58 Economic Applications of the Shapley Value

1 Introduction

In the previous chapters, the concept of value was presented in a very abstract way. It has proved, however, to be a powerful tool in modelling some economic problems. In fact, since the Shapley value can be interpreted in terms of "marginal worth," it is closely related to traditional economic ideas. To illustrate this, we first present the Value Equivalence Theorem—the analogue of the Core Equivalence Theorem. Though other important applications exist, we then focus on three applications of the value concept to economic models other than the general equilibrium model. Each of them describes a way of departing from the market model environment. The first two are economic-political models dealing with taxation. Taxation has (at least) two purposes: redistribution and the raising of funds to finance public goods. The classical literature assumes that a benevolent government takes decisions so as to maximize some social utility function. On the contrary, analysing the government as subject to the influence of those who elected it brings new light on both aspects. Value appears to be a natural tool to deal with the voting games that are part of the two corresponding models. In the last section, we deal with economies with fixed prices.

All along this chapter, we will try to provide intuitions on why one would expect the results to hold (or not) rather than to give detailed proofs. For the real proofs, the reader is referred to the original papers.

2 The Value Equivalence Theorem

First, we present the Value Equivalence Theorem (see Hart [1994] and Aumann [1975]) similar to the Core Equivalence Theorem (see Allen & Sorin [1994] and Aumann [1964]); under certain assumptions, *in a competitive economy, every value allocation is a Walrasian allocation, and conversely*.

Define a competitive economy as $(T, \ell, e, (u_t)_{t \in T})$ where $T = [0, 1]$ stands for the set of traders, endowed with Lebesgue measure μ. $\mathbb{R}_+^\ell$ is the commodity space, $e : T \to \mathbb{R}_+^\ell$, integrable, is the initial allocation. An allocation is a (measurable) map $\mathbf{x} : T \to \mathbb{R}_+^\ell$ such that $\int_T \mathbf{x}_t \mu(dt) = \int_T e_t \mu(dt)$. $u_t : \mathbb{R}_+^\ell \to \mathbb{R}$ is the utility function of t.

An individual trader is best viewed as an "infinitesimal subset" dt of T. Hence, $e_t\mu(dt)$ is trader dt's initial endowment, and $\mathbf{x}_t\mu(dt)$ denotes what

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he gets under an allocation $\mathbf{x}$, while $u_t(\mathbf{x}_t)\mu(dt)$ is the utility he derives from it. It is assumed that $u_t(0) = 0$.

Value Allocations

Consider a weight function $\lambda : T \to \mathbb{R}_+$ (integrable). The worth $v_\lambda(S)$ of coalition $S \subset T$ with respect to λ is the maximum "weighted total utility" it can get on its own; *i.e.* by properly reallocating the total initial endowment of members of S among themselves. In the previous setting,

$$v_\lambda(S) = \max\left\{ \int_S \lambda_t u_t(\mathbf{x}_t)\mu(dt) \text{ such that } \int_S \mathbf{x}_t\mu(dt) = \int_S e_t\mu(dt) \right\}$$

The value of a trader is his average marginal contribution to the coalitions to which he belongs, where "average" means expectation with respect to a distribution induced by a random order of the players. Thus, the value of trader dt is

$$(\varphi v_\lambda)(dt) = E[v_\lambda(S_{dt} \cup dt) - v_\lambda(S_{dt})],$$

where S_{dt} is the set of players "before dt" in a random order (see Aumann [1994] and Neyman [1994]).

An allocation $\mathbf{x}$ is called a *value allocation* with respect to λ if

$$(\varphi v_\lambda)(dt) = \lambda_t u_t(\mathbf{x}_t)\mu(dt).$$

We want to prove that $\mathbf{x}$ is also a Walrasian allocation.

First, remark that, using $u_t'(x)$ for the gradient of u_t at x,

$$\forall(t_1, t_2) \in T^2, \lambda_{t_1} u_{t_1}'(\mathbf{x}_{t_1}) = \lambda_{t_2} u_{t_2}'(\mathbf{x}_{t_2});$$

otherwise society could always profitably reallocate its initial endowment. This would contradict the fact that $\mathbf{x}$ is a value allocation. Call p this common value.

Second, the same applies to S_{dt}. Further S_{dt} can be considered as a perfect sample of T, in the sense that in S_{dt} the distribution of players corresponds to that in T. Therefore, the common value of the gradients for S_{dt} is also p.

Hence, dt's contribution to S_{dt} is twofold: dt's weighted utility and the change in other traders' aggregate utility. Under the new optimal allocation of the initial endowment of $S_{dt} \cup dt$, dt gets $\mathbf{x}_t\mu(dt)$ and therefore its weighted utility is $\lambda_t u_t(\mathbf{x}_t)\mu(dt)$. Hence, $e_t\mu(dt) - \mathbf{x}_t\mu(dt)$ has to be distributed among the traders in S_{dt}. Their increase in utility is then $p \cdot [e_t - \mathbf{x}_t]\mu(dt)$.

Hence,

$$(\varphi v_\lambda)(dt) = p \cdot [e_t - \mathbf{x}_t]\mu(dt) + \lambda_t u_t(\mathbf{x}_t)\mu(dt)$$

and

$$p \cdot [e_t - \mathbf{x}_t] = 0$$

Now, $(\mathbf{x}_t, u_t(\mathbf{x}_t))$ is on the boundary of the convex hull of the graph of u_t. The idea is that otherwise, it would be possible to split trader dt (meaning his endowment) into several players so that the weighted sum of their utilities would be greater than dt's utility. This transformation taking place inside T, this would be contradictory to $\mathbf{x}$ being a value allocation. This result and the equality of gradients lead to:

$$\forall x \in \mathbb{R}_+^{\ell}, u_t(\mathbf{x}_t) - u_t(x) \geqslant p \cdot (\mathbf{x}_t - x).$$

Hence, $\mathbf{x}_t$ maximizes $u_t(x)$ under the constraint $p \cdot x \leqslant p \cdot \mathbf{x}_t$, and hence under the constraint $p \cdot x \leqslant p \cdot e(t)$.

So $\mathbf{x}_t$ is a Walrasian allocation corresponding to the price system p.

3 Taxation and Redistribution

Before 1977, perhaps the most fundamental element in the theory of the public sector was that the government was regarded as an exogenous benevolent economic agent who tried to maximize some social utility, usually the sum of individual utilities (see Arrow & Kurz [1970]). On the other hand, within a democratic system, a person can vote and try to influence the government's decision according to its own utility. This section, based on Aumann & Kurz [1977], aims at taking this idea into account in what concerns taxation and redistribution. It introduces a model in which each agent's power is reflected in two spheres: politics and economics. This *Income Redistribution Game* is very simple: each agent has an initial endowment and a utility function, a tax and redistribution policy is decided by majority voting but every agent can destroy part or all of his endowment. The idea is that while any majority can expropriate the corresponding minority, anyone can, for example, decide not to work so that the others get nothing from expropriating him. Though he does not feel better in this case (no utility of leisure is assumed), he can use this as a threat to make the majority compromise. This will influence the nature of the majority coalition formed and the tax policy it enforces.

We start from the previous model but with a single commodity. Since we want to accomodate for threats and for non-transferable utility among agents, we are using Harsanyi-Shapley NTU value. Suppose a weight function λ has been fixed somehow. Then the aggregate utility of T is

$$v_\lambda(T) = \max\left\{ \int_T \lambda_t u_t(\mathbf{x}_t)\mu(dt) \text{ such that } \int_T \mathbf{x}_t\mu(dt) = \int_T e_t\mu(dt) \right\}$$

Suppose now that two complementary coalitions S and $T\backslash S$ have formed. Think of $v_\lambda(S)$ as being the aggregate utility of S if it forms and bargains against $T\backslash S$. As in Nash [1953], suppose that the two parties can commit to carry out *threat strategies* if no satisfactory agreement is reached. If under these strategies S and $T\backslash S$ get respectively f and g, the two parties are bargaining for $v_\lambda(T) - f - g$ and, under the symmetry assumption, this is split evenly. Hence, S gets $1/2(v_\lambda(T) + f - g)$ and $T\backslash S$ gets $1/2(v_\lambda(T) + g - f)$ so that the derived game between S and $T\backslash S$ is a constant-sum game.

The optimal threat strategy for the majority coalition is 100% tax since it can at least ensure its own endowment while the optimal threat strategy for the minority is precisely to destroy all of its endowment so that the majority cannot ensure more than its endowment. Hence, the reduced game value is $q(S) =$

$$\begin{cases} \max\{\int_S \lambda_t u_t(\mathbf{x}_t)\mu(dt) \text{ s.t. } \int_S \mathbf{x}_t\mu(dt) = \int_S e_t\mu(dt)\} & \text{if } \mu(S) > \frac{1}{2} \\ 0 & \mu(S) < \frac{1}{2} \end{cases}$$

and we have

$$v_\lambda(S) = \tfrac{1}{2}[q_\lambda(T) + q_\lambda(S) - q_\lambda(T\backslash S)]$$

It can be shown that

$$(\varphi v)(dt) = (\varphi q)(dt) = E[q(S_{dt} \cup dt) - q(S_{dt})]$$

As in the previous section, S_{dt} is almost certainly a perfect sample of T. We can then use the self-explanatory notation $S_{dt} = \theta T$, with $\theta \in [0, 1]$ being the size of the sample. So θ can also be viewed as the random time when dt enters the room, thus is uniformly distributed. Hence,

$$(\varphi q_\lambda)(dt) = \int_0^1 [q(\theta T \cup dt) - q(\theta T)]d\theta$$

We may have three different situations:

1. θT is a majority, $\theta T \cup dt$ is still majority;
2. $\theta T \cup dt$ is minority (and so is θT);
3. θT is minority, and $\theta T \cup dt$ is majority, i.e. dt is pivotal.

Suppose $q_\lambda(T)$ is achieved at $\mathbf{x}$. Then, for each case, we have dt's expected contribution—case 1 being as in the last section:

1. $\frac{1}{2}\lambda_t[u_t(\mathbf{x}_t) - u_t'(\mathbf{x}_t)\cdot(\mathbf{x}_t - e_t)]\mu(dt)$ $[= \int_{1/2}^{1}[q(\theta T \cup dt) - q(\theta T)]d\theta]$
2. $\frac{1}{2}(0-0)$ $[= \int_0^{(1/2)-\mu(dt)}(0)d\theta]$
3. $\frac{1}{2}q_\lambda(T)\mu(dt)$ $[= \int_{(1/2)-\mu(dt)}^{1/2}[q(\frac{1}{2}T) - 0]dt]$

By definition, $\varphi q_\lambda(dt) = \lambda_t u_t(\mathbf{x}_t)\mu(dt)$ and, as in the last section, $\lambda_t u_t'(\mathbf{x}_t)$ is constant in t, say p. We get thus

$\lambda_t u_t(\mathbf{x}_t) = q_\lambda(T) - p\cdot(\mathbf{x}_t - e_t)$ μ-a.e., or (single commodity . . .):

$$e_t - \mathbf{x}_t = \frac{u_t(\mathbf{x}_t)}{u_t'(\mathbf{x}_t)} - C \quad \text{where } C = [q_\lambda(T)]/p \text{ is a constant.}$$

It can be shown that satisfying a condition of the kind

$$e_t - \mathbf{x}_t = \frac{u_t(\mathbf{x}_t)}{u_t'(\mathbf{x}_t)} - c$$

is a necessary and sufficient condition for $\mathbf{x}$ to be a value allocation (meaning that λ is allowed to vary). Moreover, there exists a single solution ($\mathbf{x}$, c) and c is positive. This constitutes the main result in Aumann & Kurz [1977].

Let us now look for the economic interpretation. The left-hand side of the equality is the (signed) tax on t. Notice that under the assumption that utility functions u_t are increasing and concave, e_t is increasing in $\mathbf{x}_t$ or, more intuitively, $\mathbf{x}_t$ is increasing in e_t—but with slope $< 1/2$. That is to say, marginal tax rates are between 50% and 100%.

Despite that no explicit uncertainty was introduced in the model, call *fear of ruin* the ratio $u_t(\mathbf{x}_t)/u_t'(\mathbf{x}_t)$. In fact, consider the reciprocal u'/u. Suppose that some player is ready to play a game in which with probability $(1-p)$ his initial fortune x is increased by a small amount ε and with probability p he is ruined and his fortune is 0. Certainly, if he is indifferent between playing the game or not, p/ε is a measure of his boldness (at x). Indifference implies:

$$u(x) = p\cdot 0 + (1-p)\cdot u(x+\varepsilon)$$

Hence, when ε goes to zero, p/ε goes to $u'(x)/u(x)$.

Thus, the tax equals the fear of ruin at the net income, less a constant tax credit.

Example Let us end this section with an example where we can explicitly calculate the tax policy. Consider $T = [0,1]$ with identical traders

$$u_t(x) = u(x) = x^\alpha \quad 0 < \alpha \leqslant 1$$

The fear of ruin is

$$\frac{u(x)}{u'(x)} = \frac{x^\alpha}{\alpha x^{\alpha-1}} = \frac{x}{\alpha}$$

By integrating we get

$$C = \int_T \frac{u(\mathbf{x})}{u'(\mathbf{x})} + 0 = \int_T \frac{\mathbf{x}}{\alpha} = \frac{\int_T e}{\alpha}$$

since $\int_T e = \int_T \mathbf{x}$. Hence,

$$\mathbf{x}_t = \frac{\alpha}{1+\alpha} \cdot e_t + \frac{\int_T e}{1+\alpha}$$

4 Public Goods without Exclusion

The second purpose of taxation is to raise funds to finance the production of public goods. As in the case of redistribution, considering government as subject to the influence of its electors instead of benevolently maximizing a given social welfare function sheds new light on the subject. A *Public Goods Economy* is modelled, where a continuum of agents, endowed with resources and voting rights, take part in the production of non-exclusive public goods. More precisely, when a coalition forms, it chooses one strategy amongst the available, which together with the complementary coalition's choice determines which bundle of public goods will be produced. A natural question is the dependence of the outcome of the game on the distribution of voting rights, which should at first sight exert a major influence. But the Harsanyi-Shapley NTU Value leads to the surprising new and pessimistic result that the distribution of voting rights has (little or) nothing to do with the choice. Aside from its proof, this result is reinforced by an economic argument based on the implicit price of a vote.

A non-atomic public goods economy is modelled by the set of agents $T = [0, 1]$, the space $\mathbb{R}_+^\ell$ of resources, and the public goods space $\mathbb{R}_+^m$. G is a correspondence from $\mathbb{R}_+^\ell$ to $\mathbb{R}_+^m$ representing the production function. G is supposed to take compact and nonempty values. $u_t : \mathbb{R}_+^m \to \mathbb{R}$ is t's utility function. v is a non-atomic voting measure with $v(T) = 1$.

Every person has resources that can be used to produce public goods. The voting measure v is not necessarily identical to the distribution of population μ. For example, it is possible that noncitizens do not have the right to vote.

A vote takes place and the majority decides which public good will be produced. The minority is not entitled to produce public goods but, as in the previous model, it has the right to destroy its resources. The important thing is that, once the public good has been produced, the minority may not be excluded from consuming it.

THEOREM 1 *In the voting game, the value outcomes are independent of the voting measure* v.

Consider an example with 2 public goods, TV and libraries. There are two kinds of people in two equally weighted sets, the ones fond of TV programmes and the others fond of books. Assume further that TV fans possess all the voting rights: you might expect TV programmes to be the only leisure available after voting, but it happens to be both TV and books in an equal manner! Whatever the voting rights, the same bundle of public goods will be chosen.

Sketch of Proof We first describe the definition of the Harsanyi-Shapley NTU value of a game, already used in the last section. As there, for a fixed weight function λ, every coalition S announces a threat strategy z_S it would carry out in case negotiations would break down with $T\backslash S$. Together with that of the complementary coalition, it yields S a total payoff: $V(S) = U(z_S, z_{T\backslash S})(S)$. After those announcements, players are thus in a fixed threat, TU game, with as solution the Shapley value. Players want thus the threats of the different coalitions they are members of to be chosen such as to maximize their own final payoff according to this Shapley value. This can be shown to imply that all members of any given coalition S unanimously want to maximize $H(S) = V(S) - V(T\backslash S)$. Since the same holds for $T\backslash S$, the *optimal threat strategies* z_S^* and $z_{t\backslash S}^*$ are the saddle points of the two-person zero-sum game $H(S)$. Let $q_\lambda(S) = U(z_S^*, z_{T\backslash S}^*)(S)$ be the total payoff to S when both S and $T\backslash S$ carry their optimal threat out. Define $w_\lambda(S) = q_\lambda(S) - q_\lambda(T\backslash S) = H(S)(z_S^*, z_{T\backslash S}^*)$. $w_\lambda(S)$ measures the bargaining power of S (its ability to threaten). Define also $v_\lambda(S) = 1/2(w_\lambda(S) + w_\lambda(T))$. As argued in the last section, $v_\lambda(S)$ is the total utility S can expect to result from an efficient compromise with $T\backslash S$. Observe that $\varphi v_\lambda = \varphi q_\lambda$ (since the difference is a game where every coalition gets the same as its complement), but whereas q_λ might depend on the particular choices of optimal threats z_S^*, v_λ no longer does. The function $v_\lambda(\cdot)$ is the Harsanyi coalitional form of the game with weight function $\lambda(\cdot)$, and we define a *value allocation* as a bundle y achieving the Harsanyi-Shapley NTU value:

$$\varphi v_\lambda(S) = \int_S \lambda_t u_t(y)\mu(dt) \text{ for every } S.$$

We know

$$(\varphi v^{v})(dt) = \int_0^1 [v(\theta T \cup dt) - v(\theta T)]d\theta$$

We want to compare this expression for two different voting measures v and ξ.

$$(\varphi v^{v} - \varphi v^{\xi})(dt) = \int_0^1 [(v^{v} - v^{\xi})(\theta T \cup dt) - (v^{v} - v^{\xi})(\theta T)]d\theta$$

We call a coalition S *even* if S is either a majority under both v and ξ or a minority under both v and ξ. If S is even, so is $T \backslash S$ and their strategic options are the same under v and $\xi : v^{v}(S) - v^{\xi}(S) = 0$. Every perfect sample of the whole population θT is even—it is determined by its size. For $\theta T \cup dt$, it is even if $\theta > 1/2$ or $\theta < 1/2 - \delta$ [with $\delta = \max(v(dt), \xi(dt))$]. The previous difference thus amounts to:

$$(\varphi v^{v} - \varphi v^{\xi})(dt) = \int_{(1/2)-\delta}^{1/2} [(v^{v} - v^{\xi})(\theta T \cup dt)]d\theta$$

$$= \frac{1}{2}\int_{(1/2)-\delta}^{1/2} [(w^{v} - w^{\xi})(\theta T \cup dt)]d\theta$$

If $w^{v}(\theta T \cup dt)$ is achieved at the outcome y, then, by additivity of the integral, and homogeneity:

$$w^{v}(\theta T \cup dt) = H^{y}(\theta T \cup dt) = U^{y}(\theta T) + 2U^{y}(dt) - U^{y}((1-\theta)T)$$

$$= (2\theta - 1)U^{y}(T) + 2\lambda_t u_t(y)\mu(dt).$$

$(2\theta - 1)$ as well as $\mu(dt)$ are infinitesimal. Independently of the voting measure v, we have shown that in the relevant range ($\theta \in [1/2 - \delta, 1/2]$), $w^{v}(\theta T \cup dt)$ is infinitesimal: the idea is that, under those circumstances, both the coalition and its complement resemble $1/2T$, thus are close to each other, and whatever the outcome, they enjoy the same utility derived from the *common* consumption of the same public good. Nobody enjoys a real bargaining advantage, and the efficient compromise induced by the Shapley value leads to equal treatment. Going back to more technical arguments, assume $u_t(y) \leqslant K$ for all feasible y and all t.

Then $U^{y}(T) \leqslant K \int \lambda_t \mu(dt)$.

Given any coalition S, and $\delta > 0$, partition S into $S_1 \cup \cdots \cup S_n$, with $(v + \xi)(S_i) \leqslant \delta$, and $n \leqslant 2/\delta$.

Then we obtain $\varphi v^{\nu}(S) - \varphi v^{\xi}(S) =$

$$\frac{1}{2}\sum_{i=1}^{n}\int_{(1/2)-\delta}^{1/2}[(2\theta-1)(U^{y_i^{\nu}}(T) - U^{y_i^{\xi}}(T))$$

$$+2\int_{S_i}\lambda_t(u_t(y_i^{\nu}) - u_t(y_i^{\xi}))\mu(dt)]d\theta$$

$$\leqslant \frac{1}{2}\cdot n \cdot \left|\int_{(1/2)-\delta}^{1/2}(2\theta-1)d\theta\right| \cdot (2K)\int \lambda_t\mu(dt) + \frac{1}{2}\cdot 4K\delta\int_S \lambda_t\mu(dt)$$

$$\leqslant 2K\delta\left[\int_T \lambda_t\mu(dt) + \int_S \lambda_t\mu(dt)\right] \leqslant 4K\delta\int_T \lambda_t\mu(dt).$$

This being true for all δ, we obtain $\varphi v^{\nu}(S) = \varphi v^{\xi}(S)$.

This being true for any weight function λ, the result follows. □

Though counter-intuitive, this result might have been guessed from a similar analysis conducted in a transferable-utility (TU) context: we allow agents to trade their votes for money. The outcome of such a TU-game is made of a public goods vector and of a side-payments vector. In the book vs TV game, it can be derived easily from the following conditions:

1. It is Pareto-optimal.
2. Under sensible assumptions, the only Pareto-optimal situation involves a production of $(1/2, 1/2)$ with the accompanying schedule of side-payments from book-lovers to TV-lovers.
3. To have this outcome approved, as book fans need only 50% of the vote and thus can play TV-fans off against each other, they drive the price of a vote down to zero and can achieve the desired outcome without effective payments.

In the case of non-exclusive public goods, every agent endowed with a voting right is potentially a free-rider: he believes his personal vote not to influence the final outcome, which he may like or dislike, and is thus ready to sell it even at a low price. This is wrong in the case of redistribution or more generally of exclusive public goods, where we assumed, as here, voting not to be secret, and where the identify of the voters is crucial in eventually determining everyone's payoff.

There remains to stress the connection between the TU and NTU situations: the TU-games that we obtain corresponding to the latter are similarly public good economies, but with in addition a single desirable private good available for transfers.

5 Economies with Fixed Prices

5.1 Introduction

In economies with fixed prices all trading must take place at exogenously given prices, which determine (together with the positive orthant) a new consumption set—the net trade set—for each trader. This model has been used to describe market failures such as unemployment. In general, price rigidities prevent market clearance: a trader's consumption set is exogenous and, under the standard assumptions, his utility function has an absolute maximum, a satiation point, generally in the interior of the consumption set; it may be the case that for any price vector, at least one of the trader's utility functions is satiated; this trader uses less than the maximum budget available to him, creating a total budget excess.

Example Consider a fixed price exchange economy with one commodity and two traders. Let their net trade sets be $X_1 = X_2 = [-5, +5]$ and their utility functions be $u_1(x) = -(x-1)^2$ and $u_2(x) = -(x+2)^2$. The satiation points being respectively 1 and -2, if $p > 0$ then $x_1^* = 0$ and $x_2^* = -2$. Hence, the market does not clear. Similarly if $p < 0$ or if $p = 0$.

This has suggested a generalization of the equilibrium concept in the general class of markets with satiation: the total budget excess is divided among all the traders, as dividends, so that supply matches demand. However, Drèze & Müller [1980] extended the First Theorem of Welfare Economics to this equilibrium concept, proving it to be too broad: with appropriate dividends, one can obtain any Pareto-optimum.

In this respect, the Shapley value leads to more specific results: the income allocated to a trader depends only and monotonically on his trading opportunities and not on his utility function! This will be formally stated. Then, a sketch of proof will be given. A formulation in the particular context of fixed price economies will then be presented.

5.2 Dividend Equilibria

Define a market with satiation as $M^1 = (T; \ell; (X_t)_{t \in T}; (u_t)_{t \in T})$, where $T = \{1, \ldots, k\}$ is a finite set of traders, $\mathbb{R}^\ell$ is the space of commodities, $X_t \subset \mathbb{R}^\ell$ is trader t's net trade set, supposed to be compact, convex, with nonempty interior and containing 0, and u_t is trader t's utility function, assumed concave and continuous on X_t.

A price vector is any element in $\mathbb{R}^\ell$.

Let $B_t = \{x \in X_t | u_t(x) = \max_{y \in X_t} u_t(y)\}$ be the set of satiation points of trader t. B_t is nonempty *i.e.* every trader has at least one satiation point. For simplicity, traders such that $0 \in B_t$ may be taken out of the economy. They are fully satisfied with their initial endowment. Thus we will suppose that $\forall t, 0 \notin B_t$.

An *allocation* is a vector $\mathbf{x} \in \prod_{t \in T} X_t$ such that $\sum_{t \in T} \mathbf{x}_t = 0$.

As noted above, competitive equilibria may fail to exist since, whatever the price vector, a trader may well refuse to make use of his entire budget, thus preventing market clearance. The idea of dividends is to let the other traders use the excess budget.

A *dividend* is a vector $c \in \mathbb{R}^k$. A *dividend equilibrium* is a triplet constituted of a price q, a dividend c and an allocation $\mathbf{x}$ such that, for all t, $\mathbf{x}_t$ maximizes $u_t(x)$ on X_t sub $q \cdot x \leqslant c_t$.

5.3 Value Allocations

This is the "finite" version of the definition in section 2.

A *comparison vector* is a non-zero vector $\lambda \in \mathbb{R}^k_+$. For each λ and each coalition $S \subset T$, the *worth of S according to* λ is

$$v_\lambda(S) = \max\left\{ \sum_{t \in S} \lambda_t u_t(\mathbf{x}_t) \text{ s.t. } \sum_{t \in S} \mathbf{x}_t = 0 \text{ and } \forall t \in S, \mathbf{x}_t \in X_t \right\}$$

$v_\lambda(S)$ is the maximum total utility that coalition S can get by internal redistribution when its members have weights λ_t.

An allocation is called a *value allocation* if there exists a comparison vector λ such that $\lambda_t u_t(\mathbf{x}_t) = \varphi v_\lambda(t)$ where φv_λ is the Shapley value of the game v_λ.

5.4 The Main Result

M^n, the n-fold replica of market M^1, is the market with satiation where every agent of M^1 has n twins. Formally stated:

$T^n = \cup_{i \in T} T_i^n$ – set of nk traders.

$\forall i \in T, |T_i^n| = n$ – there are n traders of type i.

$\forall t \in T_i^n, u_t = u_i$ and $X_t = X_i$.

ELBOW ROOM ASSUMPTION $\forall J \subset \{1, \ldots, k\}$

$$0 \notin bd\left[\sum_{i \in J} B_i + \sum_{i \notin J} X_i\right]$$

To put this in words, if it is possible to satiate simultaneously all traders in any J then it is also possible to do so when they are restricted to the relative interior of their satiation sets, and the others to that of their net trade sets. Note that since the right-hand side is the boundary of a convex subset of $\mathbb{R}^{\ell}$, its dimension is at most $(\ell - 1)$. Since the possible J are finite in number, the assumption holds for all but an $(\ell - 1)$-dimensional set of total endowments. In that sense, it is generic.

An allocation $\hat{\mathbf{x}}$ of M^n is an *equal treatment* allocation if traders of the same type are assigned the same net trade. Trivially, there is then a corresponding allocation $\mathbf{x}$ in M^1.

THEOREM 2 *Consider a sequence* $(\mathbf{x}^n)_{n \in N}$ *where* $\mathbf{x}^n$ *is an allocation corresponding to an equal treatment value allocation in* M^n. *Let* $\mathbf{x}^{\infty}$ *be a limit of a subsequence of* $(\mathbf{x}^n)_{n \in N}$. *Then, there is a dividend (vector)* c *and a price vector* q *such that* $(q, c, \mathbf{x}^{\infty})$ *is a dividend equilibrium where*

c *is nonnegative* i.e. $\forall i, c_i \geqslant 0$
c *is monotonic* i.e. $\forall i, X_i \subset X_j \Rightarrow c_i \leqslant c_j$.

What gives substance to the theorem is the following existence result.

PROPOSITION 1 *There exists an equal treatment value allocation for every* M^n.

Sketch of Proof of the Theorem This proof is very informal. To make things simpler, suppose that:

- $\sum_i \lambda_i^n = 1$ (normalization);
- $\forall i, \mathbf{x}_i^n$ and λ_i^n converge;
- $\forall i, \mathbf{x}_i^n$ and $\mathbf{x}_i^{\infty}$ are in $\mathrm{int}(X_{it})$;
- $\forall i, u_i$ is strictly concave and continuously differentiable on X_i.

Call *lightweight* those types i such that $lim_{n \to \infty} \lambda_i^n = 0$ and *heavyweight* the rest. Suppose that all lightweight types' weights converge to 0 at the same speed.

We have

$$\forall (i,j), \forall n, \lambda_i^n u_i'(\mathbf{x}_i^n) = \lambda_j^n u_j'(\mathbf{x}_j^n) \ (:= q^n)$$

In the limit,

$$\forall (i,j), \lambda_i^{\infty} u_i'(\mathbf{x}_i^{\infty}) = \lambda_j^{\infty} u_j'(\mathbf{x}_j^{\infty}) \ (:= q^{\infty})$$

Now, consider the contribution of a trader to a coalition S.

If S is "large enough," it is very likely to be a good sample of the population T^n. Thus an optimal allocation for S is approximately the optimal $\mathbf{x}^n$ for T^n. The first term of the new trader's contribution is what he gets for himself. Since he does not change the optimal allocation by much this is $\lambda_i^n u_i(\mathbf{x}_i^n)$. The second term is his influence on the other traders' utility. Since the net trade must equal zero and all gradients are equal, this is approximately $-q^n \cdot \mathbf{x}_i^n$. Thus, the contribution is approximately $\Delta = \lambda_i^n u_i(\mathbf{x}_i^n) - q^n \cdot \mathbf{x}_i^n$.

If S is "too small," the previous considerations do not hold. However, a new trader's contribution to a small coalition is uniformly bounded. This follows from the continuous differentiability of utilities on compact net trade sets. Moreover, the probability P^n of S being a small coalition goes to zero as n goes to infinity. Denote by δ_i^n the expected contribution of t conditional on the coalition being small.

Now we have (very roughly):

$$\varphi v_\lambda^n(t) \approx (1 - P^n)\Delta + P^n \delta_i^n$$

Since $\varphi v_\lambda^n(t) = \lambda_i^n u_i(\mathbf{x}_i^n)$ we have:

$$\lambda_i^n u_i(\mathbf{x}_i^n) \approx (1 - P^n)\Delta + P^n \delta_i^n$$

Hence,

$$q^n \cdot \mathbf{x}_i^n \approx \frac{P^n}{1 - P^n}(\delta_i^n - \lambda_i^n u_i(\mathbf{x}_i^n)) \tag{A}$$

Note that $P^n/(1 - P^n) \to 0$.

Suppose there is no lightweight type. If simultaneous satiation of all traders is possible, then this is the Shapley value for all n sufficiently large, since then $\lambda_i^n > 0\ \forall i$. It is clear that the theorem holds then, for any price system q, and $c_i = C$ sufficiently large. Otherwise, for at least some $i, u_i'(\mathbf{x}_i^\infty) \neq 0$. Hence, because of the equality of the gradients, $\forall j, u_j'(\mathbf{x}_j^\infty) \neq 0$. Hence $q^\infty \neq 0$ and for all i, the gradient of u_i at $\mathbf{x}^\infty$ is in the direction of q^∞. Going to the limit in (A) gives $q^\infty \cdot \mathbf{x}_i^\infty = 0$. Hence, $\mathbf{x}_i^\infty$ maximizes $u_i(x)$ on X_i sub $q^\infty \cdot x \leqslant 0$.

Hence, it is an ordinary competitive equilibrium and trivially a dividend equilibrium.

Suppose now that type i is lightweight. We have $q^\infty = 0$. Hence, for any heavyweight type j, $u_j'(\mathbf{x}_j^\infty) = 0$ that is to say $\mathbf{x}_j^\infty$ satiates j. Before letting n go to infinity, divide equality (A) by $\|q^n\|$. We shall see that δ's order of magnitude is greater than that of $\lambda_i^n u_i(\mathbf{x}_i^n)$. Assume, for simplicity, that the sequence $q^n/\|q^n\|$ converges to some point q. We get:

$$q \cdot \mathbf{x}_i^\infty = \lim_{n\to\infty} \frac{P^n}{(1 - P^n)\|q^n\|}(\delta_i^n - \lambda_i^n u_i(\mathbf{x}_i^n))$$

Denote this quantity c_i.

If $u_i'(\mathbf{x}_i^\infty) = 0$ then $\mathbf{x}_i^\infty$ maximizes u_i over X_i. Hence it maximizes u_i over $\{x \in X_i, q \cdot x \leqslant c_i\}$. The same holds if $u_i'(\mathbf{x}_i^\infty) \neq 0$, because then q is in the direction of $u_i'(\mathbf{x}_i^\infty)$.

We claim that c_i is non-negative, depends monotonically on X_i, and not at all on u_i.

A lightweight trader t's joining a "small" coalition S contibutes to three utilities: (i) his own, (ii) that of other lightweight traders, and (iii) that of heavyweight traders.

Roughly (again), since we consider a small coalition, probably all the heavyweight traders are not simultaneously satiated. Since the weights in (i) and (ii) tend to zero when n goes large, when joining the coalition, trader t's resources are best used if distributed to unsatiated heavyweight traders.

His ability to give his resources depends only and monotonically on X_t, and not on u_t.

Though the optimal redistribution of t's resources may involve several heavyweight traders, giving it all to only one trader gives a lower bound to (iii): δ_i^n is of larger order than λ_i^n.

This establishes our claim about c_i.

If trader t is heavyweight, contributions (i) and (ii) in δ_i^n are at least cancelled out by $\lambda_i^n u_i(\mathbf{x}_i^n)$, since $\lambda_i^\infty u_i(\mathbf{x}_i^\infty)$ is the maximum t can get. Thus, the rest of the argument is as before with

$$q \cdot \mathbf{x}_i^\infty \leqslant \lim_{n\to\infty}\left(\frac{\varepsilon^n}{\|q^n\|}\right) \text{ [component (iii) of } \delta]$$

and c_i defined as the right side of this inequality.

Since i is heavyweight, $\mathbf{x}_i^\infty$ satiates. By the inequality above, it also satisfies the budget constraint. □

5.5 Concluding Remarks

A question has been eluded until now: "What about the core in an economy that allows satiation?" Remember that an allocation is in the core if it cannot be improved upon by any coalition. This can be understood either in a strong or a weak way. The strong version requires that no "weak improvement" is possible. That is to say that it is not possible that some members are strictly better off while others are not worse off. With this definition the Core Equivalence Theorem for replica economies applies. Since the set of competitive equilibria may be empty, the same holds for the core. On the other hand, the "weak" core may be too large and not even enjoy the equal treatment property.

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