70 A Variational Problem Arising in Economics

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Introduction

Let $u(x, t)$ be a real function of the two real variables x and t, where $x \geqslant 0$ and $0 \leqslant t \leqslant 1$. Suppose that u is continuous and monotone increasing in x for each fixed t. Consider the following problem:

Find an integrable function $\mathbf{x}(t)$ that maximizes $\int_0^1 u(\mathbf{x}(t), t)\, dt$, subject to the conditions $\mathbf{x}(t) \geqslant 0$ for each t, and $\int_0^1 \mathbf{x}(t)\, dt = 1$.

Unfortunately, the maximum in this problem need not be attained, even when the supremum of the integral is finite, and even when u is very "regular." In fact, $u(x, t) = xt$ provides a counterexample; we have $\int_0^1 t\mathbf{x}(t)\, dt < 1$ for any feasible choice of $\mathbf{x}$, but the supremum is 1.

Intuitively, the maximum is not attained in this problem because it is "worthwhile" to concentrate all the area $\int_0^1 \mathbf{x}(t)\, dt$ at our disposal on a t-interval that is close to 1; i.e., to choose $\mathbf{x}(t)$ large for t close to 1, and 0 elsewhere. If we would assume $u(x, t) = o(x)$ as $x \to \infty$, this might no longer be worthwhile. This assumption, when made for each t separately, is still not sufficient to ensure that the maximum is attained; some kind of uniformity condition is needed. Uniform convergence of $u(x, t)/x$ to 0 is sufficient, but not necessary; it turns out that the proper condition is that of *integrable* convergence, which we shall now define.

Definition 1. Let $f(x, t)$ be a real valued function for $x \geqslant 0$ and $0 \leqslant t \leqslant 1$. Then $f(x, t) = o(x)$ as $x \to \infty$, *integrably* in t, if for each $\epsilon > 0$ there is an integrable function $\eta(t)$, such that $|f(x, t)| \leqslant \epsilon x$ whenever $x \geqslant \eta(t)$.

Integrable convergence reduces to uniform convergence when $\eta(t)$ is a constant, or equivalently, when it is bounded. The two concepts are not equivalent; $x^{1/2}/t^{1/4} = o(x)$ integrably, but not uniformly. The relation

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between uniform and integrable convergence is roughly similar to that between a uniformly bounded and an integrably bounded sequence.

The main result of this paper holds when x may have values in an arbitrary euclidean space E^n (rather than E^1). Let us recall that a vector x in E^n is called nonnegative ($x \geqslant 0$) if all its coordinates are nonnegative,[1] and that a real function $g(x)$ defined for $x \geqslant 0$ is called *nondecreasing* (*increasing*) if it is nondecreasing (increasing) in each variable separately. Also, g is called *upper-semicontinuous* if $g(x) = \lim\sup_{y \to x} g(y)$ for all x in the domain of definition of g. The unit interval [0, 1] will be denoted by T.

MAIN THEOREM. *Let $u(x, t)$ be a Borel-measurable*[2] *nonnegative real-valued function defined for $x \geqslant 0$ in E^n and t in T, which is nondecreasing and upper-semicontinuous in x for each fixed t. Assume further that*

(A) $u(\xi, \cdots, \xi, t) = o(\xi)$ *as* $\xi \to \infty$, *integrably in* t.

Let $a \geqslant 0$ be in E^n, and let $\mathscr{P}(u, a)$ be the problem:

Maximize $\int_T u(\mathbf{x}(t), t)\, dt$ *subject to* $\mathbf{x}(t) \geqslant 0$ *for all* $t \in T$ *and* $\int_T \mathbf{x}(t)\, dt = a$.

Then $\mathscr{P}(u, a)$ has a solution.

The asymptotic condition (A) may be replaced by a similar condition along any *positive* ray. Because u is nondecreasing, any one of these conditions is equivalent to the condition that $u(x, t) = o(\| x \|)$ as $\| x \| \to \infty$, integrably in t, where $\| x \|$ is any one of the usual norms on E^n.

The condition that u be nondecreasing can be dispensed with, if the condition $\int_T \mathbf{x}(t)\, dt = a$ is replaced by $\int_T \mathbf{x}(t)\, dt \leqslant a$, and certain other slight changes are made (cf. Section 6).

The proof will be in two stages. First we will prove (Section 2) that the main theorem holds when u is concave. This proof depends on arguments involving weak compactness. For nonconcave u, we define (Section 3) the concavified function u^* in a manner similar to that of Shapley and Shubik [1], and show (Section 4) that the problems $\mathscr{P}(u, a)$ and $\mathscr{P}(u^*, a)$ have the same value and have solutions in common (where the *value* of $\mathscr{P}(u, a)$ is the supremum of $\int_T u(\mathbf{x}(t), t)\, dt$ under the restrictions on $\mathbf{x}$). In particular, we construct a solution of $\mathscr{P}(u, a)$ from a solution of $\mathscr{P}(u^*, a)$. In this part of the proof we make use of the integral of a set-valued function, as studied in [2].

The proof is an existence proof, and does not yield a characterization of the solution. A characterization is given in Section 5; it is relatively easily obtained—much more easily than the existence. The problem is really an infinite dimensional analogue of a nonlinear programming problem in the sense of Kuhn and Tucker [3], and the characterization is derived from reasoning similar to that leading to their characterizations.

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[1] Note that this *differs* from the standard usage, in which this relation is denoted $\geqq$.

[2] In all variables simultaneously.

Suppose that u is actually increasing. If we assume $u(\xi, \cdots, \xi, t) = o(\xi)$ uniformly, then we may conclude that $\mathscr{P}(u, a)$ has a bounded solution. Indeed, for any p with $1 \leqslant p \leqslant \infty$ we may define the notion $f(\xi, t) = o(\xi)$ *p-integrably* by adding the condition $\eta(t) \in L^p$ to Definition 1. Then if it is assumed that $u(\xi, \cdots, \xi, t) = o(\xi)$ p-integrably, it may be concluded (Section 6) that $\mathscr{P}(u, a)$ has a solution in L^p, and indeed all solutions are in L^p. This conclusion does not hold if u is merely nondecreasing (Section 6). Various other counterexamples are presented in Section 6, and also the generalization of the main theorem to u that are not even nondecreasing that we discussed above.

In Section 1 we explain the notations, terminology, and conventions used in the paper.

Throughout the paper, the unit interval T may be replaced by the real line $(-\infty, \infty)$, the half line $(0, \infty)$, or indeed any Borel set on the real line. The proofs are not affected.

Following are indications of two of the economic applications. The problem treated here arose in connection with an investigation of markets with a continuum of traders and transferable utilities, being conducted by L. S. Shapley and one of the authors. There x and t stand for a commodity bundle and a trader respectively; $\int_T u(\mathbf{x}(t), t)\, dt$ represents the aggregate utility of the coalition T under the commodity-assignment $\mathbf{x}$. If a is the aggregate (initial) commodity bundle held by the coalition, then the value of $\mathscr{P}(u, a)$—if it is attained—is the maximum aggregate utility that the coalition T can assure itself by trading among its own members.

In other economic applications, t stands for "time" rather than "trader." One of the interpretations possible in this direction is that x stands for a vector of resources, $u(x, t)$ is the (discounted) return from using the vector x of resources at time t, and a represents the total amount of resources available. Then $\int_T u(\mathbf{x}(t), t)\, dt$ is the total value of a program $\mathbf{x}$ of resource use, and the problem $\mathscr{P}(u, a)$ is that of finding a program that will maximize this value.[3]

1. Preliminaries

The Borel-measurability of u will be assumed throughout the paper. This is needed mainly to assure the Lebesgue-measurability of $u(\mathbf{x}(t), t)$ for each Lebesgue-measurable $\mathbf{x}$. Throughout Sections 1-5 it will be assumed that u is nonnegative.

[3] In a recent publication [7], M. Yaari has treated such a problem for the case in which x is one-dimensional and $u(x, t) = \alpha(t)\, g(x)$, where α is bounded and continuous and g is concave. Such problems have also been discussed by Arrow, Chakravarty, Karlin, Koopmans, Strotz, and others.

A number of conventions: $x \cdot y$ is the scalar product of x and y. The symbol $\mathbf{0}$ denotes the origin of E^n as well as the number zero. If $\mathbf{y}$ is a function on T, we write $\int \mathbf{y}$ for $\int_T \mathbf{y}(t)\,dt$. Abusing our notation, we write $u(\mathbf{x})$ for the function on T whose value at t is $u(\mathbf{x}(t), t)$; in particular, therefore, $\int u(\mathbf{x})$ means $\int_T u(\mathbf{x}(t), t)\,dt$. The nonnegative orthant $\{x \in E^n : x \geqslant 0\}$ is denoted P. Superscripts will be used exclusively to denote coordinates. $x \geqslant y$ means $x^i \geqslant y^i$ for[4] all i. We will use the phrase "all t in T" to mean the same as "almost all t in T"; the two phrases will be used interchangeably. The closure of a set B is denoted cl (B). The vector $(1, \cdots, 1)$ in E^n will be denoted e, and the vector $(0, \cdots, 0, 1, 0, \cdots, 0)$, where 1 is in the ith place, will be denoted e_i. For $x \in P$, we will write $\sum x$ instead of $\sum_{i=1}^n x^i$. When we say that u is "increasing," "continuous," etc., we mean "increasing in x for each fixed t," "continuous in x for each fixed t," etc. The symbol $\setminus$ denotes set-theoretic subtraction, and μ denotes Lebesgue measure on T. val $\mathscr{P}(u, a)$ denotes the value of $\mathscr{P}(u, a)$.

It is convenient to view the space of all integrable functions from T to E^n as a Banach space, with norm $\int \sum |\mathbf{x}|$. If we write $x^i(t) = x(t; i)$, then we see that this space is precisely $L^1(T \times \{1, \cdots, n\})$. All references to weak convergence, strong convergence, etc. will refer to this space.

A real function f on P will be called *concave* if

$$f(\theta x + (1 - \theta) y) \geqslant \theta f(x) + (1 - \theta) f(y)$$

for all x, y in P and θ in $[0, 1]$. A concave function must be continuous in the interior of P, but may have jumps on the boundary. If it is upper-semicontinuous, then it is a fortiori continuous everywhere.

2. The Concave Case

Lemma 2.1. *Let a be given, and suppose that u is continuous and nondecreasing, and satisfies the asymptotic condition* (A). *Let $\mathbf{x}$, $\mathbf{y}_1$, $\mathbf{y}_2, \cdots$ be integrable functions from T to P such that $\int \mathbf{y}_j \leqslant a$ and $\mathbf{y}_j \to \mathbf{x}$ almost everywhere (a.e.). Then*

$$\int u(\mathbf{y}_j) \to \int u(\mathbf{x}).$$

Proof. Without loss of generality (w.l.o.g.) let $\Sigma a = 1$. Since $\mathbf{y}_j \to \mathbf{x}$ a.e., it follows that $u(\mathbf{y}_j)$ converges a.e. to $u(\mathbf{x})$. Let $\epsilon > 0$ be given, and choose an integrable η such that $\mathbf{x}(t) < \eta(t)e$ for all t, and $u(\xi e, t) < \epsilon\xi$ whenever $\xi \geqslant \eta(t)$. Let $U = U_k = \{t : \mathbf{y}_k(t) \leqslant \eta(t)e\}$. Since convergence

[4] Note that this *differs* from the standard usage, in which this relation is denoted $\geqq$.

a.e. implies convergence in measure,[5] it follows that $\mu(T \setminus U_k) \to 0$ as $k \to \infty$. Let χ_U denote the characteristic function of U. Then

$$\int |u(\mathbf{y}_k) - u(\mathbf{x})| = \int_U + \int_{T \setminus U}$$

$$\leqslant \int \chi_U |u(\mathbf{y}_k) - u(\mathbf{x})| + \int_{T\setminus U} u(\mathbf{x}) + \int_{T\setminus U} u(\mathbf{y}_k).$$

The integrand in the first term of the last line is bounded by $u(\eta e)$, which in turn is $\leqslant \epsilon\eta$; so we may apply Lebesgue's dominated convergence theorem and deduce that the first term tends to 0. The second term tends to 0 because $\mu(T \setminus U) \to 0$ as $k \to \infty$. The third term is

$$\leqslant \epsilon \int_{T\setminus U} \max_i \mathbf{y}_k^i \leqslant \epsilon \int \sum \mathbf{y}_k \leqslant \epsilon \sum a = \epsilon.$$

Hence

$$\limsup_{k\to\infty} \int |u(\mathbf{y}_k) - u(\mathbf{x})| \leqslant \epsilon,$$

and hence it vanishes. This proves the lemma.

PROPOSITION 2.2. *Suppose that u is continuous, concave and nondecreasing, and satisfies the asymptotic condition (A). Then* val $\mathscr{P}(u, a)$ *is attained.*

PROOF. Assume w.l.o.g. that $u(0, t) = 0$ for all t, and that $\Sigma a = 1$. Let $\alpha = \text{val}\, \mathscr{P}(u, a)$. Clearly val $\mathscr{P}(u, y)$ is a nondecreasing function of y; let $D = \{y \in P : y \leqslant a \text{ and } \text{val}\, P(u, y) = \alpha\}$. Let $\lambda = \inf\{\Sigma y : y \in D\}$, $\{b_k\}$ a sequence of points in D such that $\sum b_k \to \lambda$, and let $\{c_k\}$ be a convergent subsequence of $\{b_k\}$; set $c = \lim_k c_k$. For each k, we have val $\mathscr{P}(u, c_k) = \alpha$; let $\{\mathbf{x}_k\}$ be a sequence of integrable functions such that $\int \mathbf{x}_k = c_k$ for all k, and $\int u(\mathbf{x}_k)$ is a nondecreasing sequence that approaches α as $k \to \infty$. We wish to show that $\{\mathbf{x}_k\}$ has a weakly convergent subsequence.

To show this, it is sufficient to show that for each i and each decreasing sequence $\{S_m\}$ of subsets of T with void intersection, we have

$$\int_{S_m} \mathbf{x}_k^i \to 0 \qquad \text{as} \qquad m \to \infty, \qquad \text{uniformly in } k \tag{2.3}$$

[5] In case T has infinite Lebesgue measure, we may use the measure $\nu(S) = \int_S u(\mathbf{x})$ rather than Lebesgue measure μ. Since $\mathbf{x}$ is integrable we always have $\nu(T) < \infty$, and can therefore deduce that convergence a.e. implies convergence in the measure ν, which is what is needed below.

(Dunford-Schwartz [4], p. 292, Theorem IV.8.9). Fix i. If $c^i = 0$, (2.3) follows easily from $\int \mathbf{x}_k^i = c_k^i \to c^i = 0$. Assume therefore that $c^i > 0$. Let $\epsilon > 0$ be given, and assume w.l.o.g. that $\frac{1}{2}\epsilon < c^i$. Let

$$b = c - \tfrac{1}{2}\epsilon e_i + \sum_{j=1}^{n} \min\left(\frac{\epsilon}{4n}, c^j - a^j\right) e_j \qquad \text{and} \qquad \beta = \operatorname{val} \mathscr{P}(u, b).$$

Since $\sum b < \lambda$, it follows that $b \notin D$, and hence $\beta < \alpha$. Let $\gamma = \frac{1}{2}(\alpha - \beta)$. Since $\int u(\mathbf{x}_k) \to \alpha$, it follows that $\int u(\mathbf{x}_k) > \beta + \gamma$ for $k > k_0 = k_0(\epsilon)$. Choose an integrable η so that $u(\xi e, t) \leqslant \gamma\xi$ whenever $\xi \geqslant \eta(t)$; a fortiori, $u(x, t) \leqslant \gamma \sum x$ whenever $x^i \geqslant \eta(t)$. For each k, let

$$U = U_k = \{t : \mathbf{x}_k^i(t) \leqslant \eta(t)\}.$$

Choose k_1 so that $k_1 \geqslant k_0$ and $c_k^j \leqslant \min(a^j, c^j + (\epsilon/4n)) = b^j$ for all $k \geqslant k_1$ and all j. Then

$$\int_{T\setminus U} \mathbf{x}_k^i < \tfrac{1}{2}\epsilon \qquad \text{for} \qquad k \geqslant k_1. \tag{2.4}$$

Indeed, if $\int_{T\setminus U} \mathbf{x}_k^i \geqslant \frac{1}{2}\epsilon$, then

$$\int_U \mathbf{x}_k^i \leqslant c_k^i - \tfrac{1}{2}\epsilon \leqslant b^i, \qquad \text{and} \quad \int_U \mathbf{x}_k^j \leqslant c_k^j \leqslant b^j \quad \text{for all} \quad j \neq i;$$

then $\int_U \mathbf{x}_k \leqslant b$, and hence $\int_U u(\mathbf{x}_k) \leqslant \beta$. Hence

$$\int u(\mathbf{x}_k) = \int_U + \int_{T\setminus U} \leqslant \beta + \int_{T\setminus U} \gamma \sum \mathbf{x}_k$$

$$\leqslant \beta + \gamma \sum \int \mathbf{x}_k = \beta + \gamma \sum c_k \leqslant \beta + \gamma \sum a = \beta + \gamma.$$

This contradicts the definitions of k_1 and k_0, and so proves (2.4).

Since η and $\mathbf{x}_1, \cdots, \mathbf{x}_{k_1}$ are integrable, we may choose m_0 so that whenever $m > m_0$ we have $\int_{S_m} \eta < \frac{1}{2}\epsilon$ and $\int_{S_m} \mathbf{x}_k^i < \epsilon$ for all $k \leqslant k_1$. Then if $m > m_0$ and $k \leqslant k_1$, then $\int_{S_m} \mathbf{x}_k^i < \epsilon$; if $k > k_1$, then by (2.4), we again have

$$\int_{S_m} \mathbf{x}_k^i = \int_{S_m \cap U} + \int_{S_m \setminus U} \leqslant \int_{S_m} \eta + \int_{T\setminus U} \mathbf{x}_k^i < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$$

This proves (2.3).

We conclude that $\{\mathbf{x}_k\}$ has a subsequence converging weakly to some $\mathbf{x}$. Then there is a sequence of functions converging strongly (i.e., in norm) to $\mathbf{x}$, each one of which is a (finite) convex combination of $\mathbf{x}_1, \mathbf{x}_2, \cdots$ [4, p. 422, corollary V.3.14]. Now every strongly convergent sequence in L^1 has a subsequence that converges a.e. to the same limit; so there is a sequence $\{\mathbf{y}_j\}$ of convex combinations of $\mathbf{x}_1, \mathbf{x}_2, \cdots$ that converges a.e. to $\mathbf{x}$.

From the concavity of u and the fact that the sequence $\int u(\mathbf{x}_k)$ is increasing, it follows that $\int u(\mathbf{y}_j) \geqslant \int u(\mathbf{x}_1)$; combining this with Lemma 2.1, we deduce that $\int u(\mathbf{x}) \geqslant \int u(\mathbf{x}_1)$. But for each k, the sequence $\{\mathbf{x}_k, \mathbf{x}_{k+1}, \cdots\}$ converges weakly to x, so we may conclude in the same way that $\int u(x) \geqslant \int u(\mathbf{x}_k)$ for each k. Hence $\int u(\mathbf{x}) \geqslant \alpha$. But $\{\mathbf{z} : \int \mathbf{z} \leqslant a\}$ is weakly closed, so $\int \mathbf{x} \leqslant a$. Let $d = \int \mathbf{x}$; then $\int [\mathbf{x} + a - d] = a$. So

$$\alpha \leqslant \int u(\mathbf{x}) \leqslant \int u(\mathbf{x} + a - d) \leqslant \operatorname{val} \mathscr{P}(u, a) = \alpha.$$

Hence equality holds throughout, and the proposition is proved.

3. Concavification

Let f be a nonnegative real-valued function on P and let

$$F = \{(\nu, x) \in E^{n+1} : x \geqslant 0, 0 \leqslant \nu \leqslant f(x)\}.$$

Let F^* be the convex hull of F. If there is a function f^* on P such that

$$F^* = \{(\nu, x) \in E^{n+1} : x \geqslant 0, 0 \leqslant \nu \leqslant f^*(x)\},$$

then f^* is called the *concavification* of f. The definite article is justified by the fact that there can be at most one concavification. There may be none, as has been shown by Shapley and Shubik [1].

When it exists, the concavification is always concave; this follows from the definition. If f is concave, then f^* exists and equals f.

Proposition 3.1. *If f is nondecreasing and upper-semicontinuous, and $f(\xi e) = o(\xi)$ as $\xi \to \infty$, then f has a nondecreasing and continuous concavification.*

Proof. A slightly weaker form of this lemma has been proved by Shapley and Shubik[6] [1, Theorem 3]. We first show that F^* is closed.

Let $(\nu, x) \in \operatorname{cl}(F^*)$. Then there is a sequence $\{(\nu_k, x_k)\}$ of members of F^* that tends to (ν, x). Now recall Caratheodory's theorem, which states that if F and F^* are subsets of E^{n+1} such that F^* is the convex hull of F, then every point of F^* is a convex combination of $n + 2$ points of F [5, p. 34 ff.]. Hence we have

$$(\nu_k, x_k) = \sum_{j=1}^{n+2} \alpha_{kj}(\lambda_{kj}, y_{kj}),$$

[6] Their theorem contains a monotonicity assumption that is slightly stronger than ours.

where $\alpha_{kj} \geqslant 0$, $\sum_{j=1}^{n+2} \alpha_{kj} = 1$, and $(\lambda_{kj}, y_{kj}) \in F$. The sequence of points $(\alpha_{k1}, \cdots, \alpha_{k,n+2})$ in E^{n+2} has a limit point, which we call $(\alpha_1, \cdots, \alpha_{n+2})$; w.l.o.g. assume it is the limit. We have $\alpha_j \geqslant 0$ and $\sum_{j=1}^{n+2} \alpha_j = 1$. Some of the α_j may vanish, but not all of them can vanish; for convenience, assume that α_j vanishes for $j \leqslant m$, and does not vanish for $j > m$.

Let $\beta = \min(\alpha_{m+1}, \cdots, \alpha_{n+2}) > 0$. For sufficiently large k, we have $\alpha_{kj} > \beta/2$ for all $j > m$, and $(\nu_k, x_k) < (\nu, x) + (1, e)$. Hence

$$(\nu, x) + (1, e) > (\nu_k, x_k) \geqslant \sum_{j=m+1}^{n+2} \alpha_{kj}(\lambda_{kj}, y_{kj}) \geqslant \frac{\beta}{2} (\lambda_{kj}, y_{kj})$$

for k sufficiently large and all $j > m$. It follows that the sequences $\{(\lambda_{kj}, y_{kj})\}$ are bounded when $j > m$. They therefore have limit points (λ_j, y_j) and we may assume w.l.o.g. that $(\lambda_{kj}, y_{kj}) \to (\lambda_j, y_j)$ as $k \to \infty$. Hence

$$(\nu, x) = \sum_{j=m+1}^{n+2} \alpha_j(\lambda_j, y_j) + \lim_{k\to\infty} \sum_{j=1}^{m} \alpha_{kj}(\lambda_{kj}, y_{kj}). \tag{3.2}$$

We conclude from this that the limit on the right-hand side of (3.2) exists; denote it by (λ, y).

Define real numbers ζ and ζ_{kj} by $\zeta = \sum y$, $\zeta_{kj} = \sum y_{kj}$. Then

$$\alpha_{kj}\zeta_{kj} \leqslant \sum_{j=1}^{m} \alpha_{kj}\zeta_{kj} < \zeta + 1$$

for sufficiently large k and all $j \leqslant m$. For given $\epsilon > 0$, let η be such that $f(\xi e) \leqslant \epsilon\xi$ for all $\xi \geqslant \eta$. Then because f is nondecreasing and $(\lambda_{kj}, y_{kj}) \in F$, we have

$$\lambda_{kj} \leqslant f(y_{kj}) \leqslant f(\zeta_{kj}e) \leqslant \begin{cases} \epsilon\zeta_{kj} & \text{if} \quad \zeta_{kj} \geqslant \eta \\ f(\eta e) & \text{if} \quad \zeta_{kj} \leqslant \eta. \end{cases}$$

In any case

$$\lambda_{kj} \leqslant \max(\epsilon\zeta_{kj}, f(\eta e)).$$

Hence

$$\alpha_{kj}\lambda_{kj} \leqslant \max(\epsilon\alpha_{kj}\zeta_{kj}, \alpha_{kj} f(\eta e)) \leqslant \max(\epsilon(\zeta + 1), \alpha_{kj} f(\eta e))$$

for k sufficiently large and all $j \leqslant m$. Since $\alpha_{kj} \to 0$ as $k \to \infty$ and $f(\eta e)$ is fixed, it follows that

$$\limsup_k \alpha_{kj}\lambda_{kj} \leqslant \epsilon(\zeta + 1).$$

But since ϵ was chosen arbitrarily, it follows that the lim sup vanishes and so the limit exists and vanishes. Hence $\lambda = 0$.

Now (λ_j, y_j) is the limit of (λ_{kj}, y_{kj}) for $j > m$; because f is upper-semicontinuous, F is closed, so from $(\lambda_{kj}, y_{kj}) \in F$ it follows that $(\lambda_j, y_j) \in F$. Now this means that $\lambda_j \leqslant f(y_j)$. Because f is nondecreasing and $y \geqslant 0$, it follows that $\lambda_j \leqslant f(y_j + y)$; hence $(\lambda_j, y_j + y) \in F$. But since $\alpha_j = 0$ for $j \leqslant m$, it follows that $\sum_{j=m+1}^{n+2} \alpha_j = 1$. Hence from (3.2) and $\lambda = 0$ we conclude

$$(\nu, x) = \sum_{j=m+1}^{n+2} \alpha_j(\lambda_j, y_j) + (0, y) = \sum_{j=m+1}^{n+2} \alpha_j(\lambda_j, y_j + y).$$

Since $(\lambda_j, y_j + y) \in F$, it follows that $(\nu, x) \in F^*$, and we have proved that F^* is closed.

If η is chosen so that $f(\xi e) \leqslant \xi$ whenever $\xi \geqslant \eta$, then for all x,

$$f(x) \leqslant f\left(e \sum x\right) \leqslant \max\left(\sum x, f(\eta e)\right) \leqslant \sum x + f(\eta e).$$

It follows that

$$F \subset \left\{(\nu, x) : 0 \leqslant x, 0 \leqslant \nu \leqslant \sum x + f(\eta e)\right\}.$$

The right side of this inclusion is convex, so F^* is also included in it. Hence if $(\nu, x) \in F^*$, then $\nu \leqslant \sum x + f(\eta e)$. Hence for each x, the set $\{\nu : (\nu, x) \in F^*\}$ is bounded, and because F^* is closed, it is compact. Hence the maximum of this set is attained, and this maximum is precisely $f^*(x)$. The monotonicity of f^* follows from that of f. To show that f^* is continuous, suppose that x is a point of discontinuity. Let $x_k \to x$ and $f^*(x_k) \to \psi \neq f^*(x)$. Since F^* is closed, f^* is upper-semicontinuous, so $\psi > f^*(x)$ is impossible. Hence $\psi < f^*(x)$; let $\theta = f^*(x) - \psi > 0$. For $z > 0$ sufficiently small, it follows from the upper-semicontinuity of f^* that $f^*(x + z) < f^*(x) + \frac{1}{2}\theta$. Now for k sufficiently large, $x - x_k < z$, and hence $\frac{1}{2}x_k + \frac{1}{2}(x + z) > x$. Hence

$$\begin{aligned} f^*(x) &\leqslant f^*(\tfrac{1}{2}x_k + \tfrac{1}{2}(x + z)) \leqslant \tfrac{1}{2}f^*(x_k) + \tfrac{1}{2}f^*(x + z) \\ &\leqslant \tfrac{1}{2}f^*(x_k) + \tfrac{1}{2}f^*(x) + \tfrac{1}{4}\theta. \end{aligned}$$

Letting $k \to \infty$, we deduce

$$\tfrac{1}{2}f^*(x) \leqslant \tfrac{1}{2}\psi + \tfrac{1}{4}\theta < \tfrac{1}{2}(\psi + \theta) = \tfrac{1}{2}f^*(x),$$

a contradiction. This completes the proof of the proposition.

LEMMA 3.3. *Suppose u is upper-semicontinuous and nondecreasing, and satisfies the asymptotic condition (A). For fixed t, let $u^*(x, t)$ be the concavification of $u(x, t)$. Then u^* also satisfies the asymptotic condition (A).*

PROOF. Let $\epsilon > 0$ be given. Set

$$\mathbf{F}(t) = \{(\nu, x) : 0 \leqslant x,\ 0 \leqslant \nu \leqslant u(x, t)\}. \tag{3.4}$$

Choose η in accordance with Definition 1 to correspond to $\epsilon/(n+1)$. Let

$$\mathbf{H}(t) = \left\{(\nu, x) : 0 \leqslant x, 0 \leqslant \nu \leqslant \frac{\epsilon}{n+1} \sum x + u(\eta(t)e, t)\right\}.$$

Then $\mathbf{F}(t) \subset \mathbf{H}(t)$, and since $\mathbf{H}(t)$ is convex it follows that also $\mathbf{F}^*(t) \subset \mathbf{H}(t)$. Hence for each ξ and t,

$$u^*(\xi e, t) \leqslant \epsilon \frac{n}{n+1} \xi + u(\eta(t)\, e, t).$$

Hence if $\xi \geqslant (n+1)\, u(\eta(t)\, e, t)/\epsilon$, then $u^*(\xi e, t) \leqslant \epsilon\xi$. From condition (A) and the integrability of η it follows that $u(\eta(t)e, t)$ is integrable. This proves the lemma.

LEMMA 3.5. *Under the conditions of Lemma 3.3, the Borel-measurability of u^* follows from that of u.*

PROOF. From the definition of u^*, it follows that

$$u^*(x, t) = \max \Big(\sum_{i=1}^{k} \alpha_i u(x_i, t) : k > 0, \alpha_i \geqslant 0 \quad \text{and}$$

$$x_i \geqslant 0 \text{ for } i = 1, \cdots, k, \sum_{i=1}^{k} \alpha_i = 1, \sum_{i=1}^{k} \alpha_i x_i = x\Big). \tag{3.6}$$

We claim that $u^*(x, t) < \gamma$ if and only if there is a positive integer m such that if k is a positive integer, $\beta_1, \cdots, \beta_k$ are nonnegative *rational* numbers summing to 1, and $y_1, \cdots, y_k$ are rational points in P such that

$$\sum_{i=1}^{k} \beta_i y_i < x + \frac{e}{m},$$

then

$$\sum_{i=1}^{k} \beta_i u(y_i, t) < \gamma - \frac{1}{m} \tag{3.7}$$

Indeed, the "only if" part of the previous sentence follows from the monotonicity of u^* and the fact that for sufficiently large m,

$$u^*\left(x + \frac{e}{m}, t\right) < \gamma - \frac{1}{m}$$

(because of the continuity of u^*). To demonstrate the "if" part, assume that $u^*(x, t) \geqslant \gamma$, and let the max in (3.6) be assumed at $k, \alpha_1, \cdots, \alpha_k, x_1, \cdots, x_k$. Choose nonnegative rational numbers $\beta_1, \cdots, \beta_k$ summing to 1 and rational points y_i such that $y_i \geqslant x_i$ for all i, with the β_i sufficiently close to the α_i and the y_i sufficiently close to the x_i so that

$$\sum_{i=1}^{k} \beta_i u(x_i, t) \geqslant \sum_{i=1}^{k} \alpha_i u(x_i, t) - \frac{1}{m} = u^*(x, t) - \frac{1}{m} \geqslant \gamma - \frac{1}{m} \tag{3.8}$$

and

$$\sum_{i=1}^{k} \beta_i y_i < \sum_{i=1}^{k} \alpha_i x_i + \frac{e}{m} = x + \frac{e}{m}. \tag{3.9}$$

Then from (3.8) and $y_i \geqslant x_i$ we deduce

$$\sum_{i=1}^{k} \beta_i u(y_i, t) \geqslant \gamma - \frac{1}{m}.$$

and from (3.9) and (3.7) we deduce

$$\sum_{i=1}^{k} \beta_i u(y_i, t) < \gamma - \frac{1}{m}.$$

This contradiction establishes our claim, and the lemma follows without difficulty.

4. Proof of the Main Theorem

Proposition 4.1. *Suppose that for each fixed t, u has a Borel-measurable[7] concavification u^*. Then $\mathscr{P}(u, a)$ and $\mathscr{P}(u^*, a)$ have the same value, and $\mathscr{P}(u, a)$ is solvable if and only if $\mathscr{P}(u^*, a)$ is solvable. Furthermore, every solution of $\mathscr{P}(u, a)$ solves $\mathscr{P}(u^*, a)$.*

Proof. We make use of the theory of integrals of set-valued functions [2]. Let $\mathbf{F}$ be a function defined on T, whose values are subsets of E^{n+1}. Then $\int_T \mathbf{F}(t)\, dt$, or $\int \mathbf{F}$ for short, is defined to be the set of all vectors of the form $\int \mathbf{f}$, where $\mathbf{f}$ is a point-valued function such that $\mathbf{f}(t) \in \mathbf{F}(t)$ for all t. The function $\mathbf{F}$ is said to be *Borel-measurable* if $\{(x, t) : x \in \mathbf{F}(t)\}$ is a Borel subset of $E^{n+1} \times T$. The fact that we need in the proof of our main theorem is that if $\mathbf{F}^*(t)$ denotes the convex hull of $\mathbf{F}(t)$, and $\mathbf{F}$ is Borel-measurable and takes only values that are subsets of P, then $\int \mathbf{F}^* = \int \mathbf{F}$ [2, Theorem 3].

[7] The Borel-measurability of u^* is needed only to assure that the problem $\mathscr{P}(u^*, a)$ is well defined (cf. Section 1).

For a given u, define $\mathbf{F}(t)$ by (3.4). Since u is Borel-measurable, so is $\mathbf{F}$. Then it may be verified that

(4.2) *$\mathscr{P}(u, a)$ has a solution $\mathbf{x}_0$ if and only if $V = \{\nu : (\nu, a) \in \int \mathbf{F}\}$ has a maximum, and then the value of $\mathscr{P}(u, a)$ is* max V.

Now let $\mathbf{F}^*(t)$ be the convex hull of $\mathbf{F}(t)$, and let $V^* = \{\nu : (\nu, a) \in \int \mathbf{F}^*\}$. Since $\mathbf{F}$ is Borel-measurable and takes only values that are subsets of P, it follows that $\int \mathbf{F} = \int \mathbf{F}^*$, and therefore $V = V^*$. Proposition 4.1 now follows from (4.2).

The main theorem follows from Propositions 2.2, 3.1, and 4.1, and Lemmas 3.3 and 3.5.

5. A Characterization of the Solution

Theorem 5.1. *Let u be nondecreasing, and let $a > 0$. Then a necessary and sufficient condition for a nonnegative $\mathbf{x}$ to solve $\mathscr{P}(u, a)$ is that $\int \mathbf{x} = a$ and there is a c in P such that*

$$u(x, t) - u(\mathbf{x}(t), t) \leqslant c \cdot (x - \mathbf{x}(t)) \tag{5.2}$$

for all t in T and $x \in P$. If u is increasing then $c \geqslant 0$ may be replaced by $c > 0$.

In the case in which u is differentiable in x, (5.2) implies

$$[\partial u / \partial x^i]_{x=\mathbf{x}(t)} \leqslant c^i$$

for all t and i, with equality holding whenever $\mathbf{x}^i(t) > 0$. That is, the partial derivatives, when evaluated at $\mathbf{x}(t)$, are constant for $\mathbf{x}^i(t) > 0$, and are at most equal to this constant for $\mathbf{x}^i(t) = 0$.

During the course of the proof we shall make use of Proposition 2.1 of [2], which states that for each Borel-measurable set valued function $\mathbf{F}(t)$, there is a point-valued Lebesgue-measurable function $\mathbf{f}(t)$ such that $\mathbf{f}(t) \in \mathbf{F}(t)$ for each t.

The proof of Theorem 5.1 is similar to the proof in Künzi and Krelle [6] of the Kuhn-Tucker theorem; we merely sketch it. Sufficiency is trivial. To prove necessity, we first show that there is a c in P such that

$$\int [u(\mathbf{y}) - u(\mathbf{x})] \leqslant c \cdot \int [\mathbf{y} - \mathbf{x}] \tag{5.3}$$

for all nonnegative and integrable $\mathbf{y}$. Define $K_1, K_2 \subset E^{n+1}$ by

$$K_1 = \left\{(y^0, y) \in E^{n+1} : \text{there is a nonnegative integrable } \mathbf{y} \text{ such that } y^0 \leqslant \int u(\mathbf{y}) \text{ and } y \leqslant a - \int \mathbf{y}\right\}$$

$$K_2 = \left\{(y^0, y) \in E^{n+1} : y^0 > \int u(\mathbf{x}) \text{ and } y \geqslant 0\right\}.$$

Then K_1 is convex; for, we may define

$\mathbf{K}_1(t) = \{(y^0, y) \in E^{n+1} : \text{there is a } z \in P \text{ such that } y^0 \leqslant u(z, t) \text{ and } y \leqslant a - z\}$,

and then by using Proposition 2.1 of [2] it may be established that $K_1 = \int \mathbf{K}_1$. Since every integral of a set-valued function is convex (Theorem 1 of [2]), it follows that K_1 is convex. K_2 is clearly convex and is disjoint from K_1, so there is a hyperplane separating K_1 and $\mathrm{cl}(K_2)$, i.e., there are d^0 and $d = (d^1, \cdots, d^n)$ such that

$$d^0 y_1{}^0 + d \cdot y_1 \leqslant d^0 y_2{}^0 + d \cdot y_2$$

whenever $(y_1{}^0, y_1) \in K_1$ and $(y_2{}^0, y_2) \in \mathrm{cl}(K_2)$. Then $d^0 > 0$ and $d \geqslant 0$, and we obtain

$$d^0 \int_0 [u(\mathbf{y}) - u(\mathbf{x})] \leqslant d \cdot \int [\mathbf{y} - \mathbf{x}];$$

dividing by d^0, we obtain (5.3). If u is increasing, it is easily established that $c > 0$.

To deduce (5.2) from (5.3), suppose that for all t in a set S of positive measure, there is an $x \in P$ such that

$$u(x, t) - u(\mathbf{x}(t), t) > c \cdot (x - \mathbf{x}(t)).$$

Define $\mathbf{y}(t)$ to be such an x when it exists, and $\mathbf{y}(t) = \mathbf{x}(t)$ otherwise. Integrating, we obtain a contradiction to (5.3). The possibility of choosing an appropriate $\mathbf{y}$ that is measurable follows from proposition 2.1 of [2].

6. Counterexamples and Generalizations

If $u(x, t)$ is not upper-semicontinuous for each fixed t, then it need not satisfy the main theorem, even if it is concave. Let $n = 1$, and let

$$u(x, t) = \begin{cases} x & \text{when} \quad x \leqslant 2 \\ 2 & \text{when} \quad x \geqslant 2 \end{cases}$$

when $0 \leqslant t \leqslant \frac{1}{2}$, and

$$u(x, t) = \begin{cases} 0 & \text{when} \quad x = 0 \\ 2 & \text{when} \quad x > 0 \end{cases}$$

when $\frac{1}{2} < t \leqslant 1$. Then the value of $\mathscr{P}(u, 1)$ is 2, but it is not achieved. It is possible to adjust this example so that u is increasing in x for each fixed t. It is also possible to construct an example of a u that satisfies all the conditions of the main theorem except that it may fail to be upper-semicontinuous at a point in the interior of P, and that does not satisfy the main theorem.

In the main theorem, the assumption that u is nonnegative may be replaced by the assumption that $u(0, t)$ is integrable.

The assumption that $u(x, t)$ is nondecreasing for each fixed t cannot be removed, as may be seen from the example $n = 1$, $u(x, t) = e^{-x}$, $a = 1$, in which the sup is 1 but is not attained. However, if we change the condition $\int \mathbf{x} = a$ to read $\int \mathbf{x} \leqslant a$, then the monotonicity assumption can be replaced by a far weaker assumption, which, roughly speaking, says that u is bounded on compact subsets of P. For $x \in E^n$, let[8]

$$\| x \| = \max (| x^1 |, \cdots, | x^n |).$$

THEOREM 6.1. *Let $a \geqslant 0$. Suppose that u is continuous,[9] that*

$$u(x, t) = o(\| x \|)$$

as $\| x \| \to \infty$, integrably in t, and that for every integrable real function η there is an integrable real function ζ such that $\| x \| \leqslant \eta(t)$ implies $| u(x, t) | \leqslant \zeta(t)$ for all x and t. Let $\mathcal{Q}$ be the problem:

$$\textit{Maximize} \int u(\mathbf{x}) \textit{ subject to } \mathbf{x}(t) \geqslant 0 \textit{ for all } t \textit{ and } \int \mathbf{x} \leqslant a.$$

Then $\mathcal{Q}$ has a solution.

REMARK. All nondecreasing u satisfying the asymptotic condition (A) for which $u(0, t)$ is integrable satisfy the boundedness condition of this theorem.

PROOF. We define a "nondecreasification" u' of u as follows:

$$u'(x, t) = \max \{u(y, t) : 0 \leqslant y \leqslant x\}.$$

The max is attained because u is continuous. Clearly u' is nondecreasing. It may be verified that u' is Borel-measurable in both variables, continuous in x, and satisfies the asymptotic condition (A). Hence $\mathcal{P}(u', a)$ has a solution $\mathbf{x}_0$. Then for each t, there is a $y \in P$ such that $y \leqslant \mathbf{x}_0(t)$ and $u'(\mathbf{x}_0(t), t) = u(y, t)$. The function $u'(\mathbf{x}_0(t), t)$ is a Borel-measurable function of t. Hence

$$\{(y, t) : 0 \leqslant y \leqslant \mathbf{x}_0(t) \text{ and } u'(\mathbf{x}_0(t), t) = u(y, t)\}$$

is Borel-measurable. According to Proposition 2.1 of [2], there is a nonnegative measurable function $\mathbf{y}_0$ such that $\int \mathbf{y}_0 \leqslant \int \mathbf{x}_0 = a$ and $u(\mathbf{y}_0(t), t) = u'(\mathbf{x}_0(t), t)$

[8] The L^∞ norm we use here may be replaced by any of the usual norms on E^n, without affecting Theorem 6.1.

[9] The theorem can also be proved when u is only assumed to be upper-semicontinuous. However, the notion of analytic set must then be used instead of Borel set, and we do not wish to get involved in those complications here.

for all t. Hence

$$\int u(\mathbf{x}) \leqslant \int u'(\mathbf{x}) \leqslant \int u'(\mathbf{x}_0) \leqslant \int u(\mathbf{y}_0)$$

for all nonnegative integrable $\mathbf{x}$ such that $\int \mathbf{x} \leqslant a$, and so $\mathbf{y}_0$ solves $\mathcal{Q}$. The proof of the theorem is complete.

Our final result deals with an extension of the main theorem to L^p.

THEOREM 6.2. *Let* $1 \leqslant p \leqslant \infty$ *and* $a \geqslant 0$. *Suppose* u *is nonnegative, increasing, and upper-semicontinuous,*[10] *and that*

(A_p) $u(\xi e, t) = o(\xi)$ *as* $\xi \to \infty$, *p-integrably in* t.

Then $\mathcal{P}(u, a)$ *has a solution in* L^p, *and indeed all solutions are in* L^p.

PROOF. Assume without loss of generality that $a > 0$.
By the main theorem, there is a solution $\mathbf{x}$ in L^1. By Theorem 5.1, every solution $\mathbf{x}$ satisfies (5.2) with $c > 0$. Let $\epsilon = \min(c^1, \cdots, c^n)$, and let η in L^p be such that $u(\xi e, t) < \epsilon\xi$ whenever $\xi \geqslant \eta(t)$. For each t, let

$$\xi(t) = \max(\mathbf{x}^1(t), \cdots, \mathbf{x}^n(t)).$$

By setting $x = 0$ in (5.2), we obtain

$$u(\xi(t)e, t) \geqslant u(\mathbf{x}(t), t) \geqslant c \cdot \mathbf{x}(t) \geqslant \epsilon\xi(t).$$

Hence $\xi(t) < \eta(t)$, and the theorem follows.

Theorem 6.2 cannot be extended to nondecreasing u when $p > 1$. For example, let $n = 1$, and

$$u(x, t) = \begin{cases} xt^{1/2} & \text{for} \quad x \leqslant t^{-1/2} \\ 1 & \text{for} \quad x \geqslant t^{-1/2} \end{cases}$$

Then u satisfies (A_p) for all p, and $\mathcal{P}(u, 2)$ has the solution $t^{-1/2}$, but no solution in L^2.

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