59 Endogenous Formation of Links between Players and of Coalitions: An Application of the Shapley Value

with R. Myerson

1 Introduction

Consider the coalitional game v on the player set $\{1,2,3\}$ defined by

$$v(S)=\begin{cases}0 & \text{if } |S|=1,\\ 60 & \text{if } |S|=2,\\ 72 & \text{if } |S|=3,\end{cases} \tag{1}$$

where $|S|$ denotes the number of players in S. Most cooperative solution concepts "predict" (or assume) that the all-player coalition $\{1,2,3\}$ will form and divide the payoff 72 in some appropriate way. Now suppose that P_1 (player 1) and P_2 happen to meet each other in the absence of P_3. There is little doubt that they would quickly seize the opportunity to form the coalition $\{1,2\}$ and collect a payoff of 30 each. This would happen in spite of its inefficiency. The reason is that if P_1 and P_2 were to invite P_3 to join the negotiations, then the three players would find themselves in effectively symmetric roles, and the expected outcome would be (24,24,24). P_1 and P_2 would not want to risk offering, say, 4 to P_3 (and dividing the remaining 68 among themselves), because they would realize that once P_3 is invited to participate in the negotiations, the situation turns "wide open"- anything can happen.

All this holds if P_1 and P_2 "happen" to meet. But even if they do not meet by chance, it seems fairly clear that the players in this game would seek to form pairs for the purpose of negotiation, and not negotiate in the all-player framework.

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The preceding example is due to Michael Maschler (see Aumann and Dreze 1974, p. 235, from which much of this discussion is cited). Maschler's example is particularly transparent because of its symmetry. Even in unsymmetric cases, though, it is clear that the framework of negotiations plays an important role in the outcome, so individual players and groups of players will seek frameworks that are advantageous to them. The phenomenon of seeking an advantageous framework for negotiating is also well known in the real world at many levels – from decision making within an organization, such as a corporation or university, to international negotiations. It is not for nothing that governments think hard – and often long – about "recognizing" or not recognizing other governments; that the question of whether, when, and under what conditions to negotiate with terrorists is one of the utmost substantive importance; and that at this writing the government of Israel is tottering over the question not of *whether* to negotiate with its neighbors, but of the *framework* for such negotiations (broad-base international conference or direct negotiations).

Maschler's example has a natural economic interpretation in terms of S-shaped production functions. The first player alone can do nothing because of setup costs. Two players can produce 60 units of finished product. With the third player, decreasing returns set in, and all three together can produce only 72. The foregoing analysis indicates that the form of industrial organization in this kind of situation may be expected to be inefficient.

The simplest model for the concept "framework of negotiations" is that of a *coalition structure,* defined as a partition of the player set into disjoint coalitions. Once the coalition structure has been determined, negotiations take place only within each of the coalitions that constitute the structure; each such coalition B divides among its members the total amount $v(B)$ that it can obtain for itself. Exogenously given coalition structures were perhaps first studied in the context of the bargaining set (Aumann and Maschler 1964), and subsequently in many contexts; a general treatment may be found in Aumann and Dreze (1974). Endogenous coalition formation is implicit already in the von Neumann – Morgenstern (1944) theory of stable sets; much of the interpretive discussion in their book and in subsequent treatments of stable sets centers around which coalitions will "form." However, coalition structures do not have a formal, explicit role in the von Neumann – Morgenstern theory. Recent treatments that consider endogenous coalition structures explicitly within the context of a formal theory include Hart and Kurz (1983), Kurz (1988), and others.

Coalition structures, however, are not rich enough adequately to capture the subtleties of negotiation frameworks. For example, diplomatic relations between countries or governments need not be transitive and, therefore, cannot be adequately represented by a partition; thus both Syria and Israel have diplomatic relations with the United States but not with each other. For another example, in salary negotiations within an academic department, the chairman plays a special role; members of the department cannot usually negotiate directly with each other, though certainly their salaries are not unrelated.

To model this richer kind of framework, Myerson (1977) introduced the notion of a *cooperation structure* (or *cooperation graph*) in a coalitional game. This graph is simply defined as one whose vertices are the players. Various interpretations are possible; the one we use here is that a link between two players (an edge of the graph) exists if it is possible for these two players to carry on meaningful direct negotiations with each other. In particular, ordinary coalition structures $(B_1,B_2, \ldots ,B_k)$ (with disjoint B_j) may be modeled within this framework by defining two players to be linked if and only if they belong to the same B_j. (For generalizations of this cooperation structure concept, see Myerson 1980.)

Shapley's 1953 definition of the value of a coalitional game v may be interpreted as evaluating the players' prospects when there is full and free communication among all of them – when the cooperation structure is "full," when any two players are linked. When this is not so, the prospects of the players may change dramatically. For an extreme example, a player i who is totally isolated – is linked to no other player – can expect to get nothing beyond his own worth $v(\{i\})$; in general, the more links a player has with other players, the better one may expect his prospects to be. To capture this intuition, Myerson (1977) defined an extension of the Shapley value of a coalitional game v to the case of an arbitrary cooperation structure g. In particular, if g is the complete graph on the all-player set N (any two players are directly linked), then Myerson's value coincides with Shapley's. Moreover, if the cooperation graph g corresponds to the coalition structure $(B_1,B_2, \ldots ,B_k)$ in the sense indicated here, then the Myerson value of a member i of B_j is the Shapley value of i as a player of the game $v|B_j$ (v restricted to B_j).

This chapter suggests a model for the endogenous formation of cooperation structures. Given a coalitional game v, what links may be expected to form between the players? Our approach differs from that of previous writers on endogenous coalition formation in two respects: First, we work with cooperation graphs rather than coalition structures, using the Myerson value to evaluate the pros and cons of a given cooperation structure for any particular player. Second, we do not use the usual myopic, here-and-now kind of equilibrium condition. When a player considers forming a link with another one, he does not simply ask himself whether he may expect to be better off with this link than without it, given the previously existing structure. Rather, he looks ahead and asks himself, "Suppose we form this new link, will other players be motivated to form further new links that were not worthwhile for them before? Where will it all lead? Is the *end result* good or bad for me?"

In Section 2 we review the Myerson value and illustrate the "look-ahead" reasoning by returning to the three-person game that opened the chapter. The formal definitions are set forth in Section 3, and the following sections are devoted to examples and counterexamples. The final section contains a general discussion of various aspects of this model, particularly of its range of application.

No new theorems are proved. Our purpose is to study the conceptual implications of the Shapley value and Myerson's extension of it to cooperation structures in examples that are chosen to reflect various applied contexts.

2 Looking ahead with the Myerson value

We start by reviewing the Myerson value. Let v be a coalitional game with N as player set, and g a graph whose vertices are the players. For each player i the *value* $\phi_i^g = \phi_i^g(v)$ is determined by the following axioms.

Axiom 1. If a graph g is obtained from another graph h by adding a single link, namely the one between players i and j, then i and j gain (or lose) equally by the change; that is,

$$\phi_i^g - \phi_i^h = \phi_j^g - \phi_j^h.$$

Axiom 2. If S is a connected component of g, then the sum of the values of the players in S is the worth of S; that is,

$$\sum_{i \in S} \phi_i^g(v) = v(S)$$

(Recall that a *connected component* of a graph is a maximal set of vertices of which any two may be joined by a chain of linked vertices.)

That this axiom system indeed determines a unique value was demonstrated by Myerson (1977). Moreover, he showed that if v is superadditive, then two players who form a new link never lose by it: The two sides of the equation in Axiom 1 are nonnegative. He also established[1] the following practical method for calculating the value: Given v and g, define a coalitional game v^g by

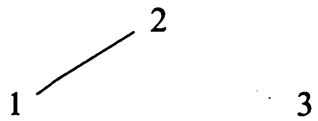
$$v^g(S) := \sum v(S_j^g), \qquad (2)$$

where the sum ranges over the connected components S_j^g of the graph $g|S$ (g restricted to S). Then

$$\phi_i^g(v) = \phi_i(v^g), \qquad (3)$$

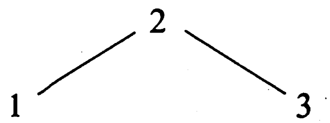
where ϕ_i denotes the ordinary Shapley value for player i.

We illustrate with the game v defined by (1). If P_1 and P_2 happen to meet in the absence of P_3, then the graph g may be represented by



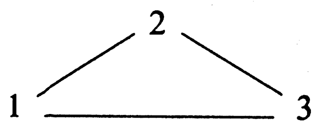
(4)

with only P_1 and P_2 connected. Then $\phi^g(v) = (30,30,0)$; we have already seen that in this situation it is not worthwhile for P_1 and P_2 to bring P_3 into the negotiations, because that would make things entirely symmetric, so P_1 and P_2 would get only 24 each, rather than 30. But P_2, say, might consider offering to form a link with P_3. The immediate result would be the graph



(5)

This graph is not at all symmetric; the central position of P_2–all communication must pass through him–gives him a decided advantage. This advantage is reflected nicely in the corresponding value, (14,44,14). Thus P_2 stands to gain from forming this link, so it would seem that he should go ahead and do so. But now in this new situation, it would be advantageous for P_1 and P_3 to form a link; this would result in the complete graph



(6)

which is again symmetric and so corresponds to a payoff of (24,24,24). Therefore, whereas it originally seemed worthwhile for P_2 to forge a new link, on closer examination it turns out to lead to a net loss of 6 (he goes from 30 to 24). Thus the original graph, with only P_1 and P_2 linked, would appear to be in some sense "stable" after all.

Can this reasoning be formalized and put into a more general context? It is true that if P_2 offers to link up with P_3, then P_1 also will, but wouldn't P_1 do this anyway? To make sense of the argument, must one assume that P_1 and P_2 explicitly agree not to bring P_3 in? If so, under what conditions would such an agreement come about?

It turns out that no such agreement is necessary to justify the argument. As we shall see in the next section, the argument makes good sense in a framework that is totally noncooperative (as far as link formation is concerned; once the links are formed, enforceable agreements may be negotiated).

3 The formal model

Given a coalitional game v with n players, construct an auxiliary *linking game* as follows: At the beginning of play there are no links between any players. The game consists of pairs of players being offered to form links, the offers being made one after the other according to some definite rule; the rule is common knowledge and will be called the *rule of order*. To form a link, *both* potential partners must agree; once formed, a link cannot be destroyed, and, at any time, the entire history of offers, acceptances, and rejections is known to all players (the game is of perfect information). The only other requirements for the rule of order are that it lead to a finite game, and that after the last link has been formed, each of the $n(n-1)/2$ pairs must be given a final opportunity to form an additional link (as in the bidding stage of bridge). At this point some cooperation graph g has been determined; the payoff to each player i is then defined as $\phi_i^g(v)$.

Most of the analysis in the sequel would not be affected by permitting the rule of order to have random elements as long as perfect information is maintained. It does, however, complicate the analysis, and we prefer to exclude chance moves at this stage.

Note that it does not matter in which order the two players in a pair decide whether to agree to a link; in equilibrium, either order (with perfect information) leads to the same outcome as simultaneous choice.

In practice, the initiative for an offer may come from one of the players rather than from some outside agency. Thus the rule of order might give the initiative to some particular player and have it pass from one player to another in some specified way.

Because the game is of perfect information, it has subgame perfect equilibria (Selten 1965) in pure strategies.[2] Each such equilibrium is associated with a unique cooperation graph g, namely the graph reached at the end of play. Any such g (for any choice of the order on pairs) is called a *natural structure* for v (or a *natural outcome* of the linking game).

Rather than starting from an initial position with no links, one may start from an exogenously given graph g. If all subgame perfect equilibria of the resulting game (for any choice of order) dictate that no additional links form, then g is called *stable.*

4 An illustration

We illustrate with the game defined by (1). To find the subgame perfect equilibria, we use "backwards induction." Suppose we are already at a stage in which there are two links. Then, as we saw in Section 2, it is worthwhile for the two players who have not yet linked up to do so; therefore we may assume that they will. Thus one may assume that an inevitable consequence of going to two links is a graph with three links. Suppose now there is only one link in the graph, say that between P_1 and P_2 [as in (4)]. P_2 might consider offering to link up with P_3 [as in (5)], but we have just seen that this necessarily leads to the full graph [as in (6)]. Because P_2 gets less in (6) than in (4), he will not do so.

Suppose, finally, that we are in the initial position, with no links at all. At this point the way in which the pairs are ordered becomes important;[3] suppose it is 12, 23, 13. Continuing with our backwards induction, suppose the first two pairs have refused. If the pair 13 also refuses, the result will be 0 for all; if, on the other hand, they accept, it will be (30,0,30). Therefore they will certainly accept. Going back one step further, suppose that the pair 12 – the first pair in the order – has refused, and the pair 23 now has an opportunity to form a link. P_2 will certainly wish to do so, as otherwise he will be left in the cold. For P_3, though, there is no difference, because in either case he will get 30; therefore there is a subgame perfect equilibrium at which P_3 turns down this offer. Finally, going back to the first stage, similar considerations lead to the conclusion that the linking game has three natural outcomes, each consisting of a single link between two of the three players.

This argument, especially its first part, is very much in the spirit of the informal story in Section 2. The point is that the formal definition clarifies what lies behind the informal story and shows how this kind of argument may be used in a general situation.

5 Some weighted majority games

Weighted majority games are somewhat more involved than the one considered in the previous section, and we will go into less detail. We start with a fairly typical example. Let v be the five-person weighted majority game [4; 3,1,1,1,1] (4 votes are needed to win; one player has three votes, the other four have one vote each). Let us say that the coalition S has *formed* if g is the complete graph on the members of S (two players are linked if both are members of S). We start by tabulating the values for the complete graphs on various kinds of coalitions, using an obvious notation.

$\{1,1,1,1\}$	$[0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}]$
$\{3,1\}$	$[\frac{1}{2},\frac{1}{2},0,0,0]$
$\{3,1,1\}$	$[\frac{2}{3},\frac{1}{6},\frac{1}{6},0,0]$
$\{3,1,1,1\}$	$[\frac{3}{4},\frac{1}{12},\frac{1}{12},\frac{1}{12},0]$
$\{3,1,1,1,1\}$	$[\frac{3}{5},\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}]$

Intuitively, one may think of a parliament with one large party and four small ones. To form a government, the large party needs only one of the small ones. But it would be foolish actually to strive for such a narrow government, because then it (the large party) would be relatively weak *within* the government, the small party could topple the government at will; it would have veto power within the government. The more small parties join the government, the less the large party depends on each particular one, and so the greater the power of the large party. This continues up to the point where there are so many small parties in the government that the large party itself loses its veto power; at that point the large party's value goes *down.* Thus with only one small party, the large party's value is $\frac{1}{2}$; it goes up to $\frac{2}{3}$ with two small parties and to $\frac{3}{4}$ with three, but then drops to $\frac{3}{5}$ with four small parties, because at that point the large party itself loses its veto power within the government. Note, too, that up to a point, the fewer small parties there are in the government, the better for those that are, because there are fewer partners to share in the booty.

We proceed now to an analysis by the method of Section 3. It may be verified that any natural outcome of this game is necessarily the complete graph on some set of players; if a player is linked to another one indirectly, through a "chain" of other linked players, then he must also be linked to him directly. In the analysis, therefore, we may restrict attention to "complete coalitions"–coalitions within which all links have formed.

As before, we use backwards induction. Suppose a coalition of type

$\{3,1,1,1\}$ has formed. If any of the "small" players in the coalition links up with the single small player who is not yet in, then, as noted earlier, the all-player coalition will form. This is worthwhile both for the small player who was previously "out" and for the one who was previously "in" (the latter's payoff goes up from $\frac{1}{12}$ to $\frac{1}{10}$). Therefore such a link will indeed form, and we conclude that a coalition of type $\{3,1,1,1\}$ is unstable, in that it leads to $\{3,1,1,1,1\}$.

Next, suppose that a coalition of type $\{3,1,1\}$ has formed. If any player in the coalition forms a link with one of the small players outside it, then this will lead to a coalition of the form $\{3,1,1,1\}$, and, as we have just seen, this in turn will lead to the full coalition. This means that the large player will end up with $\frac{3}{5}$ (rather than the $\frac{2}{3}$ he gets in the framework of $\{3,1,1\}$) and the small players with $\frac{1}{10}$ (rather than the $\frac{1}{6}$ they get in the framework of $\{3,1,1\}$). Therefore none of the players in the coalition will agree to form any link with any player outside it, and we conclude that a coalition of type $\{3,1,1\}$ is stable.

Suppose next that a coalition of type $\{3,1\}$ has formed. Then the large player does have an incentive to form a link with a small player outside it. For this will lead to a coalition of type $\{3,1,1\}$, which, as we have seen, is stable. Thus the large player can raise his payoff from the $\frac{1}{2}$ he gets in the framework of $\{3,1\}$ to the $\frac{2}{3}$ he gets in the framework of $\{3,1,1\}$. This is certainly worthwhile for him, and therefore $\{3,1\}$ is unstable.

Finally, suppose no links at all have as yet been formed. If the small players all turn down all offers of linking up with the large player but do link up with each other, then the result is the coalition $\{1,1,1,1\}$, and each one will end up with $\frac{1}{4}$. If, on the other hand, one of them links up with the large player, then the immediate consequence is a coalition of type $\{3,1\}$; this in turn leads to a coalition of type $\{3,1,1\}$, which is stable. Thus for a small player to link up with the large player inevitably leads to a payoff of $\frac{1}{6}$ for him, which is less than the $\frac{1}{4}$ he could get in the framework of $\{1,1,1,1\}$. Therefore considerations of subgame perfect equilibrium lead to the conclusion that starting from the initial position (no links), all small players reject all overtures from the large player, and the final result is that the coalition $\{1,1,1,1\}$ forms.

This conclusion is typical for weighted majority games with one "large" player and several "small" players of equal weight. Indeed, we have the following general result.

Theorem A. In a superadditive weighted majority game of the form $[q; w,1, \ldots, 1]$ with $q > w > 1$ and without veto players, a cooperation structure is natural if and only if it is the complete graph on a minimal winning coalition consisting of "small" players only.

The proof, which will not be given here, consists of a tedious examination of cases. There may be a more direct proof, but we have not found it.

The situation is different if there are two large players and many small ones, as in [4; 2,2,1,1,1,] or [6; 3,3,1,1,1,1,1]. In these cases, either the two large players get together or one large player forms a coalition with *all* the small ones (not minimal winning!). We do not have a general result that covers all games of this type.

Our final example is the game [5; 3,2,2,1,1]. It appears that there are two types of natural coalition structure: one associated with coalitions of type {2,2,1,1}, and one with coalitions of type {3,2,1,1}. Note that neither one is minimal winning.

In all these games some coalition forms; that is, the natural graphs all are "internally complete." As we will see in the next section, that is not the case in general. For simple games, however, and in particular for weighted majority games, we do not know of any counterexample.

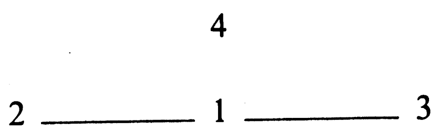
6 A natural structure that is not internally complete

Define v as the following sum of three voting games:

$$v := [2;\ 1,1,1,0] + [3;\ 1,1,1,0] + [5;\ 3,1,1,2].$$

That is, v is the sum of a three-person majority game in which P_4 is a dummy, a three-person unanimity game in which P_4 is again a dummy, and a four-person voting game in which the minimal winning coalitions are {1,2,3} and {1,4}. The *sum* of these games is defined as any sum of functions, so the worth $v(S)$ of a coalition S is the number of component games in which S wins. For example, $v(\{2,3\}) = 1$ and $v(\{1,2,4\}) = 2$.

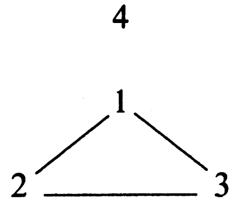
The unique natural structure for this game is



That is, P_1 links up with P_2 and P_3, but P_2 and P_3 do not link up with each other, and no player links up with P_4. The Myerson value of this game for this cooperation structure is $(\frac{5}{3},\frac{5}{6},\frac{5}{6},0)$.

The Shapley value of this game, which is also the Myerson value for the complete graph on all the players, is $(\frac{5}{4},\frac{3}{4},\frac{3}{4},\frac{1}{4})$. Notice that P_1, P_2, and P_3 all do strictly worse with the Shapley value than with the Myerson value for the natural structure described earlier. It can be verified that for any other graph either the value equals the Shapley value or there is at least one pair of players who are not linked and would do strictly better with the Shapley value. This implies inductively that if any pair of players forms a link that is not in the natural structure, then additional links will continue to form until every player is left with his Shapley value. To avoid this outcome, P_1, P_2, and P_3 will refuse to form any links beyond the two already shown.

For example, consider what happens if P_2 and P_3 add a link so that the graph becomes



The value for this graph is (1,1,1,0), which is better than the Shapley value for P_2 and P_3, but worse than the Shapley value for P_1. To rebuild his claim to a higher payoff than P_2 and P_3, P_1 then has an incentive to form a link with P_4.

Intuitively, P_1 needs both P_2 and P_3 in order to collect the payoff from the unanimity game [3; 1,1,1,0]. They, in turn, would like to keep P_4 out because he is comparatively strong in the weighted voting game [5; 3,1,1,2], whose Shapley value is $(\frac{7}{12},\frac{1}{12},\frac{1}{12},\frac{3}{12})$. With P_4 out, all three remaining players are on the same footing, because all three are then needed to form a winning coalition. Therefore P_1 and P_2 may each expect to get $\frac{1}{3}$ from this game, which is $\frac{1}{4}$ more than the $\frac{1}{12}$ they were getting with P_4 in. On the other hand, excluding P_4 *lowers* P_1's value by $\frac{1}{4}$, from $\frac{7}{12}$ to $\frac{1}{3}$, and P_1 will therefore want P_4 in.

This is where the three-person majority game [2; 1,1,1,0] enters the picture. If P_2 and P_3 refrain from linking up with each other, then P_1's centrality makes him much stronger in *this* game, and his Myerson value in it is then $\frac{2}{3}$ (rather than $\frac{1}{3}$, the Shapley value). This gain of $\frac{1}{3}$ more than makes up for the loss of $\frac{1}{4}$ suffered by P_1 in the game [5; 3,1,1,2], so he is willing to keep P_4 out. On the other hand, P_2 and P_3 also gain thereby, because the $\frac{1}{4}$ each gains in [5; 3,1,1,2] more than makes up for the $\frac{1}{6}$ each loses in the three-person majority game. Thus P_2 and P_3 are motivated to refrain from forming a link with each other, and all are motivated to refrain from forming links with P_4.

In brief, P_2 and P_3 gain by keeping P_4 isolated; but they must give P_1 the central position in the {1,2,3} coalition so as to provide an incentive for him to go along with the isolation of P_4, and a credible threat if he doesn't.

7 Natural structures that depend on the rule of order

The natural outcome of the link-forming game may well depend on the rule of order. For example, let u be the majority game [3; 1,1,1,1], let $w := [2; 1,1,0,0]$, and let $w' := [2; 0,0,1,1]$. Let $v := 24u + w + w'$. If the first offer is made to {1,2}, then either {1,2,3} or {1,2,4} will form; if it is made to {3,4}, then either {1,3,4} or {2,3,4} will form.

The underlying idea here is much like in the game defined by (1). The first two players to link up are willing to admit one more player in order to enjoy the proceeds of the four-person majority game u; but the resulting coalition is not willing to admit the fourth player, who would take a large share of those proceeds and himself contribute comparatively little. The difference between this game and (1) is that here each player in the first pair to get an *opportunity* to link up is positively motivated to seize that opportunity, which was not the case in (1).

The nonuniqueness in this example is robust to small changes in the game. That is, there is an open neighborhood of four-person games around v such that, for all games in this neighborhood, if P_1 and P_2 get the first opportunity to form a link then the natural structures are graphs in which P_1, P_2, and P_3 are connected to each other but not to P_4; but if P_3 and P_4 get the first opportunity to form a link, then the natural structures are graphs in which P_2, P_3, and P_4 are connected to each other but not to P_1. (Here we use the topology that comes from identifying the set of n-person coalitional games with euclidean space of dimension $2^n - 1$.)

Each example in this chapter is also robust in the phenomenon that it is designed to illustrate. That is, for all games in a small open neighborhood of the example in Section 4, the natural outcomes will fail to be Pareto optimal; and for all games in a small open neighborhood of the example in Section 6, the natural outcomes will not be complete graphs on any coalition.

8 Discussion

The theory presented here makes no pretense to being applicable in all circumstances. The situations covered are those in which there is a preliminary period that is devoted to link formation only, during which, for one reason or another, one cannot enter into binding agreements of any kind (such as those relating to subsequent division of the payoff, or even conditional link-forming, or nonforming, deals of the kind "I won't link up with Adams if you don't link up with Brown"). After this preliminary period one carries out negotiations, but then new links can no longer be added.

An example is the formation of a coalition government in a parliamentary democracy in which no single party has a majority (Italy, Germany, Israel, France during the Fifth Republic, even England at times). The point is that a government, once formed, can only be altered at the cost of a considerable upheaval, such as new elections. On the other hand, one cannot really negotiate in a meaningful way on substantive issues before the formation of the government, because one does not know what issues will come up in the future. Perhaps one does know something about some of the issues, but even then one cannot make *binding* deals about them. Such deals, when attempted, are indeed often eventually circumvented or even broken outright; they are to a large extent window dressing, meant to mollify the voter.

An important assumption is that of perfect information. There is nothing to stop us from changing the definition by removing this assumption – something we might well wish to try – but the analysis of the examples would be quite different. Consider, for example, the game [4; 3,1,1,1,1] treated at the beginning of Section 5. Suppose that the rule of order initially gives the initiative to the large player. That is, he may offer links to each of the small players in any order he wants; links are made public once they are forged, but rejected offers do not become known. This is a fairly reasonable description of what may happen in the negotiations for formation of governments in parliamentary democracies of the kind described here. In this situation the small players lose the advantage that was conferred on them by perfect information; formation of a coalition of type {3,1,1} becomes a natural outcome. Intuitively, a small player will refuse an offer from the large player only if he feels reasonably sure that all the small players will refuse. Such a feeling is justified if it is common knowledge that all the others have already refused, and from there one may work one's way backward by induction. But the induction is broken if refused offers do not become known; and then the small players may become suspicious of each other – quite likely rightfully, as under imperfect information, mutual suspicion becomes an equilibrium outcome. We hasten to add that mutual trust – all small players refusing offers from the large one–remains in equilibrium; but unlike in the case of perfect information, where everything is open and aboveboard, it is no longer the *only* equilibrium. In short, secrecy breeds mistrust–justifiable mistrust.

Which model is the "right" one (i.e., perfect or imperfect information) is moot. Needless to say, the perfect information model is not being suggested as a universal model for all negotiations. But one may feel that the secrecy in the imperfect information model is a kind of irrelevant noise that muddies the waters and detracts from our ability properly to analyze power relationships. On the other hand, one may feel that the backwards induction in the perfect information model is an artificiality that overshadows and dominates the analysis, much as in the finitely repeated Prisoner's Dilemma, and again obscures the "true" power relationships. Moreover, the outcome predicted by the perfect information model in the game [4; 3,1,1,1,1] (formation of the coalition of all small players) *is* somewhat strange and anti-intuitive. On the contrary, one would have thought that the large player has a better chance than each individual small player to get into the ruling coalition; one might expect *him* to "form the government," so to speak.

In brief, there is no single "right" model. Each model has something going for it and something going against it. You pay your money, and you take your choice.

We end with an anecdote. This chapter is based on a correspondence that took place between the authors during the first half of 1977. That spring, there were elections in Israel, and they brought the right to power for the first time since the foundation of the state almost thirty years earlier. After the election, one of us used the perfect information model proposed here to try to predict which government would form. He was disappointed when the government that actually did form after about a month of negotiations did not conform to the prediction of the model, in that it failed to contain Professor Yigael Yadin's new "Democratic Party for Change." Imagine his delight when Yadin did after all join the government about four months later!

Appendix

We state and prove here the main result of Myerson (1977).

For any graph g, any set of players S, and any two players j and k in S, we say that j and k are connected in S by g if and only if there is a path in g that goes from j to k and stays within S. That is, j and k are connected in S by g if there exists some sequence of players $i_1, i_2, \ldots, i_M$ such that

$i_1 = j, i_M = k, \{i_1, i_2, \ldots, i_M\} \subseteq S$, and every pair (i_n, i_{n+1}) corresponds to a link in g. Let S/g denote the partition of S into the sets of players that are connected in S by g. That is,

$$S/g = \{\{k \mid j \text{ and } k \text{ are connected in } S \text{ by } g\} \mid j \in S\}.$$

With this notation, the definition of v^g from (2) becomes

$$v^g(S) = \sum_{T \in S/g} v(T) \tag{A1}$$

for any coalition S. Then the main result of Myerson (1977) is as follows.

Theorem. Given a coalitional game v, Axioms 1 and 2 (as stated in Section 2) are satisfied for all graphs if and only if, for every graph g and every player i,

$$\phi_i^g(v) = \phi_i(v^g), \tag{A2}$$

where ϕ_i denotes the ordinary Shapley value for player i. Furthermore, if v is superadditive and if g is a graph obtained from another graph h by adding a single link between players i and j, then $\phi_i(v^g) - \phi_i(v^h) \geq 0$, so the differences in Axiom 1 are nonnegative.

Proof: For any given graph g, Axiom 1 gives us as many equations as there are links in g, and Axiom 2 gives us as many equations as there are connected components of g. When g contains cycles, some of these equations may be redundant, but it is not hard to show that these two axioms give us at least as many independent linear equations in the values ϕ_i^g as there are players in the game. Thus, arguing by induction on the number of links in the graph (starting with the graph that has no links), one can show that there can be at most one value satisfying Axioms 1 and 2 for all graphs.

The usual formula for the Shapley (1953) value implies that

$$\phi_i(v^g) - \phi_j(v^g) = \sum_{S \subseteq N \setminus \{i,j\}} \frac{|S|!(|N| - |S| - 2)!}{(|N| - 1)!} (v^g(S \cup \{i\}) - v^g(S \cup \{j\})).$$

Notice that a coalition's worth in v^g depends only on the links in g that are between two players both of whom are in the coalition. Thus, when S does not contain i or j, the worths $v^g(S \cup \{i\})$ and $v^g(S \cup \{j\})$ would not be changed if we added or deleted a link in g between players i and j. Therefore, $\phi_i(v^g) - \phi_j(v^g)$ would be unchanged if we added or deleted a link in g between players i and j. Thus, (A2) implies Axiom 1.

Given any coalition S and graph g, let the games u^S and w^S be defined by $u^S(T) = v^g(T \cap S)$ and $w^S(T) = v^g(T \backslash S)$ for any $T \subseteq N$. Notice that S is a carrier of u^S, and all players in S are dummies in w^S. Furthermore, if S is a connected component of g, then $v^g = u^S + w^S$. Thus, if S is a connected component of g, then

$$\sum_{i \in S} \phi_i(v^g) = \sum_{i \in S} \phi_i(u^S) = u^S(S) = v^g(S),$$

and so (A2) implies Axiom 2.

Now suppose that the graph g is obtained from the graph h by adding a single link between players i and j. If v is superadditive and $i \in S$, then $v^g(S) \geq v^h(S)$, because S/g is either the same as S/h or a coarser partition than S/h. On the other hand, if $i \notin S$, then $v^g(S) = v^h(S)$. Thus, by the monotonicity of the Shapley value, $\phi_i(v^g) \geq \phi_i(v^h)$ if v is superadditive.

Q.E.D.

NOTES

1 These statements are proved in the appendix, and they imply the assertions about the Myerson value that we made in the introduction.

2 Readers unfamiliar with German and the definition of subgame perfection will find the latter repeated, in English, in Selten (1975), though this reference is devoted mainly to the somewhat different concept of "trembling hand" perfection (even in games of perfect information, trembling hand perfect equilibria single out only *some* of the subgame perfect equilibria).

3 For the analysis, not the conclusion.

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