54 Core and Value for a Public Goods Economy: An Example

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1. Introduction

One can approach the problem of distribution in an economy in at least three different ways: competitive equilibrium; core; and Shapley value. In an economy with a continuum of agents, these three approaches are equivalent in that they lead to the same imputation of utility [1, 2]. The concept of Lindahl equilibrium [7] has been advanced to fill the role of competitive equilibrium when public goods are present in the economy, the other approaches remaining the same. Foley [4] has shown that the Lindahl equilibrium is in the core, regardless of the size of the economy. Muench [8] has given an example of a continuum economy whose core and Lindahl equilibrium are not equivalent, but whose value and Lindahl equilibrium are. Gardner [5] and Rosenthal [9] have given examples in which the value and Lindahl equilibrium are different, but both are in the core. This leaves open the question as to whether the value must always be in the core. This note answers that question in the negative, by presenting an example in which the core consists of a single point, which is different from the value.

2. The Example

The economy consists of two types of agents, of measure 1 each, one public good and one private good. Formally, the space of agents is the measure space $(T, \mathbf{B}, \mu)$, where $T = [0, 2]$, $\mathbf{B}$ consists of the Borel subsets of T, and μ is Lebesgue measure. The types are $T_1 = [0, 1]$ and $T_2 = (1, 2]$. Each infinitesimal agent dt in T_1 has utility function $u_i(x, y) = 2x - x^2$, where x is the aggregate amount of the public good produced, and $y\mu(dt)$ is the amount of private good consumed by dt (i.e., y is his consumption density). The utility functions for the agents in T_2 are identically zero. Agents in T_1 have no initial endowment, whereas the endowment of an agent dt in T_2 is $\mu(dt)$ (i.e., his endowment density is 1). The public good can be produced from the private good in the ratio 1 : 1; i.e., if each agent dt

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inputs the amount $\mathbf{y}(t)$ $\mu(dt)$ into the production process, then a total of $\int \mathbf{y}(t)$ $\mu(dt)$ of the public good will be produced. Side payments are permitted (as is well known, this is equivalent to assuming the existence of another private good with a linear utility that enters additively into the utility function of both types). Note that the utilities of both types are nondecreasing over the set of possible consumption vectors.

For each coalition S, denote by $\mu_i(S)$ the measure of traders in S of type i. Consider a coalition S with $\mu_1(S) = a$, $\mu_2(S) = b$. Since the private good is useless except for producing the public good, this coalition will produce an amount b of the public good. Each agent of type 1 has utility $2b - b^2$ for the amount of public good, so that the total utility of S is $\int_s (2b - b^2)\, d\mu_1 = (2b - b^2)a$. Thus if ν is the characteristic function of this game, then

$$\nu(S) = (2\mu_2(S) - \mu_2^2(S))\, \mu_1(S).$$

The Shapley value of this game may be calculated by using the diagonal formula [3, Theorem B]. It turns out to be $\frac{2}{3}\mu_1 + \frac{1}{3}\mu_2$. As for the core, we assert that it consists precisely of μ_1. Indeed, it is clear that μ_1 is in the core. Suppose another measure, μ_3, also were. We must have $\mu_3(T) = 1$, and μ_3 must be nonatomic [3, Proposition 27.8]. Since $\mu_3 \neq \mu_1$ there is a coalition U with $\mu_3(U) < \mu_1(U)$. Set $\alpha_i = 1 - \mu_i(U)$; then $\alpha_3 > \alpha_1$. By Lyapunov's theorem, there is for each small positive ϵ a coalition S such that $\mu_i(S) = (1 - \epsilon) + \epsilon\mu_i(U) = 1 - \alpha_i\epsilon$. Hence

$$\begin{aligned}\nu(S) - \mu_3(S) &= (1 - \alpha_2^2\epsilon^2)(1 - \alpha_1\epsilon) - (1 - \alpha_3\epsilon)\\ &= (\alpha_3 - \alpha_1)\,\epsilon - \alpha_2^2\epsilon^2 + \alpha_2^2\alpha_1\epsilon^3.\end{aligned}$$

The linear term is positive for all ϵ under consideration, and hence the whole expression is positive for ϵ sufficiently small. That means $\nu(S) > \mu_3(S)$, and so μ_3 is not in the core.

The interpretation is that the core gives everything to type 1, whereas the value gives twice as much to type 1 as to type 2.

3. Conclusion

In the past, strong claims have been made for both the core [10] and Lindahl equilibrium [7] as principles of just taxation. Yet on intuitive grounds, in the example we have considered the value appears rather more just. Now one may wonder whether the special nature of the utility functions or the initial endowments is responsible for the result; but it is easy to adjust this example so that all endowments are positive and every agent in the economy gets positive utility from the public good and still the value is not in the core. Again, one may wonder whether the von Neumann–Morgenstern characteristic function, with its conservative treatment of threats, is responsible for the result. If one tries the Harsanyi characteristic function [6], one gets a different value, but still a positive imputation to type 2 individuals.

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