72 Orderable Set Functions and Continuity III: Orderability and Absolute Continuity

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1. Introduction. This paper is one of a series of studies (cf. [1], [5], [6]) in which orderability and various continuity notions for set functions are investigated and related to each other. Throughout we assume familiarity with the concepts summarized in § 2 of [6]. Our main result (§ 5) concerns the *absolute continuity* of set functions (see [1, § 5] or § 2 of this paper). In [1, Prop. 12.8] it was shown that every absolutely continuous set function is orderable; here (§ 5) we construct an example to show that the converse is false. The example is a function of two nonatomic measures, and is in a sense "simplest possible": In § 4 we show that for functions of a single nonatomic measure, orderability and absolute continuity are equivalent.

2. Notations and definitions. We refer the reader to § 2 of [6] for a summary of some notations and definitions from [1] and [5] that will be used in this paper. Familiarity with the above section will be assumed throughout our discussions.

For x in the Euclidean space E^n, $\|x\|$ will always mean the summing norm, i.e., $\|x\| = \sum_{i=1}^n |x_i|$. If x, $y \in E^n$, write $x \leqq y$ if $x_i \leqq y_i$ for all i. If μ is a vector measure $(\mu_1, \cdots, \mu_n)$, then $\sum \mu$ will denote $\sum_{i=1}^n \mu_i$.

We next summarize some definitions and conventions from [1] which were not used in [6] and will be needed in this paper. The norm on BV is the *variation norm*, defined by

$$\|v\| = \inf \{u(I) + w(I) \mid u - w = v, \text{ where } u \text{ and } w \text{ are monotonic}\}.$$

A *chain* is a nondecreasing sequence of sets of the form $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = I$. A *link* of this chain is a pair of successive elements. A *subchain* is a set of links. A chain will be identified with the subchain consisting of all links. If v is a set function and Λ is a subchain of a chain, then the *variation of v over* Λ is defined by $\|V\|_\Lambda = \sum |v(S_i) - v(S_{i-1})|$, where the sum ranges over $\{i \mid \{S_{i-1}, S_i\} \in \Lambda\}$. For a fixed Λ, $\|\cdot\|_\Lambda$ is a pseudonorm on BV, i.e., it enjoys all the properties of a norm except $\|v\|_\Lambda = 0 \Rightarrow v = 0$. It is known (see [1, Prop. 4.1]) that for every $v \in \text{BV}$, $\|v\| = \sup \|v\|_\Lambda$, where the supremum is taken over all subchains Λ. It is also known that

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the linear subspaces M, NA, WC and ORD are closed subspaces of BV [5, Prop. 4.2 and 4.3].

A set function v is said to be *absolutely continuous with respect to* a set function w (written $v \ll w$) [1, p. 35] if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every chain Ω and every subchain Λ of Ω, $\|w\|_\Lambda \leqq \delta$ implies $\|v\|_\Lambda \leqq \varepsilon$. Note that this relation is transitive, and that if v and w are measures, it coincides with the usual notion of absolute continuity. A set function is *absolutely continuous* if there is a measure $\mu \in \mathrm{NA}^+$ such that $v \ll \mu$. The set of all absolutely continuous set functions forms a closed linear subspace of BV [1, Prop. 5.2], denoted AC. Finally, pNA denotes the closed subspace of BV spanned by all powers of nonatomic measures.

3. Weak continuity and absolute continuity. A real-valued function on a subset of E^n is said to be *monotonically absolutely continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x_1 \leqq y_1 \leqq x_2 \leqq \cdots \leqq x_n \leqq y_n$, then

$$\sum_{i=1}^{n} \|y_i - x_i\| \leqq \delta \Rightarrow \sum_{i=1}^{n} |f(y_i) - f(x_i)| \leqq \varepsilon.$$

If the domain of f is one-dimensional, then monotonic absolute continuity coincides with the usual absolute continuity.

PROPOSITION 1. *Let μ be an n-dimensional σ-additive measure whose components are in* NA^+ *and are mutually singular. Let f be a real-valued function on the range of μ with $f(0)=0$. Let $v = f \circ \mu$. Then $v \ll \sum \mu \Leftrightarrow f$ is monotonically absolutely continuous.*

Proof. The direction $\Leftarrow$ is obvious. To prove the direction $\Rightarrow$, recall Lyapunov's theorem [4], according to which the range of a nonatomic σ-additive vector measure is convex and compact. From this and the mutual singularity it follows that if $x_1 \leqq y_1 \leqq x_2 \leqq \cdots \leqq x_n \leqq y_n$, then there exist $S_1, T_1, \cdots, S_n, T_n$ in $\mathscr{C}$ such that $\mu(S_i) = x_i$, $\mu(T_i) = y_i$, and $S_1 \subseteq T_1 \subseteq \cdots \subseteq S_n \subseteq T_n$, completing the proof of Proposition 1.

PROPOSITION 2. *Let $v \in \mathrm{BV}$ and $\mu, \xi \in M^+$. If $v \ll \xi$, then $v \underset{w}{\lesssim} \mu$ if and only if $v \ll \mu$.*

Proof. Sufficiency of the condition is obvious. To see the necessity, let $\xi = \xi^{ac} + \xi^{\perp}$ be the Lebesgue decomposition of ξ with respect to μ, i.e., $\xi^{\perp}$ and ξ^{ac} are nonnegative measures such that $\xi^{ac} \leqq \mu$ and $\xi^{\perp} \perp \mu$ [3, Thm. C, p. 134]. Let $A \in \mathscr{C}$ be such that $\xi^{\perp}(A) = 0$ and $\mu(I \setminus A) = 0$.

We shall show that $v \ll \xi^{ac}$, and since $\xi^{ac} \ll \mu$ it will follow that $v \ll \mu$. Let $\delta > 0$ correspond to a given ε in accordance with the absolute continuity $v \ll \xi$; i.e.,

$$\text{(3.1)} \qquad \text{for any subchain } \Lambda, \qquad \|\xi\|_\Lambda \leqq \delta \Rightarrow \|v\|_\Lambda \leqq \varepsilon.$$

We shall prove that $v \ll \xi^{ac}$ by showing that

$$\text{(3.2)} \qquad \text{for any subchain } \Lambda, \qquad \|\xi^{ac}\|_\Lambda \leqq \delta \Rightarrow \|v\|_\Lambda \leqq \varepsilon.$$

If we intersect each set in each link of Λ with A then we get a subchain Λ^* such that $\|\xi\|_{\Lambda^*} = \|\xi^{ac}\|_\Lambda \leqq \delta$, and therefore by (3.1), $\|v\|_{\Lambda^*} \leqq \varepsilon$. But because $v \underset{w}{\lesssim} \mu$ and $\mu(I \setminus A) = 0$, it follows that $\|v\|_\Lambda = \|v\|_{\Lambda^*} \leqq \varepsilon$. This proves (3.2).

COROLLARY 1. *Let* $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ *be an n-dimensional vector of measures in* NA^+. *Let f be a real-valued function on the range of* μ, *such that* $v = f \circ \mu \in \mathrm{BV}$. *Then* $v \in \mathrm{AC}$ *if and only if* $v \ll \sum \mu$.

Proof. Sufficiency of the condition is obvious. To verify the necessity note that $f \circ \mu \underset{w}{\lesssim} \sum \mu$ and use Proposition 2.

COROLLARY 2.[1] *Let* $v = f \circ \mu$, *where* $\mu \in \mathrm{NA}^+$; *then* $v \in \mathrm{pNA}$ *if and only if* $v \in \mathrm{AC}$.

Proof. The fact that $\mathrm{pNA} \subseteq \mathrm{AC}$ has been proved in [1, Cor. 5.3]. Now let $f \circ \mu \in \mathrm{AC}$. Then by Corollary 1, $f \circ \mu \ll \mu$, and hence by Proposition 1 and Theorem C in [1], $f \circ \mu \in \mathrm{pNA}$.

COROLLARY 3. *The inclusions* $\mathrm{BV} \supseteq \mathrm{WC} \supseteq \mathrm{AC}$ *are strict.*

Proof. The unanimity game v defined by

$$v(S) = \begin{cases} 1, & S = I, \\ 0, & \text{otherwise}, \end{cases}$$

shows that $\mathrm{BV} \neq \mathrm{WC}$. Next, let λ be Lebesgue measure, and let g be the Cantor function, which is not absolutely continuous; then $g \circ \lambda \in \mathrm{WC}$, and by Propositions 1 and 2, $g \circ \lambda \notin \mathrm{AC}$.

4. Ordered absolute continuity. Let $\mathcal{R}$ be a measurable order. A chain $\emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_m = I$ is called an *$\mathcal{R}$-chain* if all the S_i are $\mathcal{R}$-initial segments. Note that an $\mathcal{R}$-chain is defined by a finite sequence of elements in I, $\infty \overset{\mathcal{R}}{\geqq} s_m \overset{\mathcal{R}}{\geqq} \cdots \overset{\mathcal{R}}{\geqq} s_1 \overset{\mathcal{R}}{\geqq} s_0 = -\infty$, such that $I(s_i, \mathcal{R}) = S_i$.

If v and w are in BV, then v is said to be *ordered absolutely continuous with respect to w* (written $v \underset{o}{\ll} w$), if for every measurable order $\mathcal{R}$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mathcal{R}$-chain Ω and every subchain Λ of Ω, $\|w\|_\Lambda \leqq \delta$ implies $\|v\|_\Lambda \leqq \varepsilon$. Note that the relation is transitive.

PROPOSITION 3. *Let* $v \in \mathrm{BV}$, $\mu \in M^+$.[2] *Then v is ordered absolutely continuous with respect to* μ *if and only if* $v \in \mathrm{ORD}$ *and* $v \underset{w}{\lesssim} \mu$.

Proof. First assume that v is ordered absolutely continuous. It is easily verified that this implies $v \underset{w}{\lesssim} \mu$. Using the argument of the proof of Proposition 12.8 of [1] we obtain that[3] $v \in \mathrm{ORD}$. This completes the proof of one direction.

To prove the second direction, let us assume $v \underset{w}{\lesssim} \mu$ and $v \in \mathrm{ORD}$. By [5, Thm. 3.2], we know that $v \underset{w}{\lesssim} \mu$ implies that $\varphi^{\mathcal{R}} v \underset{w}{\lesssim} \mu$ for all measurable orders $\mathcal{R}$. Recall that weak continuity and absolute continuity between members of M coincide [2, § III. 4.3, p. 131]; hence $\varphi^{\mathcal{R}} v \ll \mu$ for all measurable orders $\mathcal{R}$. But then it follows that $v \underset{o}{\ll} \mu$.

A set is said to be *ordered absolutely continuous* if there is a measure $\mu \in \mathrm{NA}^+$ such that v is ordered absolutely continuous with respect to μ. The set of all ordered absolutely continuous functions in BV is denoted OAC.

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[1] Cf. [1, Thm. C].

[2] One may extend this theorem and require only $\mu \in M$, and not $\mu \in M^+$. This would slightly complicate the proof.

[3] In Proposition 12.8 of [1] one assumes $v \ll \mu$ and obtains in addition to $v \in \mathrm{ORD}$, also that $\varphi^{\mathcal{R}} v \ll \mu$ uniformly in $\mathcal{R}$. Here we assume only $v \underset{o}{\ll} \mu$, and can also obtain $\varphi^{\mathcal{R}} v \ll \mu$, but not uniformly.

COROLLARY 4. ORD $\cap$ WC = OAC

COROLLARY 5. OAC *is a closed linear subspace of* BV.

Remark. One may conjecture that if $v \in \text{ORD}$ and every point in I is v-null then there exists a measure $\mu \in \text{NA}^+$ such that $v \underset{w}{\lesssim} \mu$. If this is true then clearly it should yield that OAC equals the set of all set functions in ORD for which every point is null.

PROPOSITION 4. *Let* $v = f \circ \mu$, *where* $\mu \in \text{NA}^+$; *then* $v \ll \mu$ *if and only if* $v \underset{o}{\lll} \mu$.

Proof. If $v \ll \mu$, then trivially $v \underset{o}{\lll} \mu$. Assume now that $v \underset{o}{\lll} \mu$. By Proposition 3, $v \in \text{ORD}$ and $v \underset{w}{\lesssim} \mu$. Let $\mathcal{R}$ be an arbitrary fixed measurable order, then by [5, Thm. 3.2], $\varphi^{\mathcal{R}} v \underset{w}{\lesssim} \mu$. Since weak continuity and absolute continuity between totally finite measures coincide, it follows that $\varphi^{\mathcal{R}} v \ll \mu$. For a given ε, let δ be given in accordance with the absolute continuity $\varphi^{\mathcal{R}} v \ll \mu$; i.e., for every subchain Λ,

$$\|\mu\|_\Lambda \leqq \delta \Rightarrow \|\varphi^{\mathcal{R}} v\|_\Lambda \leqq \varepsilon. \tag{4.1}$$

We shall show that $v \ll \mu$ by showing that for every subchain Λ,

$$\|\mu\|_\Lambda \leqq \delta \Rightarrow \|v\|_\Lambda \leqq \varepsilon. \tag{4.2}$$

Let Λ be a subchain satisfying $\|\mu\|_\Lambda \leqq \delta$ whose links are $\{S_j, T_j \mid 1 \leqq j \leqq m\}$, where $\emptyset \subseteq S_1 \subseteq T_1 \subseteq S_2 \subseteq \cdots \subseteq S_m \subseteq T_m \subseteq I$. Let

$$\bar{S}_j = \cap \{I(s, \mathcal{R}) \mid s \in I, \mu(I(s, \mathcal{R})) > \mu(S_j)\},$$

$$\bar{T}_j = \cap \{I(s, \mathcal{R}) \mid s \in I, \mu(I(s, \mathcal{R})) > \mu(T_j)\}.$$

By [1, Lem. 12.15] it follows that for $1 \leqq j \leqq m$, $\bar{S}_j$, $\bar{T}_j$ are measurable and that

$$\mu(\bar{S}_j) = \mu(S_j) \quad \text{and} \quad \mu(\bar{T}_j) = \mu(T_j). \tag{4.3}$$

Note also that $\bar{S}_j$ and $\bar{T}_j$ are $\mathcal{R}$-initial sets; hence, by [6, Lem. 2], it follows that for $1 \leqq j \leqq m$,

$$(\varphi^{\mathcal{R}} v)(\bar{T}_j) = v(\bar{T}_j) \quad \text{and} \quad (\varphi^{\mathcal{R}} v)(\bar{S}_j) = v(\bar{S}_j). \tag{4.4}$$

Let $\bar{\Omega}$ be the chain $\emptyset \subseteq \bar{S}_1 \subseteq \bar{T}_1 \subseteq \bar{S}_2 \subseteq \cdots \subseteq \bar{S}_m \subseteq \bar{T}_m \subseteq I$ and let $\bar{\Lambda}$ be a subchain of $\bar{\Omega}$ whose links are $\{\bar{S}_j, \bar{T}_j\}$, $1 \leqq j \leqq m$. Note that (4.3) implies that $\|\mu\|_{\bar{\Lambda}} = \|\mu\|_\Lambda \leqq \delta$. Hence, by (4.1), $\|\varphi^{\mathcal{R}} v\|_{\bar{\Lambda}} \leqq \varepsilon$, and therefore (4.4) and (4.3) imply that

$$\varepsilon \geqq \|\varphi^{\mathcal{R}} v\|_{\bar{\Lambda}} = \|v\|_{\bar{\Lambda}} = \sum_{j=1}^{m} |f(\mu(\bar{T}_j)) - f(\mu(\bar{S}_j))|$$

$$= \sum_{j=1}^{m} |f(\mu(T_j)) - f(\mu(S_j))| = \|v\|_\Lambda.$$

We have established (4.2), thus completing the proof of Proposition 4.

COROLLARY 6. *Let* $v = f \circ \mu$ *where* $\mu \in \text{NA}^+$; *then*

$$v \in \text{AC} \Leftrightarrow v \ll \mu \Leftrightarrow v \in \text{OAC} \Leftrightarrow v \underset{o}{\lll} \mu \Leftrightarrow v \in \text{ORD} \Leftrightarrow v \in \text{pNA}.$$

Proof. The above follows from Proposition 2, Corollary 2, Proposition 3 and Proposition 4.

Remark. It clearly follows from Corollary 6 that if we wish to construct an example of the form $v = f \circ \mu$ that is in ORD$\setminus$AC, then μ has to be at least two-dimensional.

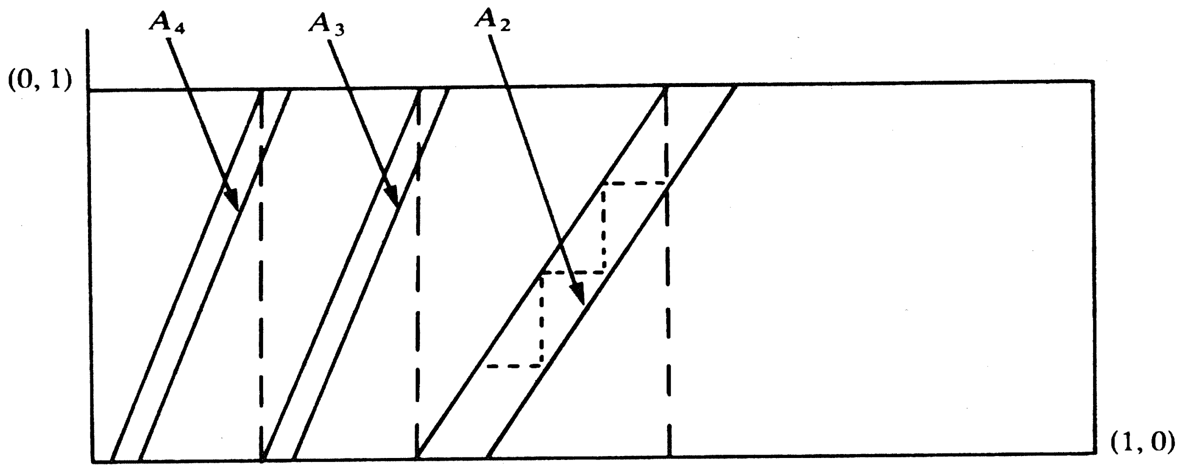
5. ORD includes AC strictly. It was proved in [1, Prop. 12.8] that ORD $\supseteq$ AC. We are now going to construct an example of a set function in ORD that is not in AC.[4] The example that we are going to describe appears, in a different context, at the beginning of § 9 in [1]. For each $k \geqq 2$ let $A_k \subset [0, 1]^2$ be the parallelogram whose vertices are: $(2^{-k}, 0)$, $(2^{-k}+4^{-k}, 0)$, $(2^{-k+1}+4^{-k}, 1)$ and $(2^{-k+1}, 1)$ (see Fig. 1). Define a nondecreasing continuous function f on the square such that for $x \in A_k$,

$$f(x) = f(x_1, x_2) = 2^k x_1 + 2^{-k+1} - 1;$$

for x between A_k and A_{k-1},

$$f(x) = 2^{-k+1} + x_2 + \frac{(x_1 - 2 \cdot 4^{-k})}{(1 - 2^{-k} + x_2)};$$

for x to the right of A_2 let f be defined by the same formula that defines f on A_2, i.e., $f(x) = 4x_1 - 1/2$; and finally for $x_1 = 0$ let $f(x) = x_2$. Let μ be any 2-dimensional vector measure on $(I, \mathscr{C})$, whose range if $[0, 1]^2$. We shall show that $v = f \circ \mu \in$ ORD$\setminus$AC.



To show that $v \notin$ AC, let

$$(2^{-k}, 0) = x_1^k \leqq x_2^k \leqq \cdots \leqq x_n^k = (2^{-k+1}, 1)$$

be a "staircase" sequence of points in A_n, i.e., each point differs from the preceding one in one coordinate only (see Fig. 1). On the vertical segments of this sequence, f does not change; all the change is concentrated on the horizontal segments. But the total length of the horizontal segments goes to 0, whereas the total change in f is 1. Therefore f is not monotonically absolutely continuous,

[4] One can easily see that by "smoothing" our example one can get a set function in MIX [1, § 13] that is not in AC.

therefore $f \circ \mu$ is not absolutely continuous with respect to $\Sigma\mu$ (Proposition 1), and therefore $f \circ \mu \notin \mathrm{AC}$ (Corollary 1).

Let us now prove that $v \in \mathrm{ORD}$. Set $\mu = \mu_1 + \mu_2$. We shall show that v is ordered absolutely continuous with respect to μ, and then use Proposition 3. Let $\mathcal{R}$ be a fixed measurable order. For a given $\varepsilon > 0$ we may choose a $1 > \delta_1 > 0$ such that

$$\|x - y\| \leqq \delta_1 \Rightarrow |f(x) - f(y)| \leqq \varepsilon/2. \tag{5.1}$$

This is possible because of the uniform continuity of f in $[0, 1]^2$.

Let J_1 denote the intersection of all $\mathcal{R}$-initial segments of μ_1-measure > 0. By [1, Lem. 12.15] it follows that J_1 is measurable and $\mu_1(J_1) = 0$. Let J denote the intersection of all $\mathcal{R}$-initial segments of μ-measure $> \mu(J_1) + \delta_1$. By the same lemma[5] we mentioned before, it follows that J is measurable and $\mu(J) = \mu(J_1) + \delta_1$, therefore $J \supseteq J_1$. Finally, observe that $\|\mu(J) - \mu(J_1)\| = \delta_1$; hence by (5.1) it follows that $|v(J) - v(J_1)| \leqq \varepsilon/2$.

Now let p be an integer $\geqq 2$ such that $2^{p-1} \geqq 1/\mu_1(J)$. Note that p depends only on $\mathcal{R}$ and ε. One can easily verify that f fulfills a Lipschitz condition on $\{x \in [0, 1]^2 | x_1 \geqq \mu_1(J)\}$ with constant 2^p, i.e., $\|f(y) - f(x)\| \leqq 2^p \|x - y\|$; this implies that if $S, T \in \mathscr{C}$ and $J \subseteq S \subseteq T$, then $\|v(T) - v(S)\| \leqq 2^p\{\mu(T) - \mu(S)\}$. Define $\delta = \min\{\delta_1, (2^p + 1)^{-1}\varepsilon/2\}$ and note that δ depends only on $\mathcal{R}$ and ε.

Let Λ be a subchain of an $\mathcal{R}$-chain Ω, with links $\{S_i, T_i\}$ $(1 \leqq i \leqq n)$, where $\emptyset \subseteq S_1 \subseteq T_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq T_n \subseteq I$. By definition of $\mathcal{R}$-chain, S_i and T_i are $\mathcal{R}$-initial segments $(1 \leqq i \leqq n)$. We shall show that $\|\mu\|_\Lambda \leqq \delta$ implies $\|v\|_\Lambda \leqq \varepsilon$, which implies that v is ordered absolutely continuous with respect to μ, and hence by Proposition 4 that $v \in \mathrm{ORD}$.

Let $\|\mu\|_\Lambda \leqq \delta$, i.e., $\|\mu\|_\Lambda = \sum_{i=1}^n \{\mu(T_i) - \mu(S_i)\} \leqq \delta$. Without loss of generality we may assume that if $T_i \supseteq J$, then $S_i \supseteq J$; otherwise split $\{S_i, T_i\}$ into two links $\{S_i, J\}$ and $\{J, T_i\}$. Similarly we may assume that if $S_i \subseteq J_1$, then $T_i \subseteq J_i$. Note that since μ and v are monotonic, $\|\mu\|_\Lambda$ and $\|v\|_\Lambda$ remain unchanged. Let

$$I_1 = \{1 \leqq i \leqq n | T_i \subseteq J_1\},$$
$$I_2 = \{1 \leqq i \leqq n | J \subseteq S_i\},$$
$$I_3 = \{1 \leqq i \leqq n | J_1 \subseteq S_i \subseteq T_i \subseteq J\}.$$

I_1, I_2 and I_3 are disjoint, and by our previous assumption $I_1 \cup I_2 \cup I_3 = \{1, 2, \cdots, n\}$. Now

$$\|v\|_\Lambda = \sum_{i=1}^n |v(T_i) - v(S_i)| = \sum_{l=1}^3 \sum_{i \in I_l} \{v(T_i) - v(S_i)\}$$

$$\leqq \sum_{i \in I_1} \{\mu_2(T_i) - \mu_2(S_i)\} + \sum_{i \in I_2} \{2^p(\mu(T_i) - \mu(S_i)\} + v(J) - v(J_1)$$

$$\leqq \delta + 2^p \cdot \delta + \frac{\varepsilon}{2} \leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

[5] The lemma must be modified to apply to measures μ in NA^+ for which $u(I) \neq 1$. Note that $\mu(J_1) + \delta_1 < \mu(I)$.

This completes the proof that $v \in \mathrm{ORD} \setminus \mathrm{AC}$. Hence we have shown

(5.2) ORD includes AC strictly.

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