply of each good exactly balances the total demand, and the allocation $\boldsymbol{x}$ that results from trading at these prices. More precisely, $\boldsymbol{x}$ is an allocation with the property that at the price structure p, no trader can, with the value of his initial bundle, buy a bundle that he prefers to his part of $\boldsymbol{x}$. If $(p, \boldsymbol{x})$ is a competitive equilibrium, then $\boldsymbol{x}$ is called an *equilibrium allocation*.

An allocation $\boldsymbol{x}$ is said to be in the *core* of the market if no coalition of traders can force an outcome that is better for them than $\boldsymbol{x}$. More precisely, $\boldsymbol{x}$ is in the core if there is no group of traders that, by its own efforts alone, without help from traders not in the group, can assure each of its members of a final commodity bundle preferred to that obtained under $\boldsymbol{x}$. What we mean by "its own efforts" is that the desired result can be obtained if the traders in the group exchange the commodities in their initial bundles among themselves only, as if the other traders were not present. The core is a generalization of Edgeworth's "contract curve" **[8]**.

It is widely recognized that the notion of competitive equilibrium makes economic sense only if perfect competition is assumed. Otherwise, a change in an individual's offer to buy or sell can easily upset prevailing prices, so that the restriction to these prices is meaningless. The notion of core, on the other hand, does not depend on perfect competition; it is perfectly valid even for markets containing only two or three traders.

The definition of competitive equilibrium assumes that the traders allow market pressures to determine prices and that they then trade in accordance with these prices, whereas that of core ignores the price mechanism and involves only direct trading between the participants. Intuitively, one feels that money and prices are no more than a device to simplify trading, and therefore the two concepts should lead to the same allocations. It is to be expected that this will not happen in finite markets, where the notion of competitive equilibrium is not really applicable; and, indeed, though every equilibrium allocation is always in the core, the core of a finite market usually contains points that are not equilibrium allocations. But, when the notion of perfect competition *is built into the model*, that is, in a continuous market, one may expect that the core equals the set of equilibrium allocations. That this is indeed the case is the main result of this paper.

It has long been conjectured that some such theorem holds; the basic idea dates back at least to Edgeworth **[8]**. The usual rough statement is that "the core approaches the set of equilibrium allocations as the number of traders tends to infinity." Unfortunately, it is extremely difficult to lend precise meaning to this kind of statement, to say nothing of proving it. Very recently, Debreu and Scarf did succeed in stating and proving a theorem of this kind in a brilliant and elegant fashion **[7]**; even so, their theorem holds only under comparatively restrictive conditions. Their work will be discussed and compared with ours in Section 6. There we will also discuss an earlier model of Scarf [18] and Debreu [6], containing a denumerable infinity of traders.

Continuous models are nothing new in economics or game theory, but it is usually parameters such as price or strategy that are allowed to vary continuously. Models with a continuum of *players* (traders in this instance) are a relative novelty, and the references can still be counted on the fingers of one hand. Milnor and Shapley [15] and Shapley alone [19] pioneered the idea in two papers dealing with power indices (Shapley values); [19] is set in a context of considerable economic interest. The only other references of which we know are Davis [4] and Peleg [17], who treat such games from the point of view of their von Neumann-Morgenstern solutions and their "bargaining sets," respectively.

The idea of a continuum of traders may seem outlandish to the reader. Actually, it is no stranger than a continuum of prices or of strategies or a continuum of "particles" in fluid mechanics. In all these cases, the continuum can be considered an approximation to the "true" situation in which there is a large but finite number of particles (or traders or strategies or possible prices). The purpose of adopting the continuous approximation is to make available the powerful and elegant methods of the branch of mathematics called "analysis," in a situation where treatment by finite methods would be much more difficult or even hopeless (think of trying to do fluid mechanics by solving n-body problems for large n).

There is perhaps a certain psychological difference between a fluid with a continuum of particles and a market with a continuum of traders. Though we are intellectually convinced that a fluid contains only finitely many particles, to the naked eye it still looks quite continuous. The economic structure of a shopping center, on the other hand, does not look continuous at all. But, for the economic policy maker in Washington, or for any professional macroeconomist, there is no such difference. He works with figures that are summarized for geographic regions, different industries, and so on; the individual consumer (or merchant) is as anonymous to him as the individual molecule is to the physicist.

Of course, to the extent that individual consumers or merchants are in fact *not* anonymous (think of General Motors), the continuous model is inappropriate, and our results do not apply to such a situation. But, in that case, perfect competition does not obtain either. In many real markets the competition is indeed far from perfect; such markets are probably best represented by a mixed model, in which some of the traders are points in a continuum, and others are individually significant (compare [19]). The purpose of this paper is to study the extreme case in which perfect competition does obtain, i.e., there are no individually significant traders.

It should be emphasized that our consideration of a continuum of traders is not merely a mathematical exercise; it is the expression of an economic idea. This is underscored by the fact that the chief result holds *only* for a continuum of traders—it is false for any finite number. It would presumably also be possible to consider a continuum of different commodities, but, in contrast to the continuum of traders, this would serve no useful purpose (in this context). A continuum of commodities is the appropriate mathematical idealization of a real situation in which there are "many" commodities, but our result holds for any number of commodities, many or few, so there is nothing to be gained by considering only the case of "many" commodities. The continuum of traders, on the other hand, captures an idea and enables the proof of a theorem that could *not* be proved in the finite set-up.

Our chief mathematical tools are Lebesgue measure and integration, but only their most elementary properties are needed. Riemann integration can *not* be substituted (see the beginning of the proof of Lemma 4.2). Much of the proof is adapted from the work of Debreu and Scarf [**6**, **7**], but becomes simpler and more natural in this context. Indeed, over and above the specific result obtained here, what we would like to stress is the power and simplicity of the continuum-of-players method in describing mass phenomena in economics and game theory. The present work should be considered primarily as an illustration of this method as applied to an area where no other treatment seemed completely satisfactory, and we hope that it may stimulate more extensive development and use of models with a continuum of players.

The mathematical model and the main theorem are presented in the following section, and the assumptions are briefly discussed in Section 3. Section 4 is devoted to the proof of the main theorem. Section 5 reviews the literature and compares the present work with that of other authors.

2. THE MATHEMATICAL MODEL AND THE MAIN THEOREM

We will be working in a Euclidean space R^n; the dimensionality n of the space represents the number of different commodities being traded in the market.

Superscripts will be used exclusively to denote coordinates. Following standard practice, for x and y in R^n we write $x>y$ to mean $x^i>y^i$ for all i; $x\geqq y$ to mean $x^i\geqq y^i$ for all i; and $x\geq y$ to mean $x\geqq y$ but not $x=y$. The integral of a vector function is to be taken as the vector of integrals of the components. The scalar product $\Sigma_{i=1}^n x^i y^i$ of two members x and y of R^n is denoted $x\cdot y$. The symbol 0 denotes the origin in R^n as well as the real number zero; no confusion will result.

A *commodity bundle* x is a point in the nonnegative orthant Ω of R^n. The set of *traders* is the closed unit interval $[0, 1]$; it will be denoted T. An *assignment* (of commodity bundles to traders) is a function $\boldsymbol{x}$ from T to Ω, each coordinate of which is Lebesgue integrable over T. As all integrals are with respect to t, we shall omit writting t, as well as the symbol dt, under the integral sign; thus $\int_T \boldsymbol{x}$ means $\int_T \boldsymbol{x}(t)\,dt$.

There is a fixed *initial assignment* $\boldsymbol{i}$. Intuitively, $\boldsymbol{i}(t)$ is the bundle with which trader t starts out. We assume

$$(2.1) \qquad \int_T \boldsymbol{i} > 0\,.$$

An *allocation* (or "final assignment" or "trade") is an assignment $\boldsymbol{x}$ for which $\int_T \boldsymbol{x} = \int_T \boldsymbol{i}$.

For each trader t there is defined a relation $\succ_t$ on Ω, which is called the *preference relation* of t and satisfies the following conditions:

(2.2) *Desirability (of the commodities)*: $x \geqq y$ implies $x \succ_t y$.
(2.3) *Continuity (in the commodities)*: For each $y \in \Omega$, the sets $\{x\colon x \succ_t y\}$ and $\{x\colon y \succ_t x\}$ are open (relative to Ω).
(2.4) *Measurability*: If $\boldsymbol{x}$ and $\boldsymbol{y}$ are assignments, then the set $\{t\colon \boldsymbol{x}(t) \succ_t \boldsymbol{y}(t)\}$ is Lebesgue measurable in T. Note specifically that $\succ_t$ is not assumed to be complete, nor even transitive.

A *coalition* of traders is a Lebesgue measurable subset of T; if it is of measure 0, it is called *null*.[3] An allocation $\boldsymbol{y}$ *dominates* an allocation $\boldsymbol{x}$ *via* a coalition S if $\boldsymbol{y}(t) \succ_t \boldsymbol{x}(t)$ for each $t \in S$, and S is *effective* for $\boldsymbol{y}$, i.e.,

$$\int_S \boldsymbol{y} = \int_S \boldsymbol{i}\,.$$

The *core* is the set of all allocations that are not dominated via any nonnull coalition.

A *price vector* p is an n-tuple of nonnegative real numbers, not all of which vanish. A *competitive equilibrium* is a pair consisting of a price vector p and an allocation $\boldsymbol{x}$, such that for almost every trader t, $\boldsymbol{x}(t)$ is maximal with respect to $\succ_t$ in t's *budget set* $\{x\colon p\cdot x \leqq p\cdot \boldsymbol{i}(t)\}$. An *equilibrium allocation* is an allocation $\boldsymbol{x}$ for which there exists a price vector p such that $(p, \boldsymbol{x})$ is a competitive equilibrium.

THEOREM: *The core coincides with the set of equilibrium allocations.*

3. DISCUSSION OF THE MODEL

The definitions of allocation, core, and competitive equilibrium are the same as the usual ones, except that where the usual definitions sum over a set of traders, we integrate. It is useful to think of the commodity bundle $\boldsymbol{x}(t)$ held by an individual trader t under an assignment $\boldsymbol{x}$ as an "infinitesimal." Only when a "significant" (i.e., nonnull) coalition pools its resources can a non-infinitesimal bundle result, namely $\int_S \boldsymbol{x}$. Thus an individual trader t—and, in general, any null coalition—is without influence in the market, and can and will always be ignored. Intuitively, this is what justifies the exclusion of null coalitions in the definitions of core and competitive equilibrium.

The intuitive meaning of "S is effective for $\boldsymbol{y}$" is that the coalition S can assure to each of its members t the bundle $\boldsymbol{y}(t)$, without any help from traders not in S.

[3] Not to be confused with the *empty* coalition, which has no members at all. The term "almost every trader" will mean "every trader except possibly for a null set."

This it can do by an appropriate redistribution of the initial bundles $\boldsymbol{i}(t)$ because of the condition $\int_S \boldsymbol{y} = \int_S \boldsymbol{i}$ that defines effectiveness.

Assumption (2.1) asserts that each of the commodities is actually present in the market. In other words, each commodity is initially held by some traders, though different comodities might be held by different traders, and it could well happen that no single trader initially holds a positive amount of all commodities. The finite analogue of this assumption has been used by McKenzie [5, p. 58, Assumption 5]. Though very weak, the assumption is not merely a normalization; it can be shown that the main theorem is false without it. On the other hand, for a given real-life market, it is always possible and, in fact, natural to build a model satisfying (2.1), namely, by counting as commodities in the model only those commodities actually present in the real-life market.

The desirability assumption (2.2) says that each trader always wants more of every commodity. One consequence is that satiation is never reached.[4] The continuity assumption (2.3) is the usual one; it asserts that if $x \succ_t y$, and x' and y' are sufficiently close to x and y, respectively, then also $x' \succ_t y'$. The measurability assumption (2.4) is of technical significance only and constitutes no real economic restriction. Nonmeasurable sets are extremely "pathological"; it is unlikely that they would occur in the context of an economic model.

We remark that our assumptions are far weaker than those usually assumed in market models.

The choice of the unit interval as a model for the set of traders is of no particular significance. A planar or spatial region would have done just as well. In technical terms, T can be any measure space *without atoms*.[5] The condition that T have no atoms is precisely what is needed to ensure that each individual trader have no influence.

4. PROOF OF THE MAIN THEOREM

First, we show that every equilibrium allocation is in the core. Let $(p, \boldsymbol{x})$ be a competitive equilibrium. Suppose, contrary to the theorem, that $\boldsymbol{x}$ is dominated via a coalition S by an allocation $\boldsymbol{y}$. Then, by the definition of competitive equilibrium, we have $p \cdot \boldsymbol{y}(t) > p \cdot \boldsymbol{i}(t)$ for almost all $t \in S$. Hence

$$p \cdot \int_S \boldsymbol{y} = \int_S p \cdot \boldsymbol{y} > \int_S p \cdot \boldsymbol{i} = p \cdot \int_S \boldsymbol{i}\,,$$

and this contradicts

$$\int_S \boldsymbol{y} = \int_S \boldsymbol{i}\,.$$

[4] The assumption can be dispensed with for $y > \boldsymbol{i}(t)$; this would permit satiation for such y.

[5] An *atom* of a measure space is a nonnull set in the space which includes no nonnull subset of smaller measure. The best example is a "mass point," i.e., a single point that carries positive measure.

Conversely, we show that every allocation in the core is an equilibrium allocation. Let $\boldsymbol{x}$ be in the core. Define

$$\boldsymbol{F}(t)=\{x : x \succ_t \boldsymbol{x}(t)\}\ ,$$
$$\boldsymbol{G}(t)=\boldsymbol{F}(t)-\boldsymbol{i}(t)=\{x-\boldsymbol{i}(t) : x\in\boldsymbol{F}(t)\}\ .$$

For each set U of traders, let $\Delta(U)$ denote the convex hull of the union $\cup_{t\in U}\boldsymbol{G}(t)$. Define U to be *full*, if its complement is null.

LEMMA 4.1: *There is a full set U of traders, such that 0 is not an interior point of $\Delta(U)$.*

PROOF: For each x in R^n, let $\boldsymbol{G}^{-1}(x)$ be the set of all traders t for whom $\boldsymbol{G}(t)$ contains x. The notation $\boldsymbol{G}^{-1}$ is suggested by the fact that $t\in\boldsymbol{G}^{-1}(x)$ if and only if $x\in\boldsymbol{G}(t)$. From $\boldsymbol{G}^{-1}(x)=\{t : x+\boldsymbol{i}(t) \succ_t \boldsymbol{x}(t)\}$ and measurability (2.4), it follows that $\boldsymbol{G}^{-1}(x)$ is measurable for each x.

Let N be the set of all those rational points r in R^n (i.e., points with rational coordinates) for which $\boldsymbol{G}^{-1}(r)$ is null. Obviously N is denumerable. Define $U=T\backslash\cup_{r\in N}\boldsymbol{G}^{-1}(r)$, where $\backslash$ denotes set-theoretic subtraction. Then U is full.

Suppose 0 is in the interior of $\Delta(U)$. Then there is a point $x > 0$ such that $-x\in\Delta(U)$; by definition of $\Delta(U)$, $-x$ is a convex combination of finitely many points in $\cup_{t\in U}\boldsymbol{G}(t)$. That is, there are traders $t_1, \ldots, t_k\in U$ (not necessarily distinct), points $x_i\in\boldsymbol{G}(t_i)$, and positive numbers $\beta_1, \ldots, \beta_k$ summing to 1, such that $\Sigma_{i=1}^k\beta_i x_i=-x<0$. By the continuity assumption (2.3), we may find rational points $r_i\in\boldsymbol{G}(t_i)$ sufficiently close to the x_i, and positive rational numbers γ_i sufficiently close to the β_i, so that we still have $\Sigma_{i=1}^k\gamma_i r_i<0$. Let $-r=\Sigma_{i=1}^k\gamma_i r_i$, and pick an arbitrary trader t_0 in U. Since $r>0$, we have $\alpha r+\boldsymbol{i}(t_0)>\boldsymbol{x}(t_0)$ for sufficiently large positive rational α. Hence by the desirability assumption (2.2), $\alpha r+\boldsymbol{i}(t_0) \succ_t \boldsymbol{x}(t_0)$, i.e., $\alpha r\in\boldsymbol{G}(t_0)$. Now set $r_0=\alpha r$, $\alpha_0=1/(\alpha+1)$, $\alpha_i=\alpha\gamma_i/(\alpha+1)$ for $i=1, \ldots, k$. Then $\alpha_i>0$ for all i, and $\Sigma_{i=0}^k\alpha_i=1$; furthermore

$$\textstyle\sum_{i=0}^k\alpha_i r_i=\frac{\alpha}{\alpha+1}r+\frac{\alpha}{\alpha+1}\sum_{i=1}^k\gamma_i r_i=0\ ,$$

and $r_i\in\boldsymbol{G}(t_i)$ for all i. Then $t_i\in\boldsymbol{G}^{-1}(r_i)$, and, since $t_i\in U$, it follows that $r_i\notin N$. Therefore $\boldsymbol{G}^{-1}(r_i)$ is of positive measure for each i. Therefore, for a sufficiently small positive number δ, we can find disjoint subsets S_i of $\boldsymbol{G}^{-1}(r_i)$ such that $\mu(S_i)=\delta\alpha_i$, where μ is Lebesgue measure. Define a coalition S by $S=\cup_{i=0}^k S_i$, and an assignment $\boldsymbol{y}$ by

$$\boldsymbol{y}(t)=\begin{cases} r_i+\boldsymbol{i}(t) & \text{for}\quad t\in S_i\ ,\\ \boldsymbol{i}(t) & \text{for}\quad t\notin S\ .\end{cases}$$

That $\boldsymbol{y}(t)\in\Omega$ is trivial for $t\notin S$, and for $t\in S_i$ it follows from $r_i\in\boldsymbol{G}(t)$ (which in turn follows from $S_i\subset\boldsymbol{G}^{-1}(r_i)$). Next,

$$\textstyle\int_S\boldsymbol{y}=\Sigma_{i=0}^n\delta\alpha_i r_i+\int_S\boldsymbol{i}=0+\int_S\boldsymbol{i}\ ,$$

and therefore S is effective for $\mathbf{y}$; since $\mathbf{y}(t)=\mathbf{i}(t)$ for $t\notin S$, it follows that $\mathbf{y}$ is an allocation. Finally, from $S_i\subset \mathbf{G}^{-1}(r_i)$ it follows that $r_i+\mathbf{i}(t) \succ_t \mathbf{x}(t)$ for $t\in S_i$; in other words, $\mathbf{y}(t) \succ_t \mathbf{x}(t)$ for $t\in S$. Since S is of positive measure, we have shown that $\mathbf{x}$ is not in the core, contrary to assumption. This proves the lemma.

Let U be as in the lemma. To avoid annoying repetitions, let us agree that in the remainder of the proof, statements about traders will refer to $t\in U$. This is sufficient because U is full.

From the lemma and the supporting hyperplane theorem we obtain a hyperplane $p\cdot x=0$ that supports $\Delta(U)$. Therefore it also supports each of the $\mathbf{G}(t)$; thus $p\cdot x\geqq 0$ for $x\in\mathbf{G}(t)$, or

(4.2) $\quad p\cdot x \geqq p\cdot\mathbf{i}(t) \quad$ for $\quad x\in\mathbf{F}(t)$.

By desirability (2.2), each $\mathbf{F}(t)$ contains a translate of the positive orthant; therefore $p\geqq 0$. We shall show that $(p,\mathbf{x})$ is a competitive equilibrium.

We first show that for almost all t, $\mathbf{x}(t)$ is in t's budget set, i.e.,

(4.3) $\quad p\cdot \mathbf{x}(t) \leqq p\cdot\mathbf{i}(t)$.

Indeed, because of desirability (2.2), there are bundles arbitrarily close to $\mathbf{x}(t)$ that t prefers to $\mathbf{x}(t)$. Therefore $\mathbf{x}(t)$ is in the closure of $\mathbf{F}(t)$, and so, by (4.2), $p\cdot\mathbf{x}(t) \geqq p\cdot\mathbf{i}(t)$. If, for a nonnull t-set, we would have $p\cdot\mathbf{x}(t) > p\cdot\mathbf{i}(t)$, then it would follow that $\int_T p\cdot\mathbf{x} > \int_T p\cdot\mathbf{i}$, contrary to the assumption that $\mathbf{x}$ is in the core and hence is an allocation. This demonstrates (4.3) for almost all t; we may and will assume that all traders satisfy it.

To complete the proof, we must show that $\mathbf{x}(t)$ is maximal in t's budget set, i.e., that (4.2) can be sharpened to

(4.4) $\quad p\cdot x > p\cdot\mathbf{i}(t) \quad$ for $\quad x\in\mathbf{F}(t)$.

To this end, we first establish $p>0$. Suppose not; let $p^1=0$, say. Since $p\neq 0$, some coordinate of p does not vanish; let $p^2>0$, say. By (2.1), $\int_T \mathbf{i}^2>0$. Since $\mathbf{x}$ is an allocation, it follows that $\int_T \mathbf{x}^2>0$, so there must be a nonnull set of traders t for whom $x^2(t)>0$. Now for any trader t, it follows from desirability (2.2) that

$$\mathbf{x}(t)+(1,0,\ldots,0) \succ_t \mathbf{x}(t) .$$

Hence choosing t so that $\mathbf{x}^2(t)>0$, we deduce from continuity (2.3) that for sufficiently small $\delta>0$,

$$\mathbf{x}(t)+(1,\,-\delta,0,\ldots,0) \succ_t \mathbf{x}(t) .$$

Then by (4.2),

$$\begin{aligned} p\cdot\mathbf{i}(t) &\leqq p\cdot[\mathbf{x}(t)+(1,\,-\delta,0,\ldots,0)] \\ &= p\cdot\mathbf{x}(t)+p^1-\delta p^2 \\ &= p\cdot\mathbf{x}(t)-\delta p^2 \\ &< p\cdot\mathbf{x}(t) , \end{aligned}$$

contradicting (4.3). This proves $p>0$.

To demonstrate (4.4), let $x \in F(t)$. Suppose first that $i(t) \geq 0$; then $p \cdot i(t) > 0$, because $p > 0$. So by (4.2), $p \cdot x > 0$; hence, there is j such that $x^j > 0$; let $j = 1$, say. From continuity (2.3), it then follows that $x - (\delta, 0, \ldots, 0) \in F(t)$ for sufficiently small $\delta > 0$. Then by (4.2),

$$p \cdot i(t) \leqq p \cdot [x - (\delta, 0, \ldots, 0)] = p \cdot x - \delta p^1 < p \cdot x,$$

proving (4.4). If $i(t) = 0$ and $x \geq 0$, then clearly $p \cdot x > 0 = p \cdot i(t)$, proving (4.4). Finally, suppose $i(t) = 0$ and $x = 0$. Since $x \in F(t)$, this means that $i(t) = 0 \succ_t x(t)$. If the set S of traders t for whom this happens is null, then it can be ignored; if, on the other hand, it has positive measure, then $\boldsymbol{i}$ dominates $\boldsymbol{x}$ via S, contradicting the membership of $\boldsymbol{x}$ in the core. This completes the proof of the main theorem.

5. DISCUSSION OF THE LITERATURE

The concept of competitive equilibrium is so well known in economics that we content ourselves with citing the papers of Walras [**23**], Wald [**22**], Arrow-Debreu [**1**], and McKenzie [**14**]; which are but a selection from an enormous literature. In discussing competitive equilibrium, Wald [**22**] acknowledged that the economic validity of his considerations is based on the assumption that "each of the participants is of the opinion that his own transactions do not influence the prevailing prices." But, like most workers in this area, he did not consider it necessary to modify either the finiteness of the model or the definition of competitive equilibrium. Von Neumann and Morgenstern saw more deeply into the problem: "The fact that every participant is influenced by the anticipated reactions of the others to his own measures . . . is most strikingly the crux of the matter in the classical problems of duopoly, oligopoly, etc. When the number of participants becomes really great, some hope emerges that the influence of every particular participant will become negligible . . . These are, of course, the classical conditions of 'free competition' . . . . The current assertions concerning free competition appear to be very valuable surmises and inspiring anticipations of results. But they are not results, and it is scientifically unsound to treat them as such" [**16**, pp. 13–14]. The results of Scarf-Debreu [**6**, **7**, **18**], Shubik [**21**], and this paper are presumably precisely the kind of "results" to which von Neumann and Morgenstern were referring.

The concept of core is well known in game theory. Though first named and intensively studied by Gillies and Shapley,[6] it had been used repeatedly already by von Neumann and Morgenstern [**16**]. This work was confined to games with side payments. An extension of the core notion to games without side payments—of which the market under consideration is an example—was made by Aumann and Peleg [**2**, **3**]. The notion of Pareto optimality is related to that of core, but only

[6] The term "core" was introduced by Gillies and Shapley during a study of the properties of the von Neumann-Morgenstern solutions (see [**9**, **10**]). The core as an independent solution concept was developed by Shapley in lectures at Princeton University in the fall of 1953.

distantly: Whereas the core consists of all those outcomes of a game with the property that no coalition of players can do better through its own efforts alone, Pareto optimality demands this only for the all-player coalition.

In economics, Edgeworth [**8**] was the first to investigate the core, albeit under a different name ("contract curve"). He, and subsequently Shubik [**21**], considered markets with two commodities and two "types" of traders, where traders of the same "type" have the same initial bundles and the same preferences. Under appropriate conditions,[7] they showed that, as the number of traders of each type tends to infinity, the core of the market in a certain sense "shrinks to a limiting point"; this point can be identified with the equilibrium allocation (which is unique in the markets they were considering). Recently, Scarf and Debreu [**7**, **18**] generalized this work considerably by allowing an arbitrary but fixed finite number k of "types" of traders, rather than just two, and an arbitrary number of commodities. As the number of traders of each type tends to ∞, the core "shrinks to a limiting set," which can be identified with the set of equilibrium allocations. In this process, the number k of types is held fixed.

These results are special cases of the rough conjecture stated in the introduction that "the core approaches the set of equilibrium allocations as the number of traders tends to infinity." Supplying a precise statement of this is a nontrivial problem. The trouble is that the core and the set of equilibrium allocations are subsets of a space whose dimension varies with the number of traders. Thus as we add traders, the space under consideration changes, and in such a context it is not clear what "approach" means. It was because of this difficulty that Debreu and Scarf [**7**] postulated a fixed finite number of types of traders; together with some other assumptions, this enabled them to work in a fixed finite dimensional space. The other assumptions are that the preferences are quasi-orders[8] and that the indifference surfaces are strictly convex (i.e., convex and contain no straight line segment).

The notion of finitely many types might not at first sight seem objectionable. But it involves the further assumption that there are "many" traders of each type; in fact the number of traders of each type must be very large compared to the number of types in order for their model to be applicable. This seems far from economic reality, where, in general, different traders cannot be expected to have the same initial bundles or the same preferences. The continuous model allows *all* traders to have different initial bundles and different preferences.[9] There is no problem of working in spaces of varying dimension, because we start with a space

[7] Which differ in the two investigations.

[8] A "quasi-order" is a transitive, reflexive, and complete relation. It is usually denoted $\succsim$, not $\gtrsim$ and is to be thought of intuitively as "preference-or-indifference."

[9] It does not, of course, insist on this; a coalition of positive measure—or even all the traders—may have the same initial bundles and preferences. The concept of "type" is simply irrelevant in our approach.

of *infinite* dimension and remain in this same space throughout the investigation. For this reason also, it is not necessary to assume that the preferences are quasi-orders or that they are convex. Incidentally, Debreu and Scarf assume that each trader starts with a positive amount of each commodity, whereas we dispense with this assumption.

As we stated in the introduction, much of our proof is adapted from that of Debreu-Scarf. The key to the proof is Lemma 4.1. Debreu-Scarf prove the corresponding lemma by using the finiteness of the number of types. Our proof uses the same basic idea, after noting that the set of traders can be partitioned into a large number of nonnull coalitions whose members, though not identical, are fairly similar. The idea involved here is the same as in the approximation of an arbitrary function by a simple function, i.e., a function taking finitely many values.

Scarf [**18**] and Debreu [**6**] have also investigated a model with a *denumerable* infinity of traders. The assumptions are similar to those of [**7**], including finitely many types, positivity of initial bundles, and preferences are quasi-orders and are convex. Other references for games with denumerably many players are Kalisch and Nering [**11**] and Shapley [**20**], but these are in a non-economic context.

There are two aspects of the continuous model that we have not discussed. The first is the question of the existence of a competitive equilibrium, or equivalently, the nonemptiness of the core. We have proved only that the core equals the set of equilibrium allocations, but it may well happen that both are empty. It is possible to prove an existence theorem if it is assumed that the preferences are quasi-orders[8] (but it is not necessary to assume convexity). We plan to publish details in a subsequent paper.

The other aspect we have not discussed is the economic significance of the Lebesgue measure of a coalition. This, too, we plan to discuss in a subsequent paper.

It is a pleasure to acknowledge inspiration received from the work of Debreu, Scarf, and Shapley, and from a stimulating correspondence with Professor Debreu. Also, we thank Mr. A. Shlein for an enlightening conversation.

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REFERENCES

[1] ARROW, K. J., AND G. DEBREU: "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, Vol. 22 (1954), pp. 265–90.

[2] AUMANN, R. J., AND B. PELEG: "Von Neumann-Morgenstern Solutions to Cooperative Games without Side Payments," *Bulletin of the American Mathematical Society*, Vol. 66 (1960), pp. 173–79.

[3] AUMANN, R. J.: "The Core of a Cooperative Game without Side Payments," *Transactions of the American Mathematical Society*, Vol. 98 (1961), pp. 539–52.

[4] DAVIS, M.: "Symmetric Solutions to Symmetric Games with a Continuum of Players," in *Recent Advances in Game Theory* [**13**], pp. 119–26.

[5] DEBREU, G.: *Theory of Value*, John Wiley, 1959.

[6] ———: "On a Theorem of Scarf," *Review of Economic Studies*, June, 1963.

[7] DEBREU, G., AND H. SCARF: "A Limit Theorem on the Core of an Economy," *International Economic Review*, September, 1963.

[8] EDGEWORTH, F. Y.: *Mathematical Psychics*, London, 1881.

[9] GILLIES, D. B.: *Some Theorems on n-Person Games*, Ph.D. Thesis, Department of Mathematics, Princeton University, 1953.

[10] ———: "Solutions to General Non-Zero Sum Games," in *Contributions to the Theory of Games IV* [12], pp. 47–85.

[11] KALISCH, G. K., AND E. D. NERING: "Countably Infinitely Many Person Games," in *Contributions to the Theory of Games IV*, pp. 43–45.

[12] LUCE, R. D., AND A. W. TUCKER (editors): *Contributions to the Theory of Games IV*, Annals of Mathematics Studies No. 40, Princeton University Press, 1959.

[13] MASCHLER, M. (editor): *Recent Advances in Game Theory*, Proceedings of the Princeton University Conference of October, 1961, privately printed for members of the conference.

[14] MCKENZIE, L. W.: "On the Existence of General Equilibrium for a Competitive Market," *Econometrica*, Vol. 27 (1959), pp. 54–71.

[15] MILNOR, J. W., AND L. S. SHAPLEY: "Values of Large Games II: Oceanic Games," *The RAND Corporation*, RM 2649, February 28, 1961 (multilithed).

[16] NEUMANN, J. VON, AND O. MORGENSTERN: *Theory of Games and Economic Behavior*, Princeton University Press, 1944 (Third edition, 1953).

[17] PELEG, B.: "Quota Games with a Continuum of Players," *Israel Journal of Mathematics*, 1 (1963), pp. 48–53.

[18] SCARF, H.: "An Analysis of Markets with a Large Number of Participants", in *Recent Advances in Game Theory* [13], pp. 127–55.

[19] SHAPLEY, L. S.: "Values of Large Games III: A Corporation with Two Large Stockholders," *The RAND Corporation*, RM 2650 PR, December, 1961 (multilithed).

[20] ———: "Values of Games with Infinitely Many Players," in *Recent Advances in Game Theory* [13], pp. 113–118.

[21] SHUBIK, M.: "Edgeworth Market Games," in *Contributions to the Theory of Games IV* [12], pp. 267–78.

[22] WALD, A.: "Ueber einige Gleichungssysteme der mathematischen Ökonomie," *Zeitschrift für Nationalökonomie*, Vol. 7 (1936), pp. 637–70 (translated as "On Some Systems of Equations of Mathematical Economics," *Econometrica*, Vol. 19 (1951) pp. 368–403).

[23] WALRAS, L.: *Mathematische Theorie der Preisbestimmung der wirtschaftlichen Güter*, Stuttgart, Ferdinand Enke, 1881.