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AN AXIOMATIZATION OF THE NON-TRANSFERABLE UTILITY VALUE

BY ROBERT J. AUMANN[1]

Shapley's Non-transferable Utility Value correspondence is characterized by a set of axioms, which combine the features of the axioms for the value of transferable utility games, and those for Nash's solution to the bargaining problem. The axioms refer to values as payoff vectors only—the comparison weights associated with a value make no explicit appearance. A key axiom is *conditional additivity*, which may be stated as follows: If the same payoff vector is a value of each of two games, then it is also a value of the half-half probability combination of these two games, unless it is Pareto-dominated there. For the axioms to work, the boundary of the set of all feasible outcomes must be smooth.

1. INTRODUCTION

THE NTU (NON-TRANSFERABLE UTILITY) VALUE is a solution concept for multi-person cooperative games in which utility is not "transferable" (games "without side payments"). Introduced by Shapley in [**12**], it generalizes the Shapley value [**11**] for TU (Transferable Utility) games.[2] Many economic contexts are more naturally modelled by NTU than by TU games; and indeed, the NTU value has been applied with some success to a variety of economic and economic-political models.[3] Two well-known applications are Nash's solution for the bargaining problem [**7**] and for two-person cooperative games [**8**], both of which are instances of the NTU value.

The original definition [**12**] of the NTU value works roughly as follows: Given an NTU game V and a vector λ of "comparison weights" for the players, one derives a TU game v_λ, and calculates its value $\varphi(v_\lambda)$; if this value is feasible in the original NTU game V, then it is defined to be a value of V. A precise definition is given in Section 4.

Technically, the definition is reminiscent of that of the competitive equilibrium, with λ playing the role of prices, and $\varphi(v_\lambda)$ the role of the demand. Historically, it grew out of successive attacks by several investigators, notably J. Harsanyi [**4, 5**], on the value problem for NTU games. The bare definition may perhaps seem a little strange and unmotivated; but when one delves deeper [**12, 1**], one finds that it is quite natural. Nevertheless, it has been the object of controversy [**9, 10, 6, 2**].

In this paper, we offer an axiomatization of the NTU value. Like any axiomatization, it should enable us to understand the concept better, and hence to focus discussion. One can now view the NTU value as *defined* by the axioms, with the treatment in [**12**] serving as a formula or method of calculation. Thus the NTU value joins the ranks of the TU value and Nash's solution to the bargaining

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[2] I.e., games with side payments, representable by a coalitional worth ("characteristic") function.

[3] See the references of [**2**].

problem, each of which is defined by axioms, but usually calculated by a formula[4]—a formula whose intuitive significance is not, on the face of it, entirely clear.

This work is an outgrowth of ideas that have been "in the air" for many years. The problem of axiomatizing the NTU value is a natural one; already in his original paper [**12**, p. 260], Shapley discusses "properties of our ... solution that ... could be used in the derivation of our definition." Our treatment owes much to that discussion, and to subsequent oral discussions with Shapley.[5]

Worthy of particular note is that the axioms refer to values as payoff vectors only—the comparison weights associated with a value make no explicit appearance in the axioms. This is important because the question of the intuitive significance of the comparison weights has often been raised in critical discussion. By contrast, Shapley's viewpoint is that his solution consists of both the payoff vector and the comparison weights [**12**, p. 259, line 20 ff.; p. 261, line 1], with the latter playing at least as important a role as the former.[6] Also worthy of note is the smoothness condition (3.1), which is indispensable for our approach (see Section 9).

The domain of the axioms—the family of games to which they apply—is described in Section 3; the axioms themselves are presented and discussed in Section 5. Section 6 is devoted to an alternative treatment, in which one of the axioms (Independence of Irrelevant Alternatives) is dropped. Proofs are presented in Sections 7 and 8. Section 10 discusses possible variations on the theme; it also contains a discussion of the implications of the axioms for our understanding of the intuitive content of the value solution.

2. SOME NOTATION AND TERMINOLOGY

Denote the real numbers by $\mathbb{R}$. If N is a finite set, denote by $|N|$ the cardinality of N, and by R^N the set of all functions from N to $\mathbb{R}$. We will think of members x of R^N as $|N|$-dimensional vectors whose coordinates are indexed by members of N; thus when $i \in N$, we will often write x^i for $x(i)$. If $x \in R^N$ and $S \subset N$, write x^S for the restriction of x to S, i.e., the member of R^S whose ith coordinate is x^i. Write 1_S for the *indicator* of S, i.e., the member of R^N whose ith coordinate is 1 or 0 according as i is or is not in S. Call x *positive* if $x^i > 0$ for all i in N. If λ and x are in R^N, define λx in R^N by $(\lambda x)^i = \lambda^i x^i$, and denote the scalar product $\sum_{i \in N} x^i y^i$ by $x \cdot y$. Write $x \geq y$ if $x^i \geq y^i$ for all i in N. Denote the origin of R^N (the vector all of whose coordinates are 0) by O.

Let $A, B \subset R^N$ and $\lambda, x \in R^N$. Write $A + B = \{a + b: a \in A \text{ and } b \in B\}$, $\lambda A = \{\lambda a: a \in A\}$, $A + x = A + \{x\}$, and $\frac{1}{2}A = \{\frac{1}{2}x: x \in A\}$. Denote the closure of A by $\bar{A}$, its complement by $\sim A$, and its frontier $\bar{A} \cap \overline{(\sim A)}$ by ∂A. If A is convex, call it

[4] The random order expected contribution formula for the TU value, and the maximum product formula for the Nash Bargaining Problem.

[5] Specifically, the idea of adding the zero-game V^0 (see Section 8) to a given NTU game in order to obtain the induced transfer game is due to him.

[6] The importance that Shapley attaches to the endogeneous determination of comparison weights is evident from the title of his paper, as well as from its introduction.

smooth if it has a unique supporting hyperplane at each point of its frontier. Call A *comprehensive* if $x \in A$ and $x \geqslant y$ imply $y \in A$.

3. NTU GAMES

Let N be a finite set, which will henceforth be fixed; set $n = |N|$. The members of N are called *players*, its non-empty subsets *coalitions*; points in R^N are called *payoff vectors*. An *NTU game on N* (or simply *game*) is a function V that assigns to each coalition S a convex comprehensive non-empty proper subset $V(S)$ of R^S, such that

(3.1) $V(N)$ is smooth;

(3.2) if $x, y \in \partial V(N)$ and $x \geqslant y$, then $x = y$; and

(3.3) for each coalition S there is a payoff vector x such that

$$V(S) \times \{O^{N \setminus S}\} \subset V(N) + x.$$

Of these three conditions, only (3.1) is a substantive restriction from the intuitive viewpoint; the others are technical in nature. Condition (3.2) says that $\partial V(N)$ has no "level" segments, i.e., segments parallel to a coordinate hyperplane; it is a familiar regularity condition in game theory. Condition (3.3) says that if one thinks of $V(S)$ as embedded in R^N by assigning 0 to players outside S, then $V(S)$ is included in some translate of $V(N)$; it can be thought of as an extremely weak kind of monotonicity.

A *TU game* (on N) is a function v that assigns to each coalition S a real number $v(S)$. The NTU game V *corresponding* to a TU game v is given by

$$V(S) = \left\{ x \in R^S : \sum_{i \in S} x^i \leqslant v(S) \right\}.$$

If T is a coalition, define a *TU* game u_T by

$$(3.4) \qquad u_T(S) = \begin{cases} 1 & \text{if } S \supset T, \\ 0 & \text{otherwise.} \end{cases}$$

The NTU game U_T corresponding to u_T is called the *unanimity game on T*.

Operations on games are defined like the corresponding operations on sets, for each coalition separately. Thus $(V+W)(S) = V(S) + W(S)$, $(\lambda V)(S) = \lambda^S V(S)$, $\bar{V}(S) = \overline{V(S)}$, and so on.

4. SHAPLEY VALUES OF NTU GAMES

Recall that the *value* of a TU game v is the vector $\varphi(v)$ in R^N given by

$$(4.1) \qquad \varphi^i(v) = \frac{1}{n!} \sum_R [v(S_i^R \cup \{i\}) - v(S_i^R)],$$

where R ranges over all $n!$ orders on N, and S_i^R denotes the set of players preceding i in the order R. The value is usually defined by a set of axioms, which are then shown [**11**] to lead to (4.1).

Let V be a game. For each positive λ in R^N, write

$$v_\lambda(S) = \sup\{\lambda^S \cdot x: x \in V(S)\}. \tag{4.2}$$

We say that the TU game v_λ is *defined* if the right side of (4.2) is finite for all S. A *Shapley value* of V is a point y in $\bar{V}(N)$ such that for some positive λ in R^N, the TU game v_λ is defined, and $\lambda y = \varphi(v_\lambda)$. The set of all Shapley values of V is denoted $\Lambda(V)$. The set of games V for which $\Lambda(V) \neq \varnothing$—i.e., that possess at least one Shapley value—is denoted Γ^N or simply Γ. The correspondence from Γ to R^N that associates the set $\Lambda(V)$ to each game V is called the *Shapley Correspondence.*

5. THE AXIOMS

A *value correspondence* is a correspondence that associates with each game V in Γ a set $\Phi(V)$ of payoff vectors, satisfying the following axioms for all games U, V, W in Γ:

AXIOM 0—Non-Emptiness: $\Phi(V) \neq \phi$.

AXIOM 1—Efficiency: $\Phi(V) \subset \partial V(N)$.

AXIOM 2—Conditional Additivity: *If* $U = V + W$, *then*

$$\Phi(U) \supset (\Phi(V) + \Phi(W)) \cap \partial U(N).$$

AXIOM 3—Unanimity: *If* U_T *is the unanimity game on a coalition* T, *then*

$$\Phi(U_T) = \{1_T/|T|\}.$$

AXIOM 4—Closure Invariance: $\Phi(\bar{V}) = \Phi(V)$.

AXIOM 5—Scale Covariance: *If* λ *in* R^N *is positive, then* $\Phi(\lambda V) = \lambda\Phi(V)$.

AXIOM 6—Independence of Irrelevant Alternatives: *If* $V(N) \subset W(N)$ *and* $V(S) = W(S)$ *for* $S \neq N$, *then* $\Phi(V) \supset \Phi(W) \cap V(N)$.

For a fixed value correspondence Φ, call x a *value* of V if $x \in \Phi(V)$. *Efficiency* says that all values are Pareto optimal. Suppose next that y and z are values of V and W respectively. We cannot in general expect $y + z$ to be a value of $V + W$, because it need not be Pareto optimal there. *Conditional additivity* says that if $y + z$ does happen to be Pareto-optimal in $V + W$, then it is a value of $V + W$; i.e., that additivity obtains whenever it does not contradict efficiency. *Unanimity* says that the unanimity game on T has a *unique* value, which provides that the

coalition T split the available amount equally. *Closure invariance* is a conceptually harmless technical assumption; we simply do not distinguish between a convex set and its closure. If the payoffs are in utilities, then *scale invariance* says that representing the same real outcome by different utility functions does not affect the value in real terms. *Independence of irrelevant alternatives* (IIA) says that a value y of a game W remains a value when one removes outcomes other than y ("irrelevant alternatives") from the wet $W(N)$ of all feasible outcomes, without changing $W(S)$ for coalitions S other than the all player coalition. (For a thorough discussion of this assumption, see the next section.)

These axioms are an amalgam of those that characterize the value for TU games [**11**] and those that characterize Nash's solution to the Bargaining Problem [**7**]. Axioms 1, 2, and 3 are fairly straightforward analogues of the TU value axioms, with the unanimity axiom combining the symmetry and dummy axioms. As we have noted, Axiom 4 is purely technical; and Axioms 5 and 6 are essentially the same as the corresponding axioms in Nash's treatment.

THEOREM A: *There is a unique value correspondence, and it is the Shapley correspondence.*

6. AN AXIOMATIC TREATMENT WITHOUT IIA

IIA is perhaps the best-known of the axioms in the preceding section. This is partly due to its key role in Nash's work, and partly to its having stirred some controversy. In this section, after discussing the axiom, we offer an axiomatic treatment that avoids using it.

Whether or not IIA is reasonable depends on how we view the value. If we view it as an expected or average outcome, then IIA is not very convincing. By removing parts of the feasible set, we decrease the range of possible outcomes, and so the average may change even if it itself remains feasible. But in NTU games, viewing the value as an average is fraught with difficulty even without IIA, because the convexity of $V(N)$ implies that in general, an average will not be Pareto optimal.

An alternative is to view the value as a group decision or arbitrated outcome; i.e., a reasonable compromise[7] in view of all the possible alternatives open to the players. In that case IIA does sound quite convincing and even compelling. An anecdote—it happens to be a true one—may serve to illustrate its force. Several years ago I served on a committee that was to invite a speaker for a fairly prestigious symposium. Three candidates were proposed; their names would be familiar to many of our readers, but we will call them Alfred Adams, Barry Brown, and Charles Clark. A long discussion ensued, and it was finally decided to invite Adams. At that point I remembered that Brown had told me about a family trip that he was planning for the period in question, and realized that he would be unable to come. I mentioned this and suggested that we re-open the

[7] We are purposely staying away from the word "fair", in order to avoid ethical connotations.

discussion. The other members looked at me as if I had taken leave of my senses. "What difference does it make that Brown can't come," one said, "since in any case we decided on Adams?" I was amazed. All the members were eminent theorists and mathematical economists, thoroughly familiar with the nuances of the Nash model. Not long before, the very member who had spoken up had roundly criticized IIA in the discussion period following a talk. I thought that perhaps he had overlooked the connection, and said that I was glad that in the interim, he had changed his mind about IIA. Everybody laughed appreciatively, as if I had made a good joke, and we all went off to lunch. The subject was never reopened, and Adams was invited.

Note that we are discussing a true game, not an individual decision problem. The members had different interests, coalitions could be formed, etc. Occasionally issues even came to a vote; and when they did not, the vote was definitely "there," in the background. If there ever was a situation in which IIA could be criticized, this was it.

Yet I think that the members were right to laugh off my suggestion. No matter how convincing such criticism may seem in the abstract, the concrete suggestion to reconsider the choice of Adams because Brown could not come sounded—and was—absurd.

Let us nevertheless examine the consequences of omitting this axiom. It turns out that IIA is not nearly as central here as in the Nash theory; something is lost, but less than might have been expected. The result is as follows:

THEOREM B: *The Shapley correspondence is the maximal correspondence from* Γ *to* R^N *satisfying Axioms* 0 *through* 5.

More explicitly:

(6.1) Λ satisfies Axioms 0 through 5.

(6.2) If Φ satisfies Axioms 0 through 5, then $\Phi(V) \subset \Lambda(V)$ for all games V in Γ.

At issue, of course, is the categoricity of the axioms. There may[8] be many correspondences Φ satisfying Axioms 0 through 5. With Axiom 6, there is only one; the system is fully determined. On a practical level, though, there isn't much difference. Many of the applications involve necessary conditions only: they assert that every value has a particular form (e.g., competitive equilibrium). This kind of result remains unchanged when Axiom 6 is omitted. The other kind of result—every outcome of a particular form is a value—*is* weakened; but if we interpret a "value" of V to mean a member of $\Phi(V)$ for *some* Φ (rather than for a particular, fixed Φ) satisfying the axioms, then this kind of result also

[8] A referee asked for an example to show that IIA is really needed; i.e., for a correspondence other than Λ satisfying Axioms 0 through 5. We don't know of one. Thus at present, it is conceivable that Axioms 0 through 5 are already categoric.

remains true. Another kind of application in which dropping IIA changes nothing is when there is only one Shapley value ($|\Lambda(V)| = 1$); for example, this is the case for 2-person superadditive games, and in [3].

7. PROOF THAT THE SHAPLEY VALUE SATISFIES THE AXIOMS

In the remainder of the paper, we abbreviate $\partial V(N)$ by ∂V. We call a member λ of R^N *normalized* if $\max_i |\lambda^i| = 1$.

Let V be a game, and let $y \in \partial V$. Since $V(N)$ is smooth (3.1), there is a unique supporting hyperplane to $V(N)$ at y. That means that there is a unique normalized λ in R^N such that $\lambda \cdot x$ is maximized over $\bar{V}(N)$ at $x = y$. By comprehensiveness and (3.2), this λ is positive; denote it $\delta(V, y)$.

LEMMA 7.1: $\Lambda(V) \subset \partial V$.

PROOF: Follows from the efficiency of the TU value.

LEMMA 7.2: *Let $y \in \Lambda(V)$, and let $\lambda = \delta(V, y)$. Then the TU game v_λ is defined, and $\lambda y = \phi(v_\lambda)$.*

PROOF: By the definition of the Shapley value (Section 4), there is a positive μ in R^N, which we may assume normalized, such that v_μ is defined and $\mu y = \varphi(v_\mu)$. By the efficiency of the TU value,

$$\mu \cdot y = \sum \mu^i y^i = \sum \varphi^i(v_\mu) = v_\mu(N) = \sup \{\mu \cdot x : x \in V(N)\}.$$

Hence $\mu \cdot x$ is maximized over $\bar{V}(N)$ at $x = y$, i.e., $\mu = \delta(V, y) = \lambda$, and the proof is complete.

PROPOSITION 7.3: *The correspondence Λ from Γ to R^N satisfies Axioms* 0 *through* 6.

PROOF: Axiom 0 follows from the definition of Γ. Axiom 1 is Lemma 7.1. To verify Axiom 2, let $y \in \Lambda(V)$, $z \in \Lambda(W)$, $y + z \in \partial U$; we wish to show $y + z \in \Lambda(U)$. Let $\lambda = \delta(U, y + z) = \delta(V + W, y + z)$. Then $\lambda \cdot x$ is maximized over $\bar{V}(N) + \bar{W}(N)$ at $x = y + z$, and hence over $\bar{V}(N)$ at $x = y$; hence $\lambda = \delta(V, y)$. Since $y \in \Lambda(V)$ it follows from Lemma 7.3 that the TU game v_λ is defined, and $\lambda y = \varphi(v_\lambda)$. Similarly the TU game w_λ (the notations w_λ and u_λ are analogous to v_λ) is defined, and $\lambda z = \varphi(w_\lambda)$. Hence the TU game u_λ is defined, and $u_\lambda = v_\lambda + w_\lambda$. Hence by the additivity axiom for the TU value,

$$\lambda(y + z) = \lambda y + \lambda z = \varphi(v_\lambda) + \varphi(w_\lambda) = \varphi(v_\lambda + w_\lambda) = \varphi(u_\lambda). \tag{7.4}$$

But $y + z \in \bar{V}(N) + \bar{W}(N) \subset \bar{U}(N)$; together with (7.4), this shows that $y + z$ is a Shapley value of U, as was to be shown. The remaining axioms are straightforward, and so the proof of the proposition is complete.

8. PROOFS OF THE THEOREMS

Throughout this section, Φ is an arbitrary but fixed correspondence from Γ to R^N satisfying Axioms 0 through 5.

LEMMA 8.1: *If V is the game corresponding to a TU game v, then $\Phi(V)=\{\varphi(v)\}$.*

PROOF: Note first that Γ contains all games corresponding to TU games, so that we can apply our axioms to all these games at will.

Let V correspond to the TU game v. For any real number α, let V^α correspond to the TU game αv. Then V^0 corresponds to the TU game that is identically 0 (i.e., vanishes on all coalitions), and hence by Axioms 1, 2, and 3,

$$\Phi(V^0)+\{1_N/n\}=\Phi(V^0)+\Phi(U_N)\subset\Phi(V^0+U_N)=\Phi(U_N)=\{1_N/n\}.$$

By Axiom 0, it follows that

$$\Phi(V^0)=\{O\}. \tag{8.2}$$

Hence by Axioms 1 and 2, $\Phi(V)+\Phi(V^{-1})\subset\Phi(V+V^{-1})=\Phi(V^0)=\{O\}$. By Axiom 0, it follows that each of $\Phi(V)$ and $\Phi(V^{-1})$ consists of a single point, and

$$\Phi(V^{-1})=-\Phi(V). \tag{8.3}$$

If α is a positive scalar, then Axiom 5 with $\lambda=(\alpha,\ldots,\alpha)$ yields

$$\Phi(V^\alpha)=\alpha\Phi(V). \tag{8.4}$$

Combining this with (8.2) and (8.3), we deduce (8.4) for *all* scalars α, no matter what their sign is. From Axiom 3 and $\varphi(\alpha u_T)=\alpha 1_T/|T|$ we then deduce that

$$\Phi(U_T^\alpha)=\{\varphi(\alpha u_T)\} \tag{8.5}$$

for all coalitions T and all real numbers α.

Now each TU game v may be expressed in the form $v=\sum_T \alpha_T u_T$ where the α_T are real. Hence for the corresponding game V we have $V=\sum_T U_T^{\alpha_T}$. By (8.5), and Axioms 1 and 2, it follows that

$$\{\varphi(v)\}=\sum_T\{\varphi(\alpha_T u_T)\}=\sum_T\Phi(U_T^{\alpha_T})\subset\Phi\left(\sum_T U_T^{\alpha_T}\right)=\Phi(V).$$

But we have already seen that $\Phi(V)$ consists of a single point. Hence $\{\varphi(v)\}=\Phi(V)$, and the proof of the lemma is complete.

LEMMA 8.6: *$\Phi(V)\subset\Lambda(V)$ for each V in Γ.*

PROOF. Let $y\in\Phi(V)$. By Axiom 1, $y\in\partial V$. Setting $\lambda=\delta(V,y)$, we deduce from (3.2) that λ is positive. By Axiom 5 (scale covariance) applied both to Φ and to Λ, we may assume without loss of generality that $\lambda=(1,\ldots,1)$. If V^0 corresponds to the game that is identically 0, then by (3.3), $V+V^0$ is a game;[9] moreover,

[9]Condition (3.3) is needed to ensure that $(V+V^0)(S)$ does not fill all of R^S.

$y \in \partial(V + V^0)$, and $\overline{V + V^0}$ corresponds to the TU game v_λ (see (4.2)). Hence by Lemma 8.1, Axioms 2 and 4, and again Lemma 8.1, we have

$$\begin{aligned}\lambda y = y \in \Phi(V) \cap \partial(V + V^0) &= (\Phi(V) + O)) \cap \partial(V + V^0) \\ &= (\Phi(V) + \Phi(V^0)) \cap \partial(V + V^0) \subset \Phi(V + V^0) = \Phi(\overline{V + V^0}) = \{\varphi(v_\lambda)\}.\end{aligned}$$

Hence $\lambda y = \varphi(v_\lambda)$, which means that $y \in \Lambda(V)$. This completes the proof of Lemma 8.6.

Theorem B follows from Proposition 7.3 and Lemma 8.6.

LEMMA 8.7: *If Φ is a value correspondence,*[10] *then $\Lambda(V) \subset \Phi(V)$ for all V in Γ.*

PROOF: Let $y \in \Lambda(V)$. Then $y \in \bar{V}(N)$, and there is a comparison vector λ such that the TU game v_λ (see (4.2)) is defined, and

$$\lambda y = \varphi(v_\lambda).$$

Let V_λ be the game corresponding to v_λ. Define a game W by

$$W(S) = \begin{cases} V_\lambda(N) & \text{when} \quad S = N, \\ \lambda \bar{V}(S) & \text{when} \quad S \neq N. \end{cases}$$

Then λy is a Shapley value of W, so $W \in \Gamma$, so $\Phi(W)$ is defined.

Let V^0 correspond to the TU game that vanishes on all coalitions. Then $V_\lambda = \overline{W + V^0}$ and $\partial(W + V^0) = \partial W$, and so by Lemma 8.1 and Axioms 4, 2, 1, and 0, we have

$$\begin{aligned}\{\varphi(v_\lambda)\} = \Phi(V_\lambda) = \Phi(\overline{W + V^0}) &= \Phi(W + V^0) \\ &\supset (\Phi(W) + \Phi(V^0)) \cap \partial(W + V^0) \\ &= (\Phi(W) + O) \cap \partial W = \Phi(W) \neq \varnothing.\end{aligned}$$

Hence

$$\Phi(W) = \{\varphi(v_\lambda)\} = \{\lambda y\}.$$

By definition, $W(N) = V_\lambda(N) \supset \lambda \bar{V}(N)$, and $W(S) = \lambda \bar{V}(S)$ for $S \neq N$. Moreover $y \in \bar{V}(N)$ yields $\lambda y \in \lambda \bar{V}(N)$. Hence by Axioms 6, 5 and 4, $\lambda y \in \Phi(\lambda \bar{V}) = \lambda \Phi(V)$. Hence $y \in \Phi(V)$, as was to be proved.

Theorem A follows from Proposition 7.3 and Lemmas 8.6 and 8.7.

[10] Satisfies Axiom 6 as well as 0 through 5.

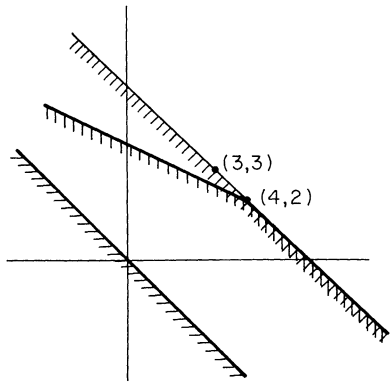

FIGURE 1.—$V(N)$, $W(N)$, and $U(N)$ are, respectively, horizontally, vertically, and diagonally hatched.

9. SMOOTHNESS

The smoothness condition (3.1) is of the essence; without it, the Shapley correspondence fails to satisfy the conditional additivity axiom, and both our theorems become irreparably false.

To see how smoothness works, let y be a Shapley value of V. The associated "comparison vector" λ always defines a supporting hyperplane to $V(N)$ at y; because of smoothness, it is the *only* supporting hyperplane. If now z is a Shapley value of W, then $y+z$ is efficient in $V+W$ if and only if the supporting hyperplanes at y and z are parallel; therefore, y and z must be associated with the *same* comparison vector, and then additivity follows from the addivity of the TU value.

Without smoothness, the reasoning breaks down. It is possible for $V(N)$ and $W(N)$ to have parallel supporting hyperplanes at y and z, by dint of which $y+z$ is efficient in $V+W$; but these need not be the hyperplanes defined by the comparison vectors that make y and z Shapley values. For example, let $N=\{1,2\}$, let V correspond to the TU game given by $v(12)=v(1)=(2)=0$, and define W by

$$W(1)=W(2)=(-\infty,0],$$

$$W(12)=\{x\in R^N: x^1+x^2\leqslant 6 \text{ and } x^1+2x^2\leqslant 8\}$$

(see Figure 1); setting $U=V+W$, we see that U corresponds to the TU game u given by $u(12)=6$, $u(1)=u(2)=0$. Then $\Lambda(V)=\{(0,0)\}$, $\Lambda(W)=\{(4,2)\}$, and $\Lambda(U)=\{(3,3)\}$; $(0,0)+(4,2)$ is efficient in U, but it is not a value. What is happening is that $V(N)$ and $W(N)$ both have hyperplanes at the respective Shapley values that are orthogonal to $(1,1)$; but the value $(4,2)$ of W is associated with the comparison vector $(1,2)$, not with $(1,1)$.

Smoothness may be interpreted as local linearity, or, if one wishes, local TU; but note that it is needed for the all-player coalition only. In the guise of

differentiability, it has played a significant role in several of the applications; so it is interesting that it makes an appearance on the foundational side as well.

10. DISCUSSION

A. *Vanishing Comparison Weights and the Non-levelness Condition*

Shapley's treatment [**12**] permits some of the comparison weights λ^i to vanish. Ours does not. Vanishing comparison weights are undesirable for several reasons. In the direct, non-axiomatic approach, their intuitive significance is murky; and in the axiomatic approach, they greatly complicate matters. In the applications,[11] vanishing λ^i have played no significant role; in most specific cases it can be shown that the λ^i must be positive, though the definition allows them to vanish.

Our definition of "Shapley value" explicitly takes λ positive; and the non-levelness condition (3.2) assures that whatever emerges from the axioms will be associated with a positive λ. A verbal statement of (3.2) is that weak and strong Pareto optimality are equivalent.

One can avoid the non-levelness condition by strengthening the efficiency axiom to read as follows:

AXIOM 1*—Strong Efficiency: *If* $y \in \Phi(V)$, *then* $\{x \in \partial V(N)\colon x \leqq y\} = \{y\}$.

This is more than strong Pareto optimality; it says that y is in the relative (to $\partial V(N)$) interior of the strongly Pareto optimal set, or equivalently that $\delta(V, y)$ is positive.[12] If we replace Axiom 1 by Axiom 1*, then one can simply drop (3.2), and our theorems remain true.

B. *The Domain*

The domain Γ of the axioms is the set of all games that possess at least one Shapley value. This might be considered an esthetic drawback, since in this way the Shapley value enters into its own axiomatic characterization (albeit only via the domain). If one wishes to avoid this, one can replace Γ by any family Δ with the following properties:

(10.1) Every game in Δ has a Shapley value.

(10.2) All games corresponding to TU games are in Δ.

(10.3) If $V \in \Delta$ and λ is a positive vector in R^N, then $\lambda V \in \Delta$.

(10.4) The game obtained from any V in Δ by replacing $V(N)$ by any one of its supporting half-spaces is also in Δ.

(10.5) $V \in \Delta$ if and only if $\bar{V} \in \Delta$.

[11] We are referring to the existing applications to economic and/or political models, not to isolated numerical examples.

[12] In effect, (3.2) asserts that no part of the efficient surface is level, whereas 1* asserts this only on a neighborhood of the value.

For example, we may take Δ to be the family of all games V such that for all coalitions S, the set of extreme points of $\bar{V}(S)$ is bounded (or equivalently, $\bar{V}(S)$ is the sum of a compact set and a cone).

That this Δ satisfies (10.2) through (10.5) is easily verified. To see that it satisfies (10.1), let $V \in \Delta$. Let C be the set of all positive λ in R^N for which v_λ is defined and $\sum_{i \in N} \lambda^i = 1$. Then C is convex and compact, and the mapping $\lambda \to v_\lambda(S)$ is continuous on C for each S; hence also $\lambda \to \varphi v_\lambda$ is continuous on C. For each λ in C, define x_λ in R^N by $\lambda x_\lambda = \varphi v_\lambda$; let y_λ be the point in $\bar{V}(N)$ closest to x_λ; and let λ' be the normal to the supporting hyperplane of $\bar{V}(N)$ at x_λ. By (3.3), $\lambda' \in C$; hence $\lambda \to \lambda'$ is a continuous mapping from C into itself, and so has a fixed point λ^*. Then x_{λ^*}, is a Shapley value of V.

Thus Δ does satisfy (10.1) through (10.5), and so can replace Γ as the domain of the value correspondence. We adopted Γ as the domain because it is the largest on which the axioms "work,", and is thus the most useful from the point of view of applications.

Note that the above existence proof may be modified so as to make no use of smoothness.

Finally, we mention that (10.1) is not gratuitous; there are indeed games not possessing any Shapley value. For example, let $N = \{1, 2\}$, and define V by

$$V(1) = V(2) = (-\infty, 0],$$

$$V(12) = \{(\xi, \eta) \in R^N : \eta < 0 \text{ and } \xi\eta^2 \leqslant -1\}.$$

If $x = (\xi, \eta)$ were a value, then the tangent to $\partial V(N)$ at x would have a slope equal in magnitude (but opposite in sign) to the slope of the line connecting x with the origin; and this can never be, since the respective slopes are $-\eta/2\xi$ and η/ξ. The example is of course highly pathological, since each player can guarantee 0 to himself, but can never achieve this in $V(N)$; but it does show that one cannot simply take the domain to be the set of *all* games.

C. *Conditional Additivity*

The Conditional Additivity Axiom can be replaced by the following pair of axioms:

AXIOM 2*a—Conditional Sure-Thing: *If* $U = \frac{1}{2}V + \frac{1}{2}W$, *then*

$$\Phi(U) \supset \Phi(V) \cap \Phi(W) \cap \partial U(N).$$

AXIOM 2*b—Translation Covariance: *For all* x *in* R^N, $\Phi(V + x) = \Phi(V) + x$.

In Section 6, we suggested that the value of a game may be viewed as a group decision, compromise, or arbitrated outcome, that is reasonable in view of the alternatives open to the players and coalitions (rather than an outcome that is itself in some sense stable, such as a core point). In these terms, Axiom 2*a says

the following: Suppose that y is a reasonable compromise both in the game V and in the game W. Suppose further that one of the games V and W will be played; at present it is not yet known which one, but it is common knowledge that the probabilities are half-half. Then y is a reasonable compromise in this situation as well,[13] unless the players can use the uncertainty to their mutual advantage.

D. *Non-Uniqueness of the Value*

Given the above view of the value as a reasonable compromise, some readers may be disturbed by the fact that a given game V may have more than one value.[14] Non-uniqueness, they may say, is all very well for stability or equilibrium concepts; but a theory of arbitration, of reaching reasonable compromises, should "recommend" a single point.

On closer examination, there seems to be no particular reason to accept such a view. Compromises may be based on many different kinds of principles and criteria. Such criteria are usually overlapping, in the sense that a given one applies to only a limited range of situations, and to a given situation several criteria may apply. This results in a multi-valued function—a correspondence.

A good analogy is to law; in fact, one can view civil law as society's way of reaching "reasonable compromises." Specific laws always have limited ranges; these ranges often overlap and yield contradictory results. An important function of a judge is to "resolve" such contradictions in each specific case brought before him, by selecting one of the applicable laws. It is no wonder that judgements are often overturned on appeal, and that different jurisdictions reach different opinions on identical cases. Law is multi-valued, not incoherent.

In much the same way, a value correspondence is a coherent system. Its coherence is expressed by the axioms, by the way that they relate values of different games to each other. The axioms say, *if* you can decide such-and-such in case (a), *then* you can decide so-and-so in case (b). There is no reason to expect such a system to be single valued.

The original definition of the NTU value is an instance of this kind of system. Here a "criterion" is a vector λ of comparison weights, which the players (or the arbitrator) use to compare utilities. Given such a criterion, a "reasonable compromise" is the TU value $\varphi(v_\lambda)$; and the criterion "applies" to the NTU game V if $\varphi(v_\lambda)$ is feasible in V.

Our results say that *every* value system that is coherent, in the sense that it satisfies the axioms, *must* be of this specific kind.

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[13] Compare [**12**, p. 261, IV].

[14] I.e., that $\Phi(V)$ may contain several points. Of course, Theorem A guarantees that the *correspondence* Φ *is* unique.

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