Stochastic Spectral Analysis

Lecture Notes

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Chapter 1

Motivation for the use of semigroups in spectral theory

1.1 Definition: Schrödinger semigroup

Let H_0 be the selfadjoint realization of $-\Delta = -\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ in $L^2(\mathbb{R}^d)$. Furthermore, let $(M_V f)(x) = V(x)f(x)$. V shall be chosen in such a way that $H_0 + M_V$ is selfadjoint and semi-bounded on $\mathrm{dom}(H_0 + M_V) = \mathrm{dom}(H_0)$.
Then $H = H_0 + M_V$ generates a semigroup denoted by $\{e^{-tH},\ t \geq 0\}$, called Schrödinger semigroup.

Having in mind the integral form of the Schrödinger equation,

$$\psi(t) = e^{-itH}\psi(0),$$

one could think that the unitary group e^{itH} is the right one to be studied. This is true if we want to study the dynamics of the system, for instance in scattering theory. However, the Schrödinger semigroup implies several interesting spectral consequences not obtained by the study of the unitary group.

1.2 Examples

1.2.1 Eigenfunctions

Let ψ be an eigenfunction of H with the eigenvalue E, i.e. $H\psi = E\psi$. Then

$$
\begin{aligned}
& e^{-tH}\psi = e^{-tE}\psi && (1.1)\\
\Leftrightarrow \qquad & \psi = e^{tE}e^{-tH}\psi && (1.2)\\
\Rightarrow \qquad & \psi \in e^{-tH}[L^2] && (1.3)
\end{aligned}
$$

Hence it is of general interest to study e^{-tH} as an operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$. For instance, if e^{-tH} maps $L^2(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ then ψ is a bounded eigenfunction.
By the way, a similar result is impossible by the use of e^{-itH} because $e^{-itH}\psi$ is always in $L^2(\mathbb{R}^d)$ if $\psi \in L^2(\mathbb{R}^d)$.

1.2.2 Long time behaviour

If $\{e^{-tH},\ t \geq 0\}$ is known to be a semigroup in $L^2(\mathbb{R}^d)$ the question arises whether we can extend this strongly continuous semigroup to further L^p-spaces. If this is possible the generator of the extended semigroup is denoted by H_p, i.e. $\{e^{-tH_p},\ t \geq 0\}$ is a semigroup in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$.
The spectrum of the H_p lies in the halfplane $\{z : \ \mathrm{Re} z > -\omega_p\}$ whereas $-\omega_p = \inf(\mathrm{Re}\sigma(H_p))$.
If

$$\sigma(H_p) = \{-\ln z : \ z \in \sigma(e^{-H_p})\},$$

then ω_p is defined by the spectral radius of e^{-H_p},

$$
\begin{aligned}
\mathrm{spr}(e^{-H_p}) &= \lim_{t\to\infty} ||e^{-tH_p}||_{p,p}^{1/t}\\
&= \lim_{t\to\infty} \exp\left(\frac{1}{t}\ln ||e^{-tH_p}||_{p,p}\right)
\end{aligned}
$$

Therefore,

$$\omega_p = \lim_{t\to\infty} \frac{1}{t}\ln ||e^{-tH_p}||_{p,p},$$

i.e. the spectrum of H_p is partly determined by the long time behaviour of e^{-tH_p}. In particular this is of interest in L^2. Then the generator $H_2 = H$ is selfadjoint and ω_p is equal to the infimum of the spectrum, i.e.

$$-E_0 = \inf \sigma(H) = -\lim_{t\to\infty} \frac{1}{t} \ln ||e^{-tH}||_{2,2}$$

1.2.3 Ground state

Let $H = H_0 + M_V$ be a positive selfadjoint operator in L^2. Assume that $\inf \sigma(H)$ is an eigenvalue $-E_0$ with the ground state ψ (i.e. $H\psi = -E_0\psi$). Then the ground state is given by

$$\psi = \lim_{t\to\infty} \frac{e^{-tH} f}{||e^{-tH} f||},$$

where f is an arbitrary, positive function in $L^2(\mathbb{R}^d)$.

1.2.4 Resolvents

The investigation of the spectra is mainly based on the study of resolvents. Take $H = H_0 + M_V$ in $L^2(\mathbb{R}^d)$. H is assumed to be selfadjoint and bounded from below. Let $-a \in \mathrm{res}(H)$ such that $-\mathrm{Re}(a) < \inf(H)$. Then the resolvent can be expressed by the semigroup using the Laplace-transform:

$$(H+a)^{-r} f = \frac{1}{\Gamma(r)} \int_0^\infty e^{-a\lambda} \lambda^{r-1} e^{-\lambda H} f \,\mathrm{d}\lambda \quad (r > 0).$$

The study of resolvents plays an essential role in spectral theory. Therefore there is a large variety of possibilities to use the Laplace-transform. For instance, if $(H+i)^{-1} - (H_0+i)^{-1}$ is a trace class operator, then the absolutely continuous spectra of H_0 and H coincide (Birman-Kuroda Theorem). Or if the resolvent difference is a compact operator, the essential spectra of H_0 and H are the same.

1.2.5 Semigroup kernels

As long as we study $H_0 = -\Delta$ in $L^2(\mathbb{R}^d)$ the semigroup consists of integral operators of the form

$$(e^{-tH_0} f)(x) = \int_{\mathbb{R}^d} p_w(t,x,y) f(y) \,\mathrm{d}y \quad (t > 0),$$

where the kernel is given by

$$p_w(t,x,y) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4t}\right),$$

the so-called Wiener density. p_w is continuous in t, x and y its total mass is

$$\int_{\mathbb{R}^d} p_w(t,x,y)\,\mathrm{d}y = 1.$$

Due to the semigroup property,

$$e^{-tH_0}e^{-sH_0}f = e^{-(t+s)H_0}f,$$

p_w satisfies the Chapman-Kolmogorov equation:

$$\int\limits_{\mathbb{R}^d} p_w(t,x,u)p_w(s,u,y)\,\mathrm{d}u = p_w(s+t,x,y)$$

On the other hand, p_w generates a stochastic process known as Wiener process. It consists of a probability space $(\Omega, \mathcal{B}, P_x)$ where Ω is the set of continuous functions from $\mathbb{R}_+ \to \mathbb{R}^d$. P_x is the Wiener measure. The trajectories $X(.)$ of the process are continuous functions starting in $X(0) = x$.
Let E_x denote the expectation with respect to P_x, then the semigroup is given by

$$(e^{-tH_0}f)(x) = E_x\{f(X(t))\}.$$

For a certain class of potentials, called Kato potentials, one obtains an analogous representation for the perturbed semigroups. This is known as Feynman-Kac formula:

$$\left(e^{-t(H_0+M_V)}f\right)(x) = E_x\left\{\exp\left(-\int\limits_0^t V(X(s))\,\mathrm{d}s\right) f(X(t))\right\}.$$

On the σ-algebra of all events $\{X(s):\ 0 \le s < t\}$ one can define a conditional measure by

$$E_x^{y,t}(A) = \lim_{s\nearrow t} E_x\{p_w(t-s, X(s), y)\mathbf{1}_A\}$$

with

$$\int_{\mathbb{R}^d} E_x^{y,t}(A)\,\mathrm{d}y = E_x(\mathbf{1}_A)$$

Its total mass is

$$E_x^{y,t}(\mathbf{1}_{\boldsymbol{\Omega}}) = p_w(t,x,y).$$

Using this conditional Wiener measure the perturbed semigroup turns out to consist of integral operators, too:

$$(e^{-t(H_0+M_V)}f)(x) = \int_{\mathbb{R}^d} E_x^{y,t}\left\{\exp\left(-\int_0^t V(X(s))\,\mathrm{d}s\right)\right\} f(y)\,\mathrm{d}y$$

or, in terms of integral kernels:

$$e^{-t(H_0+M_V)}(x,y) = E_x^{y,t}\left\{\exp\left(-\int_0^t V(X(s))\,\mathrm{d}s\right)\right\}.$$

Because of these representations one can use the theory of integral operators to obtain spectral results. For instance, let $D_t = e^{-t(H_0+M_V)} - e^{-tH_0}$. Then the essential spectra of the generators are the same if D_t is a Hilbert-Schmidt operator. For that it suffices that

$$\int\limits_{\mathbb{R}^d} \mathrm{d}x \int\limits_{\mathbb{R}^d} \mathrm{d}y \;\; \left|e^{-t(H_0+M_V)}(x,y) - e^{-tH_0}(x,y)\right|^2 < \infty,$$

where $e^{-tH_0}(x,y) = p_w(t,x,y)$.

1.2.6 Large coupling limits

The semigroups can be used also for studying singular perturbations of the form

$$V(x) = \begin{cases} \infty & \text{for} \quad x \in \quad \Gamma \subset \mathbb{R}^d \\ 0 & \text{for} \quad x \in \quad R^d \setminus \Gamma \end{cases}$$

Γ is a closed set called obstacle region . We set $\Sigma = \mathbb{R}^d \setminus \Gamma$.
The corresponding semigroups can be defined by large coupling limits: Let

$$M_\beta f = \beta \mathbf{1}_\Gamma f, \quad \beta > 0.$$

$\beta\mathbf{1}_\Gamma$ is a bounded potential such that the Feynman-Kac formula holds:

$$\left(e^{-t(H_0+M_\beta)}f\right)(x) = E_x\left\{\exp\left(-\beta\int\limits_0^t \mathbf{1}_\Gamma(X(s))\,\mathrm{d}s\right) f(X(t)\right\}.$$

The integral $\int\limits_0^t \mathbf{1}_\Gamma(X(s))\,\mathrm{d}s =: T_{t,\Gamma}$ is the spending time of the trajectory in Γ. It is zero if $X(s) \notin \Gamma$ for all $s \in [0,t]$.
One obtains

$$\begin{aligned} & E_x\left\{\exp\left(-\beta\int\limits_0^t \mathbf{1}_\Gamma(X(s))\,\mathrm{d}s\right) f(X(t)\right\} \\ = \quad & E_x\left\{f(X(t)):\ \ T_{t,\Gamma}=0\right\} + E_x\left\{e^{-\beta T_{t,\Gamma}} f(X(t)):\ \ T_{t,\Gamma}>0\right\}. \end{aligned}$$

The second term tends to zero if $\beta \to \infty$.
Therefore, the large coupling limit is

$$\lim_{\beta\to\infty} e^{-t(H_0+M_\beta)}f =: U_\Sigma(t)f \tag{1.4}$$

with $(U_\Sigma(t)f)(x) = E_x\{f(X(t)):\ \ T_{t,\Gamma}=0\}$.
$U_\Sigma(t)$ restricted to $L^2(\Sigma)$ turns out to be a strongly continuous semigroup in $L^2(\Sigma)$. Its generator is denoted by H_Σ . H_Σ is a model for $H_0 + M_V$, $V = \infty$ on Γ and $V = 0$ on Σ.

1.2.6.1 Remark

Further examples are given by Simon in [14], p. 447-526

Chapter 2

Kolmogorov construction

2.1 Definition: Stochastic process

Let $(\Omega, \mathcal{B}_\Omega, P)$ be a probability space. Let $\mathcal{B}_{\mathbb{R}^d}$ be the Borel-σ-algebra of $\mathbb{R}^d$. Any measurable function $\zeta : \Omega \to \mathbb{R}^d$ is called a random variable, i.e.

$$\zeta^{-1}(A) \in \mathcal{B}_\Omega \quad \text{if} \quad A \in \mathcal{B}_{\mathbb{R}^d}.$$

A random variable defines a measure μ on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$ by

$$\mu(A) := P(\zeta^{-1}(A)), \quad A \in \mathcal{B}_{\mathbb{R}^d}.$$

Let I be a finite index set. $n(I)$ denotes the number of indices in I. Consider a family of random variables ζ_i denoted by

$$\zeta_I := \{\zeta_i, \;\; i \in I\}.$$

Take $A = (A_1, \dots, A_{n(I)}) \in \mathcal{B}_{(\mathbb{R}^d)^{n(I)}}$ and $\zeta^{-1}(A) = \{\zeta_1^{-1}(A_1), \dots, \zeta_{n(I)}^{-1}(A_{n(I)})\}$. Correspondingly we obtain a measure on $(\mathbb{R}^d)^{n(I)}$ by

$$\mu_I(A) := P(\zeta_I^{-1}(A)).$$

That can be generalized to „infinite index sets“ e.g. to intervals $T \subset \mathbb{R}$. Later, T will be equal to $\mathbb{R}_+$.

A stochastic process is a quintupel

$$\big(T, (\Omega, \mathcal{B}_\Omega, P), \zeta(t,\omega)\big),$$

where $T \subset \mathbb{R}$ is an interval, $(\Omega, \mathcal{B}_\Omega, P)$ is a probability space, ζ is a function mapping $T \times \Omega \to \mathbb{R}^d$. For any fixed t $\zeta(t,.)$ is a random variable from $\Omega \to \mathbb{R}^d$. Let $\theta = \{t_1, \dots, t_n\}$ be a finite set of pairwise different values in T. Then, one can define a measure on $(\mathbb{R}^d)^n$ by

$$\mu_\theta(A_1, \dots, A_n) = P\left(\bigcap_{j=1}^{n} \zeta_{t_j}^{-1}(A_j)\right),$$

with $A_j \in \mathcal{B}_{\mathbb{R}^d}$. μ_θ is called a finite dimensional distribution of the process. These distributions fulfill the so-called Kolmogorov consistency conditions.

2.2 Kolmogorov consistency conditions

2.2.1 First condition

Let π be a permutation of the numbers $\{1, \dots, n\}$. Then it is required

$$\mu_{t_{\pi(1)},\dots,t_{\pi(n)}}(A_{\pi(1)}, \dots, A_{\pi(n)}) = \mu_{t_1,\dots,t_n}(A_1, \dots, A_n)$$

2.2.2 Second condition

$$\mu_{t_1,\dots,t_n}(A_1, \dots, A_{n-1}, \mathbb{R}^d) = \mu_{t_1,\dots,t_{n-1}}(A_1, \dots, A_{n-1})$$

and

$$\mu_t(\mathbb{R}^d) \leq 1$$

2.2.3 Remarks for the Kolmogorov construction

The definition of a stochastic process implies at least the property that its finite dimensional distributions fulfill the consistency conditions. This consideration can be reversed: One defines a distribution fulfilling the consistency conditions and constructs a stochastic process.

At first we construct a measurable space $(\Omega, \mathcal{B}_\Omega)$. Let $T \subseteq \mathbb{R}_+$ and μ_θ be a family of distributions fulfilling the consistency conditions. Set $\Omega = (\mathbb{R}^d)^T = \prod_{t\in T} \mathbb{R}^d$, Ω is the space of functions $X: \ T \to \mathbb{R}^d$. Let $\theta = \{t_1, \dots, t_n\}$ be a collection of values from T. For any θ one defines a projection from $\Omega \to (\mathbb{R}^d)^n$ by

$$p_\theta = \big((X(t_1), \dots, X(t_n)\big).$$

Then $\mathcal{B}_\Omega$ is defined as the smallestσ-algebra generated by $p_\theta^{-1}(A)$ for all Borel sets $A \in \mathcal{B}_{(\mathbb{R}^d)^n}$ and for all $\theta = \{t_1, \dots, t_n\}$.

2.3 Theorem of Kolmogorov

Let μ_θ be a consistent family of distributions, then there is a measure P on $(\Omega, \mathcal{B}_\Omega)$ that fulfills:

$$P \circ p_\theta^{-1} = \mu_\theta$$

i.e. $\forall A \in \Omega$

$$P(p_\theta^{-1}(A)) = \mu_\theta(A).$$

That means there is a stochastic process $(T, (\Omega, \mathcal{B}_\Omega, P), X(.))$, which realizes the given distribution.

We will assume that the trajecories start in x, i.e. that $P(X(0) = x) = 1$. Hence, we will write $P_x(.)$ for the measure and $E_x(.)$ for the expectation.

Thus for constructing a process it suffices to find a distribution satisfying the consistency conditions. Finite dimensional distributions can be implemented by products of functions

$$p: \ T \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+.$$

Define:

$$\mu_{t_1,\dots,t_n}(A_1, \dots, A_n) := \\ \int_{A_1} \mathrm{d}x_1 \dots \int_{A_n} \mathrm{d}x_n \quad p(t_1, x_1, x)p(t_2 - t_1, x_2, x_1) \dots p(t_n - t_{n-1}, x_n, x_{n-1}),$$

then the consistency conditions are fulfilled if

$$\int_{\mathbb{R}^d} p(s,x,u)p(t,u,y) \;\; \mathrm{d}u = p(s+t,x,y) \quad \forall s,t>0, \;\; \forall x,y \in \mathbb{R}^d,$$

and

$$\int_{\mathbb{R}^d} p(s,x,u) \;\; \mathrm{d}u \leq 1.$$

The one-dimensional distribution is

$$P\{X(t) \in B\} = \int\limits_B p(t,x,y) \;\; \mathrm{d}y$$

More details are given by Ginibré in [9], pp. 327-427.

Chapter 3

Basic assumptions of stochastic spectral analysis

3.1 The Wiener density

As we have seen that the finite distributions are based on the transition function $p: \ \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$.
On one hand p can be used to generate processes, on the other hand p is the kernel of an integral operator.
The Wiener density function forms the basis of our consideration. It was defined by

$$p_w(t,x,y) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{3.1}$$

This function has the following properties:

1) It satisfies the Chapman-Kolmogorov equation

$$\int\limits_{\mathbb{R}^d} p_w(t,x,u)p_w(s,u,y)\,\mathrm{d}u \quad = \quad p_w(t+s,x,y), \tag{3.2}$$

and

$$\int\limits_{\mathbb{R}^d} p_w(t,x,y)\,\mathrm{d}y \quad = \quad 1. \tag{3.3}$$

2) The corresponding semigroup is given by

$$(S(t)f)(x) = \int\limits_{\mathbb{R}^d} p_w(t,x,y) f(y)\,\mathrm{d}y, \tag{3.4}$$

defined here for all $f \in L^2(\mathbb{R}^d)$. The semigroup is strongly continuous. Its generator is denoted by H_0.

3) The Wiener density is symmetric in x and y, i.e.

$$p_w(t,x,y) = p_w(t,y,x). \tag{3.5}$$

Hence, $S(t) = e^{-tH_0}$ is selfadjoint. H_0 is known to be the selfadjoint realization of the Laplace operator $-\Delta$.

4) The Wiener density is continuous in t, x, y. We have

$$p_w(t,x,y) \quad \leq \frac{c}{|x-y|^d} \quad, \forall t > 0, \tag{3.6}$$

where c is independent of $t \geq 0$.
On the other hand

$$p_w(t,x,y) \quad \leq \frac{1}{(4\pi t)^{d/2}}, \quad \forall x, y \in \mathbb{R}^d. \tag{3.7}$$

It decreases exponentially in all directions as $|x| \to \infty$.

If $f \in C_c(\mathbb{R}^d)$ (continuous functions with compact support) then

$$\lim_{|x|\to\infty} \int\limits_{\mathbb{R}^d} p_w(t,x,y) f(y)\,\mathrm{d}y = 0. \tag{3.8}$$

This remains true for $f \in C_\infty(\mathbb{R}^d)$ (continuous functions vanishing at infinity).

5) For $t \to 0$ we get

$$\lim_{t\to 0} \int\limits_{\mathbb{R}^d} p_w(t,x,y) f(y)\,\mathrm{d}y = f(x) \tag{3.9}$$

for almost all $x \in \mathbb{R}^d$.

6) By the Kolmogorov construction p_w generates a stochastic process, called Wiener process, $\{\mathbb{R}_+, (\Omega, \mathcal{B}_\Omega, P_x), X(.)\}$. The trajectories are continuous functions starting in x almost surely. The generator of the Wiener process is the selfadjoint realization of the Laplacian (given by $H_0 f = \mathcal{F}^{-1} M_{k^2} \mathcal{F}$, $\mathcal{F}$ Fourier-transformation, M_{k^2} multiplication operator with k^2 in $L^2(\mathbb{R}^d, dk)$).
We have

$$\left(e^{-tH_0} f\right)(x) = E_x\{f(X(t))\}, \quad \forall f \in L^2(\mathbb{R}^d). \tag{3.10}$$

$E_x\{.\}$ is the Wiener expectation, $P_x\{.\}$ is the Wiener measure.

7) The Wiener trajectories are continuous functions starting in x. A subclass of them are functions with $X(0) = x$ and $X(t) = y$, i.e. they start in x and are in y at time t.
Consider the time interval $T = [0, t]$. Set

$$\Omega_x^{y,t} := \{X, \ \ X \in \Omega_x, \ \ X(t) = y\} = \Omega$$

for all $x, y \in \mathbb{R}^d$. Then there is a finite dimensional distribution

$$\begin{aligned} \mu_{t_1 \dots t_k}^{y,t}(A_1, \dots, A_k) = & \int_{A_1} \mathrm{d}x_1 \dots \int_{A_k} \mathrm{d}x_k p_w(t_1, x, x_1) p_w(t_2 - t_1, x_1, x_2) \dots \\ & \dots p_w(t_k - t_{k-1}, x_{k-1}, x_k) p_w(t - t_k, x_k, y) \end{aligned}$$

Normalizing the distribution by $p_w(t, x, y)$ is satisfies the consistency conditions. Hence we can implement a new stochastic process $\{T, (\Omega, \mathcal{B}_\Omega, P_x^{y,t}), X(.)\}$. $P_x^{y,t}$ is known as conditional Wiener measure. It is not a probability measure. Its total mass is given by

$$\begin{aligned} & \int_{\mathbb{R}^d} 1_\Omega \ \ \mathrm{d}P_x^{y,t} = E_x^{y,t}(1_\Omega) \\ = \ \ & p_w(t, x, y) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \end{aligned} \tag{3.11}$$

That allows to write (3.10) in the form

$$\left(e^{-tH_0}f\right)(x) = \int\limits_{\mathbb{R}^d} p_w(t,x,y)f(y)\,\mathrm{d}y = \int\limits_{\mathbb{R}^d} E_x^{y,t}\{f(X(t)\}\,\mathrm{d}y, \tag{3.12}$$

$\forall f \in L^2(\mathbb{R}^d)$, or if we denote the kernel of e^{-tH_0} by $e^{-tH_0}(x,y)$, we obtain

$$e^{-tH_0}(x,y) = E_x^{y,t}\{\mathbf{1}_\Omega\}. \tag{3.13}$$

3.2 BASSA - Basic Assumptions of Stochastic Spectral Analysis

The motivation is to enlarge the class of selfadjoint generators and to study their spectral behaviour if they are perturbed by regular or singular perturbations. Not all properties of the Wiener density function are necessary for this objective. Hence we modify and diminish the conditions for a transition density function p.

A1) Let $p(.,.,.)$ be a continuous function from $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}_+$.

A2) p is assumed to satisfy the Chapman-Kolmogorov equation, i.e.

$$\int_{\mathbb{R}^d} p(t,x,u)p(s,u,y)\,\mathrm{d}u = p(t+s,x,y).$$

A3) Let

$$\int\limits_{R^d} p(t,x,y)\,\mathrm{d}y \leq 1. \tag{3.14}$$

A4) Assume p to be symmetric, i.e.

$$p(t,x,y) = p(t,y,x). \tag{3.15}$$

A5) For $f \in C_\infty(\mathbb{R}^d)$ we assume

$$\lim_{t\searrow 0} \int\limits_{R^d} p(t,x,y)f(y)\,\mathrm{d}y = f(x) \tag{3.16}$$

pointwise for almost all $x \in \mathbb{R}^d$.

A6) Feller property:
Let $f \in C_\infty(\mathbb{R}^d)$. Then we assume that the function

$$x \mapsto \int\limits_{R^d} p(t,x,y) f(y)\,\mathrm{d}y$$

is also in $C_\infty(\mathbb{R}^d)$, i.e.

$$\int\limits_{R^d} p(t,.,y) f(y)\,\mathrm{d}y \in C_\infty(\mathbb{R}^d). \tag{3.17}$$

3.2.1 Remarks

The conditions A1) - A4) are not very restrictive. The continuity of p is not necessary but the proofs are simpler in this case. A2) is necessary to establish semigroups and processes. A3) is necessary to introduce probability measures. A4) is equivalent to say that we have selfadjoint generators. Because we will study only selfadjoint operators A4) is no restriction.
A5) and A6) are more restrictive. In particular the Feller property characterizes the possible free Feller generators. If K_0 is the Feller generator then A6) means that $e^{-tK_0} f \in C_\infty(\mathbb{R}^d)$ if $f \in C_\infty(\mathbb{R}^d)$. On the other hand it is much more general than the behaviour of the Wiener density at infinity. Hence, in particular A6) allows to study a large variety of operators. And their associated semigroups can not be estimated by $e^{t\Delta}$.

3.3 Consequences

3.3.1 Feller semigroups

We can define the Feller semigroup by

$$(U(t)f)(x) = \int\limits_{R^d} p(t,x,y) f(y)\,\mathrm{d}y. \tag{3.18}$$

This is defined for all f, as long as the right hand side is finite. For instance for $f \in L^\infty(\mathbb{R}^d)$ or for $f \in L^2(\mathbb{R}^d)$. If we assume additionally that p is $L^1 - L^\infty$-smoothing, i.e.

$$\sup_{x,y} \; p(t,x,y) < \infty, \tag{3.19}$$

then $U(t)$ is defined on $L^1(\mathbb{R}^d)$, too.

3.3.2 Feller processes

Following the construction of Kolmogorov we get a strong Markov process

$$\{\mathbb{R}_+, (\Omega, \mathcal{B}_\Omega, P_x), X(.)\}. \tag{3.20}$$

The one-dimensional distribution of this process is

$$P_x\{X(t) \in B\} = \int_B p(t,x,y)\,\mathrm{d}y \tag{3.21}$$

for $t > 0$ and $B \in \mathcal{B}_{\mathbb{R}^d}$. The trajectories are cadlag, i.e. they are P_x-almost surely right continuous and possess left hand limits. As p satisfies A6), the Feller property, the process is called a Feller process.

3.3.2.1 Definition: (Free) Feller generator

$\{U(t), t \geq 0\}$, defined in (3.18) is a contractive strongly continuous semigroup in $L^2(\mathbb{R}^d)$. Due to the Theorem of Hille-Yosida it has a generator. We denote it by K_0, such that

$$e^{-tK_0} f = U(t)f, \quad \forall f \in L^2. \tag{3.22}$$

Because of A6) the semigroup possesses the Feller property

$$e^{-tK_0} C_\infty \subset C_\infty. \tag{3.23}$$

Therefore K_0 is called the Feller generator, or free Feller Operator

3.3.2.2 Remark

Following the theory described in Section 3.1 we have

$$\left(e^{-tK_0} f\right)(x) = E_x\{f(X(t))\} = \int\limits_{\mathbb{R}^d} p(t,x,y) f(y)\,\mathrm{d}y. \tag{3.24}$$

The semigroup consists of integral operators. Denoting its kernels by $e^{-tK_0}(x,y)$ we obtain

$$e^{-tK_0}(x,y) = p(t,x,y). \tag{3.25}$$

e^{-tK_0} is a contraction semigroup in $L^2(\mathbb{R}^d)$. The spectrum of K_0 is in $\mathbb{R}_+$. Let $a > 0$.
Then $-a$ is in the resolvent set of K_0 and we have the following representation for the resolvent

$$\left((a+K_0)^{-1} f\right)(x) = \int\limits_0^\infty \mathrm{d}\lambda\ e^{-a\lambda} \int\limits_{\mathbb{R}^d} p(\lambda,x,y) f(y)\,\mathrm{d}y. \tag{3.26}$$

Again, this is defined as long as the right hand side makes sense. The resolvent is also an integral operator. Its kernel is given by

$$(a+K_0)^{-1}(x,y) = \int\limits_0^\infty e^{-a\lambda} p(\lambda,x,y)\,\mathrm{d}\lambda. \tag{3.27}$$

3.4 Conditional measures

In the Markov process (3.24) Ω is the set of functions with $X(0) = x$. Set $T = [0,t]$ and $\Omega_x^{y,t} = \{X \in \Omega,\ \ X(t) = y\}$. Let $\theta = (t_1,\ldots,t_k)$ with $0 < t_1 < \cdots < t_k < t$. Then we define the finite dimensional distribution

$$\mu_\theta(x_1 \ldots x_k) = p(t_1,x,x_1) p(t_2 - t_1, x_1, x_2) \ldots p(t - t_k, x_k, y). \tag{3.28}$$

Thus $\mu_\theta(x_1 \ldots x_k)$ divided by $p(t,x,y)$ satisfies the Kolmogorov consistency conditions. Then μ_θ in (3.28) establishes a new process on $\Omega_x^{y,t}$ denoted by

$$\{T, (\Omega, \mathcal{B}_\Omega, P_x^{y,t}), X(.)\}. \tag{3.29}$$

Here $\Omega_x^{y,t}$ is denoted shortly by $\tilde{\Omega}$. $P_x^{y,t}$ is a measure on $\mathcal{B}_{\tilde{\Omega}}$. It is not a probabality measure. Its total mass is $p(t,x,y)$. $P_x^{y,t}$ is called the conditional (Feller) measure. Let $\mathcal{B}_s$, $s < t$ be the σ-algebra generated by all $\{X(\rho),\ 0 \le \rho \le s\}$. Let $\mathcal{B}_{t^-} = \bigcup_{0\le s<t} \mathcal{B}_s$. If A is an event in $\mathcal{B}_{t^-}$ the conditional expectation of it is

$$E_{0,x}^{y,t}\{\mathbf{1}_A\} := \lim_{s\nearrow t} E_x\{p(t-s, X(s), y)\mathbf{1}_A\}. \tag{3.30}$$

If the original process is conservative we get

$$\int_{\mathbb{R}^d} E_{0,x}^{y,t}\{\mathbf{1}_A\}\,\mathrm{d}y = E_x\{\mathbf{1}_A\} \tag{3.31}$$

for $A \in \mathcal{B}_{t^-}$.
The total mass of the conditional Feller measure is

$$\begin{aligned} E_{0,x}^{y,t}(\mathbf{1}_{\tilde{\Omega}}) &= E_x\{p(t-s, X(s), y)\} \\ &= \int_{\mathbb{R}^d} p(s,x,u)p(t-s,u,y)\,\mathrm{d}u \\ &= p(t,x,y). \end{aligned} \tag{3.32}$$

Hence, instead of (3.25) we may write

$$e^{-tK_0}(x,y) = E_{0,x}^{y,t}(\mathbf{1}_{\tilde{\Omega}}). \tag{3.33}$$

Here this looks artificial. The advantage of this representation becomes clear in the perturbed cases.

3.5 Examples for free Feller operators and transition densities

3.5.1 Diffusion process

Let $p(t,x,y)$ be a transition density function of a Markov process satisfying

$$\lim_{t\searrow 0} \frac{1}{t} \int_{|x-y|>\epsilon} p(t,x,y)\ \ \mathrm{d}y = 0 \tag{3.34}$$

for all $\epsilon > 0$ and $x \in \mathbb{R}^d$. Let

$$(T_t f)(x) := \begin{cases} \int\limits_{\mathbb{R}^d} p(t,x,y) f(y) \ \ \mathrm{d}y & \text{for} \quad t > 0 \\ f(x) & \text{for} \quad t = 0. \end{cases} \tag{3.35}$$

Then T_t is a contractive semigroup in $C(\mathbb{R}^d)$, the set of uniformly continuous functions on $\mathbb{R}^d$. We denote its generator by K_0. For $f \in C^2(\mathbb{R}^d) \cap \mathrm{dom}(K_0)$ the generator has the form

$$(K_0 f)(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial f}{\partial x_j \partial x_j}(x) + \sum_{j=1}^{d} b_j(x) \frac{\partial f}{\partial x_j}(x). \tag{3.36}$$

The diffusion matrix elements are given by

$$a_{ij}(x) = \lim_{t \searrow 0} \frac{1}{t} \int\limits_{|x-y|<1} (x_i - y_i)(x_j - y_j) p(t,x,y) \ \ \mathrm{d}y. \tag{3.37}$$

The drift coefficients are

$$b_j(x) = \lim_{t \searrow 0} \frac{1}{t} \int\limits_{|x-y|<1} (x_j - y_j) p(t,x,y) \ \ \mathrm{d}y. \tag{3.38}$$

For more details see Dynkin: [6], p.167 and Yosida [16], p. 398ff.

The diffusion process is called canonical if $p(.)$ can be estimated by the Wiener density,

$$p(t,x,y) \leq \frac{c_1}{(4\pi t)^{d/2}} e^{-c_2 \frac{|x-y|^2}{4t}}, \quad 0 < t < t_0, \tag{3.39}$$

where c_1, c_2 are positive constants.

For canonical diffusion processes the diffusion matrix elements have to be bounded. That is too restrictive for our purposes.

Kochubei studied these kind of operators in some detail. He gave the following sufficient condition (see [13]):

a) Let $\frac{\partial^2 a_{ij}}{\partial x_k \partial x_l}$ be Hölder continuous and in $L^1_{loc}(\mathbb{R}^d)$. Assume

$$\sum_{i,j=1}^{d} \frac{\partial^2 a_{ij}}{\partial x_k \partial x_l} \leq 0$$

for all $x \in \mathbb{R}^d$.

b) Let $\sum_{j=1}^d \frac{\partial a_{ij}}{\partial x_j}$ be Hölder continuous.

c) Let $b_j \in L^d_{loc}(\mathbb{R}^d)$ and $\sum_{j=1}^d \frac{\partial b_j}{\partial x_j}$ in $L^{\max(1,d/2)}_{loc}(\mathbb{R}^d)$.

Then the operator K_0, given in (3.36), is a generator of a Feller process. The transition density function is continuous in $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$.
Of course, if we want to extend the semigroup to $L^2(\mathbb{R}^d)$ and if we want to concentrate on selfadjoint K_0 we need some further condition for $a_{ij}(.)$ and $b_j(.)$. For that the reader may consult Weidmann [15], Berezanskii [2] or Glazman [10].

3.5.2 K_0 on manifolds

Here we have studied the theory on $\mathbb{R}^d$. It can be extended to second countable locally compact Hausdorff spaces E (see [5]). In particular, E can be a locally finite Riemannian manifold. Davies [3] gave examples of the form

$$K_0 f = -\frac{1}{\sigma^2}\nabla(\sigma^2 \nabla f)$$

with fairly general conditions on σ. He showed that $\{e^{-tK_0},\ t \geq 0\}$ is positivity preserving, strongly continuous and a contraction semigroup on L^p, $1 \leq p < \infty$. The semigroup possesses a kernel $e^{-tK_0}(x,y)$. Under certain geometric conditions the kernel has the Feller property.

3.5.3 Pseudo-differential operators

Jacob ([11],[12]) considered pseudo-differential operators of the form

$$[p(x,D)f](x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot k} p(x,k) \hat{f}(k) \;\, \mathrm{d}k. \tag{3.40}$$

He gave sufficient conditions for the symbols $p(x,k)$ such that the pseudo-differential operator $p(x,D)$ is defined on the Sobolev space $H^\infty(\mathbb{R}^d) = \bigcap_{q\geq 0} H^q(\mathbb{R}^d)$. Further conditions on $p(x,D)$ ensure that $p(x,D)$ possesses a selfadjoint extension K_0 to

$L^2(\mathbb{R}^d)$. K_0 generates a Feller semigroup. e^{-tK_0} are integral operators. Their kernels satisfy BASSA.

3.5.4 Functions of the Laplacian

Instead of the pseudo-differential operators as in Section 3.5.3 we can restrict ourselves to functions of the Laplacian.
Let $p(.)$ be a function from $\mathbb{R}^d \to \mathbb{R}$ with the property

$$\int_{\mathbb{R}^d} e^{-tp(k)} \,\mathrm{d}k < \infty \tag{3.41}$$

and $p(0) = 0$.
Then

$$p_L(t,x,y) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-tp(k)+ik\cdot(x-y)} \,\mathrm{d}k \tag{3.42}$$

is a transition density function of a Markov process, ore more specific, of a Lévy process. It is conservative because $\int_{\mathbb{R}^d} p_L(t,x,y)\,\mathrm{d}y = 1$. If $p(k) = |k|^\alpha$, $0 < \alpha < 2$, then

$$\frac{c_1}{t^{d/\alpha}|x-y|^{d+\alpha}} \le p_L(t,x,y) \le \frac{c_2}{t^{d/\alpha}|x-y|^{d+\alpha}} \tag{3.43}$$

as long as $t \ge 1$ and $|x-y| \ge 1$.
In general, there is no explicit analytic expression for the density function. One exception is the Wiener density, another one is the density for the relativistic Hamiltonian, given by

$$p(k) = \sqrt{k^2+m^2} - m, \quad m > 0. \tag{3.44}$$

The density is given by

$$p(t,x,y) = \frac{1}{(2\pi)^d} \frac{t}{\sqrt{|x-y|^2+t^2}} \int\limits_{\mathbb{R}^d} e^{mt} e^{-\sqrt{|x-y|^2+t^2}\sqrt{k^2+m^2}}. \tag{3.45}$$

Further examples and more details are given in [5].

Chapter 4

Perturbations

4.1 Potential perturbations

In this section the selfadjoint operator K_0 is assumed to be perturbed by a deterministic potential $V : \mathbb{R}^d \to \mathbb{R}$. given by a multiplication operator.

$$(M_V f)(x) := V(x)f(x), \quad \forall f \in L^2(\mathbb{R}^d). \tag{4.1}$$

Assume that V is Lebesgue measurable and take $f \in L^2$ so that M_V is selfadjoint. Then we restrict the class of potentials in such a way that the form sum $K_0 \dot{+} M_V$ becomes selfadjoint. This class will be called Kato-Feller class.

4.1.1 Definition: The class $\mathbf{KF}(K_0)$

Let $W : \mathbb{R}^d \to [0, \infty)$ be a measurable function on $\mathbb{R}^d$. Let K_0 be a free Feller operator as discussed in the previous chapter. The function W is said to belong to the class $K = K(K_0)$ if

$$\lim_{t \searrow 0} \sup \ \sup_{x \in \mathbb{R}^d} \int_0^t (e^{-sK_0} V)(x) \, \mathrm{d}s = 0. \tag{4.2}$$

W is said to belong to the class $K_{loc}(K_0)$ if $\mathbf{1}_B W \in K(K_0)$ for any compact set $B \subset \mathbb{R}^d$. Set $W_+ = \max(W, 0)$, $W_- = \max(-W, 0)$.
The potential $W = W_+ - W_-$ is said to belong to the Kato-Feller class KF =

$\mathrm{KF}(K_0)$ if $W_+ \in K_{loc}(K_0)$ and $W_- \in K(K_0)$.
M_W is called a Kato-Feller (multiplication) operator if $W \in \mathrm{KF}$.

4.1.2 Definition: Kato-Feller norm

Let $V : \mathbb{R}^d \to \mathbb{R}$. We define the so-called Kato-Feller norm by

$$\|V\|_{\mathrm{KF}} := \sup_{x\in\mathbb{R}^d} \int_0^1 (e^{-sK_0}|V|)(x) \ \ \mathrm{d}s, \tag{4.3}$$

whenever the right-hand side is finite.
Without proofs we summarize some properties of Kato-Feller potentials. The proofs are given in the lecture.

4.1.3 Properties

(i) $\|V\|_{\mathrm{KF}}$ is finite for $V \in K(K_0)$.

(ii) $V \in K(K_0)$ if and only if

$$\lim_{a\to\infty} \|(K_0 + a)^{-1} M_V\|_{\infty,\infty} = 0, \tag{4.4}$$

where $-a$ is assumed to be real and in the resolvent set of K_0.

(iii) Moreover, we have

$$\|(K_0 + a)^{-1} M_V\|_{\infty,\infty} \le \frac{1}{1 - e^{-a}} \|V\|_{\mathrm{KF}}. \tag{4.5}$$

(iv) Let $(\mathbb{R}_+, (\Omega, \mathcal{B}_\Omega, P_x), X(.))$ be the Feller process associated to K_0. $E_x\{.\}$ denotes its expectation. Let $V = V_+ - V_-$ be a Kato-Feller potential. Then $V_- \in K(K_0)$ and we have

$$\sup_{x\in\mathbb{R}^d} E_x \left\{ \int_0^{t_0} V_-(X(s))\,\mathrm{d}s \right\}$$
$$= \sup_{x\in\mathbb{R}^d} \int_0^{t_0} (e^{-sK_0} V_-)(x)\,\mathrm{d}s =: \alpha.$$

If t_0 is small enough we get $\alpha < 1$. The Lemma of Kashminskii says that if $\alpha < 1$ then

$$\sup_{x\in\mathbb{R}^d} E_x\left\{e^{-\int_0^{t_0} V(X(s))\,\mathrm{d}s}\right\} \le \frac{1}{1-\alpha}. \tag{4.6}$$

That can be extended to arbitrary t. Hence, for Kato-Feller potentials we obtain

$$\sup_{x\in\mathbb{R}^d} E_x\left\{e^{-\int_0^{t} V(X(s))\,\mathrm{d}s}\right\} \le ce^{ct} \tag{4.7}$$

for any $t \ge 0$.

(v) If $V_+ \in K_{loc}(K_0)$ then $V_+ \in L^1_{loc}(\mathbb{R}^d)$.

4.1.4 Remarks and consequences

- Because of (ii) the KLMN-theorem is applicable.
- From (v) it follows that $C_c(\mathbb{R}^d) \subseteq \mathrm{dom}(M_{V_+}^{1/2})$. Assume that $C_c(\mathbb{R}^d) \subseteq \mathrm{dom}(K_0)$. Then the form

$$\langle K_0^{1/2} f, K_0^{1/2} g\rangle + \langle M_{V_+}^{1/2} f, M_{V_+}^{1/2} g\rangle$$

on $\mathrm{dom}(K_0^{1/2}) \cap \mathrm{dom}(M_{V_+}^{1/2})$ is densely defined. Moreover it is nonnegative and symmetric. Thus, there is a unique selfadjoint operator, denoted by $K_0 \dot{+} M_{V_+}$ which is associated to this form.

- If $V \in K(K_0)$ then $\mathrm{dom}(M_{V_+}^{1/2}) \subseteq \mathrm{dom}(K_0^{1/2})$.
- Let $V \in \mathrm{KF}(K_0)$. Then M_{V_-} is relatively form bounded with respect to $K_0 \dot{+} M_{V_+}$. The form bound can be estimated by $\sup\limits_{x\in\mathbb{R}^d} \int_0^\eta (e^{-sK_0}V)(x)\,\mathrm{d}s$. This becomes smaller than one for small η. Using the KLMN-Theorem we get a semibounded selfadjoint operator $K_0 \dot{+} M_V$ on $\mathrm{dom}(K_0^{1/2}) \cap \mathrm{dom}(M_{V_+}^{1/2})$.
- The selfadjoint operator $K_0 \dot{+} M_V$ generates a strongly contonuous semigroup $\{e^{-t(K_0 \dot{+} M_V)},\ t \ge 0\}$ at first on $C_\infty(\mathbb{R}^d)$, then on $L^2(\mathbb{R}^d)$. The semigroup can be extended to $L^p(\mathbb{R}^d)$, $1 \le p < \infty$.

4.1.5 Theorem

For any $t > 0$ the operator $e^{-t(K_0 \dot{+} M_V)}$ is an integral operator (due to the Dunford-Pettis Theorem). We denote its kernel by $e^{-t(K_0 \dot{+} M_V)}(x, y)$, i.e.

$$\left(e^{-t(K_0 \dot{+} M_V)} f\right)(x, y) = \int_{\mathbb{R}^d} \left(e^{-t(K_0 \dot{+} M_V)} f\right)(x, y) f(y)\, \mathrm{d}y \tag{4.8}$$

for $f \in L^2(\mathbb{R}^d)$. The functions $(t, x, y) \mapsto e^{-t(K_0 \dot{+} M_{V_+})}(x, y)$ are continuous on the set $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

4.1.6 Theorem

Let $(\mathbb{R}_+, (\Omega, \mathcal{B}, P_x), X(.))$ be the Feller process associated to K_0. Let V be a Kato-Feller potential.
Then for the perturbed semigroup holds the Feynman-Kac formula

$$\left(e^{-t(K_0 \dot{+} M_V)} f\right)(x) = E_x \left\{ e^{-\int_0^t V(X(u))\, \mathrm{d}u} f(X_t) \right\}, \tag{4.9}$$

for $f \in L^2(\mathbb{R}^d)$.

4.1.7 Corollary

For the integral kernels $\left(e^{-t(K_0 \dot{+} M_V)}\right)(x, y)$ there is the corresponding Feynman-Kac representation given by

$$\begin{aligned} \left(e^{-t(K_0 \dot{+} M_V)}\right)(x, y) &= \lim_{s \nearrow t} E_x \left\{ e^{-\int_0^s V(X(u))\, \mathrm{d}u} e^{-(t-s)K_0}(X(s), y) \right\} \\ &= E_x^{y,t} \left\{ e^{-\int_0^t V(X(u))\, \mathrm{d}u} \right\} \end{aligned} \tag{4.10}$$

with the pinned or conditional measure defined in Section 3.4.
For the perturbed semigroups and perturbed kernels there are a series of interesting estimates which can be used in several spectral theoretical criteria.

4.1.8 Proposition

Let M_V be a Kato-Feller operator. Assume that the free semigroup $\{e^{-tK_0}, t \geq 0\}$ consists of $L^1 - L^\infty$-smoothing operators, i.e.

$$\operatorname*{ess\,sup}_{x,y} e^{-tK_0}(x,y) < c_t \tag{4.11}$$

for $t > 0$.

Then we have the following estimates of the perturbed kernels:

$$e^{-t(K_0 \dot{+} M_V)}(x,y) \leq \|e^{-\frac{t}{2}(K_0 \dot{+} 4M_V)}\|_{\infty,\infty}^{\frac{1}{2}} \|e^{-\frac{t}{2}K_0}\|_{1,\infty}^{\frac{1}{2}} \left(e^{-tK_0}(x,y)\right)^{1/2} \tag{4.12}$$

and

$$e^{-t(K_0 \dot{+} M_V)}(x,y) \leq c e^{ct} \left[\sup_{x \in \mathbb{R}^d} e^{-\frac{t}{2}K_0}(x,x)\right]^{1/2} \left(e^{-tK_0}(x,y)\right)^{1/2}. \tag{4.13}$$

4.1.9 Remarks

- The denotation of the norm of an operator T from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ used here is

  $$\|T\|_{p,q} = \sup_{\|f\|_p = 1} \{\|Tf\|_q, f \in L^p(\mathbb{R}^d)\}$$

  for $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$

- In the proof of Proposition 4.1.8 one uses further essential estimates, namely

$$\begin{aligned}
\|e^{-t(K_0 \dot{+} M_V)}\|_{2,\infty}^2 &\leq \|e^{-t(K_0 \dot{+} 2M_V)}\|_{\infty,\infty} \|e^{-tK_0}\|_{1,\infty}, &(4.14)\\
\|e^{-t(K_0 \dot{+} M_V)}\|_{2,\infty} &\leq c e^{ct} &(4.15)\\
\|e^{-tK_0}\|_{1,\infty} &= \sup_{x,y} e^{-tK_0}(x,y) \\
&\leq \sup_{x} e^{-tK_0}(x,x). &(4.16)
\end{aligned}$$

  For the examples of the form (see Section 3.5.4)

$$e^{-tK_0}(x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-tp(k) + ik\cdot(x-y)}\, \mathrm{d}k \tag{4.17}$$

we have

$$e^{-tK_0}(x,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-tp(k)}\,\mathrm{d}k. \tag{4.18}$$

Thus, we get

$$e^{-t(-\Delta)}(x,x) \leq ct^{-d/2}, \tag{4.19}$$

$$e^{-t(-\Delta)^\alpha}(x,x) \leq ct^{-\frac{d}{2\alpha}} \quad \text{for } \alpha \in (0,1), \tag{4.20}$$

$$e^{-t\sqrt{-\Delta+m^2}-m}(x,x) \leq ct^{-d}. \tag{4.21}$$

- If one has a Gaussian estimate for the free semigroups, i.e. if

$$e^{-tK_0}(x,y) \leq \frac{c_1}{t^{d/2}} e^{-c_2\frac{|x-y|^2}{t}}, \tag{4.22}$$

then (4.12) implies a Gaussian estimate for the perturbed operator. Due to the general theory of one parametric semigroups, Gaussian estimates imply several features of the generators. For instance, if $\{e^{-tB_2}, t \geq 0\}$ satisifies a Gaussian estimate in $L^2(\mathbb{R}^d)$ then there are consistent semigroups in $L^r(\mathbb{R}^d)$, $1 \leq r < \infty$ and $B_2 = B_r$. Here B_r denotes the generator of the semigroup in L^r.
Or if e^{-tB_r} is holomorphic for one fixed r then e^{-tB_s} is holomorphic for any $s \in [1,\infty)$ and the spectra of B_r and B_s coincide. In particular they are equal to the spectrum of B_2, the generator in $L^2(\mathbb{R}^d)$.

However, as explained in Section 3.5 Gaussian estimates are too restrictive for our objectives. For diffusion processes it implies the boundedness of the diffusion matrix elements. For the simplest pseudo-differential operators like in Section 3.5.4 an inequality like (3.43) prevents a Gaussian estimate for $e^{-t(-\Delta)^\alpha}$. In the sequel we will never assume a Gaussian estimate for e^{-tK_0}. Nevertheless, (4.12) is important because the perturbed kernel can be estimated by the free one. The estimate in (4.12) is due to the Hölder inequality for the Feynman-Kac formulae (4.9) or (4.10). One can estimate the perturbed kernel also by

$$e^{-t(K_0\dot{+}M_V)}(x,y) \leq ce^{ct}\sup_{x\in\mathbb{R}^d}\left(e^{-\frac{t}{2}K_0}(x,y)\right)^{1/p}\left(e^{-tK_0}(x,y)\right)^{1/q} \tag{4.23}$$

with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p, q < \infty$.

4.2 Obstacle perturbations

From the physical point of view obstacles are perturbations of the free system due to impenetrable potential barriers. There are several possibilities to model such a physical situation.

Let Γ be a closed region in $\mathbb{R}^d$ with the complement Σ, i.e. $\Sigma = \mathbb{R}^d \setminus \Gamma$. On Γ we can construct a a potential barrier $\beta\mathbf{1}_\Gamma$ with the height $\beta > 0$. The Feynman-Kac formula holds for $K_0 + \beta\mathbf{1}_\Gamma$, such that one can investigate the limit of $e^{-t(K_0+\beta\mathbf{1}_\Gamma)}$ if β tends to infinity. If the limit exists and forms again a strongly continuous semigroup its generator is an appropriate model for such an obstacle perturbation (see also Section 1.2.6).

Another elegant method is based on the theory of regular Dirichlet forms. Hence, a short summary on these forms seems to be helpful.

4.2.1 Dirichlet forms and capacity

4.2.1.1 Definition: Dirichlet form

Let $(\mathfrak{a}, \mathrm{dom}(\mathfrak{a}))$ be a non-negative closed form on $L^2(\mathbb{R}^d)$ associated with the selfadjoint operator K_0. It is called a Dirichlet form if it satisfies the Markov property, i.e. if

$$0 \leq \left(e^{-tK_0} f\right)(x) \leq 1 \tag{4.24}$$

for a.e. $x \in \mathbb{R}^d$ as long as $0 \leq f(x) \leq 1$ for a.e. $x \in \mathbb{R}^d$. The domain, $\mathrm{dom}(\mathfrak{a})$, is called a Dirichlet space.

A Dirichlet space is regular, and correspondingly the Dirichlet form is regular, if $\mathrm{dom}(\mathfrak{a}) \cap C_c(\mathbb{R}^d)$ is dense in the Hilbert space $\{\mathrm{dom}(\mathfrak{a}), \langle ., . \rangle_\mathfrak{a}\}$ and also dense in $C_c(\mathbb{R}^d)$. with respect to the L^∞-norm. The scalar product in the Hilbert space above is

$$\langle f, g \rangle_\mathfrak{a} := \mathfrak{a}(f, g) + \langle f, g \rangle. \tag{4.25}$$

It is known that there is a one-to-one correspondence between that processes and regular Dirichlet forms and Markov processes (see Fukushima et. al. [7]). In our case the (free) regular Dirichlet form is

$$\left(\langle K_0^{1/2} f, K_0^{1/2} g\rangle, \mathrm{dom}(K_0^{1/2})\right)$$

and the scalar product is

$$\mathfrak{a}_1(f,g) = \langle K_0^{1/2} f, K_0^{1/2} g\rangle + \langle f, g\rangle.$$

K_0 is the generator of the process and we have

$$(e^{-tK_0} f)(x) = E_x\{f(X_t)\}. \tag{4.26}$$

An essential role for the obstacle perturbation plays the notion of the capacity. It seems to be a right place to introduce it.

4.2.1.2 Definition: Capacity

1) Let $(\mathfrak{a}, \mathrm{dom}(\mathfrak{a}))$ be a regular Dirichlet form in $L^2(\mathbb{R}^d)$. Set

   $$\mathfrak{a}_1[f] = \mathfrak{a}[f] + \|f\|^2$$

   for all $f \in \mathrm{dom}(\mathfrak{a}) = \mathrm{dom}(K_0^{1/2})$.
   Let $\mathcal{O}$ be an open set in $\mathbb{R}^d$. Its capacity is defined by

   $$\mathrm{cap}(\mathcal{O}) = \inf\{\mathfrak{a}_1[f], f \in \mathrm{dom}(\mathfrak{a}) : f(x) \geq 1 \text{ for a.e. } x \in \mathcal{O}\}. \tag{4.27}$$

   If there is no such f we write $\mathrm{cap}(\mathcal{O}) = \infty$. If Γ is an arbitrary subset of $\mathbb{R}^d$ then

   $$\mathrm{cap}(\Gamma) := \inf\{\mathrm{cap}(\mathcal{O}) : \mathcal{O} \supset \Gamma, \mathcal{O} \text{ open}\}. \tag{4.28}$$

2) A statement is said to hold quasi-everywhere (q.e.) if it holds outside of sets with capacity zero.

3) A function f is called quasi-continuous (q.c.) if for each $\epsilon > 0$ there is an open set $\mathcal{O} \subset \mathbb{R}^d$ with $\mathrm{cap}(\mathcal{O}) < \epsilon$ such that f is continuous on $\mathbb{R}^d \setminus \mathcal{O}$.

4.2.1.3 Theorem

Let $\big(\mathfrak{a}, \mathrm{dom}(\mathfrak{a})\big)$ be a regular Dirichlet form. Then for each $f \in \mathrm{dom}(\mathfrak{a})$ there is a function $\tilde{f}$ which is quasi continuous and coincides with f almost everywhere with respect to the Lebesgue measure. $\tilde{f}$ is called a quasi continuous version of f.

It turns out that the capacity can be computed, i.e. the infimum in its definition is attained.

4.2.1.4 Theorem

Let $\big(\mathfrak{a}, \mathrm{dom}(\mathfrak{a})\big)$ a regular Dirichlet form in $L^2(\mathbb{R}^d)$ and let Γ be an arbitrary set in $\mathbb{R}^d$. Then

$$\mathrm{cap}(\Gamma) = \ \inf \, \{\mathfrak{a}_1[f], \ f \in \mathrm{dom}(\mathfrak{a}), \ \tilde{f} \geq 1 \quad \text{q.e. on} \quad \Gamma\}. \tag{4.29}$$

If the set $\{ \ f \in \ \mathrm{dom}(\mathfrak{a}), \ \tilde{f} \geq 1 \quad \text{q.e. on} \quad \Gamma\}$ is not empty there is a minimizing element $v_\Gamma \in \mathrm{dom}(\mathfrak{a})$ such that

$$\mathrm{cap}(\Gamma) = \mathfrak{a}_1[v_\Gamma]. \tag{4.30}$$

4.2.1.5 Definition: Equilibrium potential

The unique element $v_\Gamma \in \mathrm{dom}(\mathfrak{a})$ which realizes the infinium in the notion of the capacity is called the equilibrium potential. (In the literature this is also called the one-equilibrium potential.) For the equilibrium potential v_Γ we have

$$\begin{aligned} 0 \leq \ v_\Gamma(x) &\leq 1 \quad \text{a.e.} \\ \text{and} \quad \tilde{v}_\Gamma(x) &= 1 \quad \text{q.e. on } \Gamma, \end{aligned}$$

where $\tilde{v}_\Gamma$ is a quasi continuous version of v_Γ. If Γ is open $\ v_\Gamma(x) = 1\ $ a.e. on Γ.
For the spectral theory in case of obstacle perturbations the equilibrium potential plays a role similar to that of the potential function for regular perturbations due to the fact that there is a stochastic representation for the equilibrium potential.

The following theorem characterizes the equilibrium potential.

4.2.1.6 Theorem

Let $\big(\mathfrak{a}, \mathrm{dom}(\mathfrak{a})\big)$ be a regular Dirichlet form with the associated process $(\mathbb{R}_+, (\Omega, \mathcal{B}_\Omega, P_x), X(.))$. Let Γ be a Borel set of finite capacity. Define by

$$\tau_\Gamma := \inf\{s, s > 0, X(s) \in \Gamma\} \tag{4.31}$$

the first hitting time of Γ.
Then the equilibrium potential can be represented by

$$v_\Gamma(x) = E_x\{e^{-\tau_\Gamma}, \tau_\Gamma < \infty\}, \tag{4.32}$$

which holds almost everywhere.

There is a close connection between equilibrium potentials and Radon measures which we will outline in the following. A positive Radon measure μ on $\mathbb{R}^d$ is said to be of finite energy integral if there is a positive constant c such that, (recall $\tilde{f}$ to be the quasicontinuous version of f),

$$\int_{\mathbb{R}^d} |\tilde{f}(x)|\ d\mu(x) \leq\ c\sqrt{\mathfrak{a}_1[f]}$$

for all $f \in \mathrm{dom}(\mathfrak{a}) \cap C_c(\mathbb{R}^d)$. The collection of such measures is denoted by S_0. For a Radon measure of finite energy integral the map $f \to \int |\tilde{f}(x)|\ d\mu(x)$ defines a continuous linear functional on the Hilbert space $\{\mathrm{dom}(\mathfrak{a}), \langle\cdot,\cdot\rangle_{\mathfrak{a}}\}$. Hence by the Theorem of Riesz there is a function $u_\mu \in \mathrm{dom}(\mathfrak{a})$ such that

$$\mathfrak{a}_1(u_\mu, f) = \int_{\mathbb{R}^d} \tilde{f}\ d\mu \tag{4.33}$$

for all $f \in \mathrm{dom}(\mathfrak{a}) \cap C_c(\mathbb{R}^d)$. u_μ is called the (one-) potential of the measure μ. For a given equilibrium potential v_Γ there is a unique Radon measure, called μ_Γ in S_0 such that $u_{\mu_\Gamma} = v_\Gamma$. μ_Γ is called the equilibrium measure of Γ. It does not charge sets of capacity zero.
For compact Γ we get

$$\mathrm{cap}(\Gamma) = \mathfrak{a}_1[v_\Gamma] = \mu_\Gamma(\Gamma). \tag{4.34}$$

For the examples of this section the equilibrium measure is

$$\mu_\Gamma(B) = \int_{\mathbb{R}^d} E_x\{e^{-\tau_\Gamma},\ X_{\tau_\Gamma} \in B,\ \tau_\Gamma < \infty\}\ dx. \tag{4.35}$$

Using Definition 4.2.1.2 we get the following corollary:

4.2.1.7 Corollary

Let Γ be a compact set in $\mathbb{R}^d$. Take the regular Dirichlet form $\big(\mathfrak{a}, \mathrm{dom}(\mathfrak{a})\big)$. Then the capacity of Γ is given by

$$\begin{aligned} \mathrm{cap}(\Gamma) &= ||K_0^{1/2} v_\Gamma||^2 + ||v_\Gamma||^2 \\ &= \mu_\Gamma(\Gamma) \\ &= \int_{\mathbb{R}^d} E_x\{e^{-\tau_\Gamma},\ \tau_\Gamma < \infty\}\ dx \\ &= \int_{\mathbb{R}^d} v_\Gamma(x)\ dx. \end{aligned} \tag{4.36}$$

4.2.2 Obstacle operators

There are several possibilities to introduce the operators which are associated to obstacle perturbations as aleady mentioned in the beginnign of Section 4.2. One can study the restriction of operators, the restriction of form domains, the restriction of processes or the restriction of the semigroups. Here we will mention only the restriction of form domains and semigroups.

4.2.2.1 Restriction of form domains

Starting with the regular form

$$\mathfrak{a}_1[f,g] = \langle K_0^{\frac{1}{2}} f, K_0^{\frac{1}{2}} g\rangle + \langle f, g\rangle,$$

where K_0 is assumed to be the selfadjoint free Feller generator. Then $\{\mathrm{dom}(K_0^{\frac{1}{2}}), \mathfrak{a}_1[.,.]\}$ forms a Hilbert space. Now we restrict the form domain by

$$(\mathrm{dom}(\mathfrak{a}))_\Sigma = \{f \in \mathrm{dom}(K_0^{\frac{1}{2}}),\ \ \tilde{f} = 0 \ \text{ q.e. on } \Gamma\}. \tag{4.37}$$

$\tilde{f}$ is the quasi-continuous version of f.
The restricted form $\{\mathfrak{a}, (\mathrm{dom}(\mathfrak{a}))_\Sigma\}$ is again a regular Dirichlet form, now defined in $L^2(\Sigma)$. That form corresponds uniquely to a selfadjoint operator in $L^2(\Sigma)$ denoted by $H_{(\mathrm{dom}(\mathfrak{a}))_\Sigma}$.

4.2.2.2 Restriction of semigroups

Let us introduce a family of operators given by

$$\left[\tilde{T}_{K_0,\Sigma}(t)f\right](x) = E_x\{f(X(t)), \tau_\Gamma > t\}, \tag{4.38}$$

where τ_Γ is the first hitting time of Γ, defined in (4.31), and $f \in L^2(\mathbb{R}^d)$. In order to find an appropriate restriction of the family we need a geometric condition for Γ.

4.2.2.2.1 Definition A point $x \in \mathbb{R}^d$ is called a τ_Γ-regular point of Γ if and only if

$$P_x\{\tau_\Gamma = 0\} = 1. \tag{4.39}$$

The set of all regular points of Γ will be denoted by Γ^r.

4.2.2.2.2 Assumption Throughout this lecture we will assume

$$\Gamma = \Gamma^r.$$

Then $\left[\tilde{T}_{K_0,\Sigma}(t)f\right](x) = 0$ if $x \in \Gamma$.
Hence we restrict this family to $L^2(\Sigma)$, i.e.

$$T_{K_0,\Sigma}(t) := \tilde{T}_{K_0,\Sigma}(t)|_{L^2(\Sigma)}. \tag{4.40}$$

In $L^2(\Sigma)$ the $T_{K_0,\Sigma}(t)$ are bounded operators for any t. They are integral operaotrs for $t > 0$ and the kernels are given by

$$\left[T_{K_0,\Sigma}(t)\right](x, y) = E_x^{y,t}\{\tau_\Gamma > t\}. \tag{4.41}$$

4.2.2.2.3 Theorem The family of operators $\{T_{K_0,\Sigma}(t), t \geq 0\}$ forms a strongly continuous semigroup in $L^2(\Sigma)$.

Proof

We write τ_Γ as $\tau_\Gamma(X)$ to denote the dependence on the paths X.

The semigroup property follows because

$$\begin{aligned}[T_{K_0,\Sigma}(t)\ T_{K_0,\Sigma}(s)f](x) = &\ E_x\left\{[T_{K_0,\Sigma}(s)f]\,(X(t))\,,\ \tau_\Gamma(X) > t\right\} \\ = &\ E_x\left\{E_{X(t)}\{f\left(\tilde{X}(s)\right),\ \tau_\Gamma(X) > t,\ \tau_\Gamma(\tilde{X}) > s\}\right\}\end{aligned}$$

where $\tilde{X}(.)$ are the trajectories starting at $X(t)$. Let ϑ_ρ be the shift operator given by

$$\vartheta_\rho(X)(\sigma) = X(\sigma+\rho) \qquad \text{for } \rho, \sigma > 0.$$

Then the last expression can be rewritten as

$$E_x\left\{E_{X(t)}\{f\left(\vartheta_t(X)(s)\right),\ \tau_\Gamma(X) > t,\ \tau_\Gamma \circ \vartheta_t(X) > s\}\right\}.$$

τ_Γ is a terminal stopping time which means that on the event $\tau_\Gamma > t$ we have

$$\tau_\Gamma + \tau_\Gamma \circ \vartheta_t = \tau_\Gamma.$$

Hence

$$\begin{aligned}[T_{K_0,\Sigma}(t)\ T_{K_0,\Sigma}(s)f](x) = &\ E_x\left\{E_{X(t)}\{f\left(X(t+s)\right),\ \tau_\Gamma > t+s\}\right\} \\ = &\ E_x\{f\left(X(t+s)\right),\ \tau_\Gamma > t+s\} \\ = &\ [T_{K_0,\Sigma}(t+s)f](x).\end{aligned}$$

The strong continuity follows finally by the dominated convergence and is based on the following pointwise consideration. Noting that the trajectories are right continuous we get for continuous functions f

$$\begin{aligned}\lim_{t\downarrow 0}|E_x\{f\left(X(t)\right),\ \tau_\Gamma > t\} - f(x)| \ \le &\ \lim_{t\downarrow 0}\ E_x\left\{|f\left(X(t)\right) - f(x)|,\ \tau_\Gamma > t\right\} \\ &\ + \lim_{t\downarrow 0}\ E_x\left\{|f(x)|,\ \tau_\Gamma \le t\right\} \\ \le &\ \lim_{t\downarrow 0}\ E_x\{|f\left(X(t)\right) - f(x)|\} \\ &\ + |f(x)|\ P_x\{\tau_\Gamma = 0\} \\ = &\ 0,\end{aligned}$$

where we used $P_x\{\tau_\Gamma = 0\} = 0$ for $x \in \Sigma$. $\square$

Hence $\{T_{K_0,\Sigma}(t), t \geq 0\}$ forms a strongly continuous semigroup in $L^2(\Sigma)$. Its generator is denoted by $(K_0)_\Sigma$.

4.2.2.2.4 Corollary .

If $\Gamma = \Gamma^r$ we have

$$\left(e^{-t(K_0)_\Sigma} f\right)(x) = E_x\{f(X(t)), \tau_\Gamma > t\}, \tag{4.42}$$

a strongly continuous semigroup in $L^2(\Sigma)$. This semigroup is contractive. It is $L^1 - L^\infty$ smoothing and it consists of integral operators with the kernels

$$e^{-t(K_0)_\Sigma}(x, y) = E_x^{y,t}\{\tau_\Gamma > t\}. \tag{4.43}$$

The kernels are symmetric such that $(K_0)_\Sigma$ is selfadjoint, $(K_0)_\Sigma \geq 0$ and $\sigma((K_0)_\Sigma) \subseteq \mathbb{R}_+$.

The associated form of $\{e^{-t(K_0)_\Sigma}, t \geq 0\}$ is given by

$$\mathfrak{a}[f, f] = \lim_{t \searrow 0} \frac{1}{t} \left| \langle f, f \rangle - \langle f, e^{-t(K_0)_\Sigma} f \rangle \right| \tag{4.44}$$

for $f \in \operatorname{dom}\left((K_0)_\Sigma^{\frac{1}{2}}\right)$. Thus $(K_0)_{(\operatorname{dom}(\mathfrak{a}))_\Sigma}$ and $(K_0)_\Sigma$ coincide.

The physical motivation of this kind of operator are perturbations by impenetrable potential barriers. Assume formally a function

$$W_\Gamma(x) = \begin{cases} \infty & x \in \Gamma \\ 0 & x \notin \Gamma. \end{cases} \tag{4.45}$$

Consider, also formally, the Feynman-Kac expression

$$E_x\{e^{-\int_0^t W_\Gamma(X(s))\,\mathrm{d}s} f(X(t))\}. \tag{4.46}$$

Then

$$e^{-\int_0^t W_\Gamma(X(s))\,\mathrm{d}s} = 1 \tag{4.47}$$

for such trajectories which are in Σ for all the time, i.e. $X(s) \in \Sigma$ for $0 \le s \le t$.
And

$$e^{-\int_0^t W_\Gamma(X(s))\,\mathrm{d}s} = 0 \tag{4.48}$$

for such trajectories where $\mathrm{meas}\{s : s \le t, X(s) \in \Gamma\} > 0$.
We define

$$T_{\Gamma,t} := \mathrm{meas}\{s : s \le t, X_s \in \Gamma\}. \tag{4.49}$$

This is the spending time of the trajectory in Γ.
That description corresponds to the killing of the process when it hits Γ. And it corresponds also to the introduction of $e^{-t(K_0)_\Sigma}$, because (4.46) is equal to

$$E_x\{f(X(t)),\;\; T_{\Gamma,t} > 0\}$$

and $T_{\Gamma,t} > 0$ iff $\tau_\Gamma > t$ for regular Γ.

4.2.2.3 Dynkin's Formula

If K_0 is not a local operator the relation between K_0 and $(K_0)_\Sigma$ has to be studied in some more detail. In doing so we define the harmonic extension operator

4.2.2.3.1 Definition Let K_0 be the free Feller generator with the associated process described above. Let $\Gamma^r = \Gamma$ and τ_Γ the first hitting time of Γ.
Then the harmonic extension operator V_Γ^a is defined by

$$\begin{aligned} \mathrm{dom}(V_\Gamma^a) &= \mathrm{dom}(K_0) \\ (V_\Gamma^a f)(x) &= E_x\{e^{-a\tau_\Gamma} f(X(\tau_\Gamma)),\;\; \tau_\Gamma < \infty\} \end{aligned} \tag{4.50}$$

with $a > 0$.

The harmonic extension operator is the correct operator to express the resolvent difference of the free and the singularly perturbed operator. This formula is known as Dynkin's formula.

4.2.2.3.2 Proposition (Dynkin's formula) Let Γ be an obstacle region with $\Gamma^r = \Gamma$. Denote by J the restriction operator of any $f \in L^2(\mathbb{R}^d)$ to $L^2(\Sigma)$, with $\Sigma = \mathbb{R}^d \setminus \Gamma$. Then

$$(a + K_0)^{-1} - J^*(a + (K_0)_\Sigma)^{-1}J = V_\Gamma(a + K_0)^{-1}, \tag{4.51}$$

which holds on $L^2(\mathbb{R}^d)$.
The proof will be explained in the lecture. J^* is the extension operator from $L^2(\Sigma)$ to $L^2(\mathbb{R}^d)$, that means any $g \in L^2(\Sigma)$ is extended to $L^2(\mathbb{R}^d)$ by setting $g(x) = 0$ on Γ.
Note that

$$JJ^*g = \mathbf{1}_{L^2(\Sigma)}g \quad , \quad g \in L^2(\Sigma),$$

and

$$J^*Jf = M_{\chi(\Sigma)}f \quad , \quad f \in L^2(\mathbb{R}^d).$$

(4.51) is the analogue of the second resolvent equation for regular potentials, i.e. for

$$(a + K_0)^{-1} - (a + K_0 + M_V)^{-1} = (a + K_0 + M_V)^{-1}M_V(a + K_0)^{-1}, \tag{4.52}$$

where a is assumed to be large enough. Here we can see that the harmonic extension operator V_Γ^a is the counterpart of $(a + K_0 + M_V)^{-1}M_V$. It contains all the perturbation which is included in the resolvent difference.

Dynkin's formula can be used to clarify the relation between JK_0J^* and $(K_0)_\Sigma$.

4.2.2.3.3 Theorem Let K_0 and Γ be given as above. Then $(K_0)_\Sigma$ is a self-adjoint extension of the operator JK_0J^*, i.e. if $f \in L^2(\Sigma) \cap \mathrm{dom}(K_0)$ then $f \in \mathrm{dom}(K_0)_\Sigma$ and

$$JK_0J^*f = (K_0)_\Sigma f. \tag{4.53}$$

The proof can be found in Demuth, Krishna [4], p.141.

Chapter 5

Resolvent and semigroup differnces for Feller operators: Operator norms

Let K_0 be a free Feller generator, M_V a Kato-Feller perturbation. Let Γ be an obstacle region with $\Gamma = \Gamma^r$. Let $(K_0)_\Sigma$ be the singularly perturbed operator, where $\Sigma = \mathbb{R}^d \setminus \Gamma$. Let J be the restriction operator from $L^2(\mathbb{R}^d)$ to $L^2(\Sigma)$.

In this chapter we will summarize results on operator norms estimates of resolvent and semigroup differences of the form

$$\begin{aligned}
R_p^{V,a} &= (a+K_0+M_V)^{-p} - (a+K_0)^{-p}, && p \in \mathbb{N}, && (5.1)\\
S_{V,t} &= e^{-t(K_0+M_V)} - e^{-tK_0}, && t > 0, && (5.2)\\
R_p^{\Sigma,a} &= (a+K_0)^{-p} - J^*(a+(K_0)_\Sigma)^{-p}J, && p \in \mathbb{N}, && (5.3)\\
S_{\Sigma,t} &= e^{-tK_0} - J^* e^{-t(K_0)_\Sigma} J, && t > 0. && (5.4)
\end{aligned}$$

The proofs are omitted here. Partly, they will be given in the lecture. The interested reader can find the proofs in Demuth, van Casteren [5], where even sandwiched operator differences are considered.

For the free operator K_0 we assumed (see 4.11) that

$$\operatorname*{ess\,sup}_{x,y \in \mathbb{R}^d} e^{-tK_0}(x,y) \le c t^{-\rho} \tag{5.5}$$

for some fixed $\rho > 0$. Then all the operators $R_p^{V,a}$, $S_{V,t}$, $R_p^{\Sigma,a}$, $S_{\Sigma,t}$ are integral operators if $p > \rho$.

We have

$$R_p^{V,a} f = \frac{1}{\Gamma(p)} \int_0^\infty \mathrm{d}\lambda \lambda^{p-1} e^{-a\lambda} S_{V,\lambda} f. \tag{5.6}$$

The kernel of $S_{V,\lambda}$ is

$$S_{V,\lambda}(x,y) = E_x^{y,\lambda} \left\{ e^{-\int_0^\lambda V(X(u))\,\mathrm{d}u} - 1 \right\}$$

All the kernels are $L^1 - L^\infty$ smoothing and $L^\infty - L^\infty$ smoothing. For $R_p^{\Sigma,a}$ and $S_{\Sigma,t}$ this is obvious. For $S_{V,t}$ and $R_p^{V,a}$ it follows from

$$\begin{aligned}
&\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} E_x^{y,\lambda} \left\{ e^{-\int_0^\lambda V(X(u))\,\mathrm{d}u} \right\} \mathrm{d}y \\
&= \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} E_x \left\{ e^{-\int_0^\lambda V(X(u))\,\mathrm{d}u} \right\} \\
&\leq \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} E_x \left\{ e^{-\int_0^\lambda V_-(X(u))\,\mathrm{d}u} \right\} \\
&\leq ce^{c\lambda}
\end{aligned} \tag{5.7}$$

by Kashminskii's Lemma (see 4.7). On the other hand they are $L^1 - L^\infty$ smoothing because

$$\begin{aligned}
e^{-t(K_0+M_V)}(x,y) &\leq ce^{ct} t^{-\frac{\varrho}{2}} (e^{-tK_0}(x,y))^{\frac{1}{2}} \\
&\leq ct^{-\rho} e^{ct}
\end{aligned} \tag{5.8}$$

(See (4.13) and (5.5).)

5.1 Regular perturbations

5.1.1 Theorem

Let K_0 be a free Feller operator. Let $e^{-tK_0}(x,y)$ satisfy BASSA. Let V and W be two Kato-Feller potentials.

Let $\gamma \in [0,1]$. Then

$$\begin{aligned}
&\|e^{-t(K_0+M_V)} - e^{-t(K_0+M_W)}\| \\
&\leq \|\int_0^t \mathrm{d}s e^{-sK_0}|V-W|^{2\gamma} e^{-(t-s)(K_0+2M_V)}\mathbf{1}\|_\infty^{1/2} \\
&\leq \|\int_0^t \mathrm{d}s e^{-sK_0}|V-W|^{2-2\gamma} e^{-(t-s)(K_0+2M_W)}\mathbf{1}\|_\infty^{1/2}. \qquad (5.9)
\end{aligned}$$

The proof will be given in the lecture.

Kashminskii's Lemma gives

$$\begin{aligned}
&\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} E_x\left\{e^{-\int_0^t V(X(u))\,\mathrm{d}u}\right\} \\
&= \|e^{-t(K_0+M_V)}\mathbf{1}\|_{L^\infty} \\
&\leq ce^{ct}.
\end{aligned}$$

The same holds for W. Take the last theorem for $\gamma = \frac{1}{2}$. Then we get the following corollary:

5.1.2 Corollary

$$\begin{aligned}
&\|e^{-t(K_0+M_V)} - e^{-t(K_0+M_W)}\| \\
&\leq ce^{ct}\|e^t \int_0^t e^{-s}e^{-sK_0}|W-V|\,\mathrm{d}s\|_\infty \\
&\leq ce^{ct}\|(K_0+1)^{-1}|W-V|\| \\
&\leq ce^{ct}\frac{1}{1-\frac{1}{e}}\|W-V\|_{\mathrm{KF}}. \qquad (5.10)
\end{aligned}$$

(see (4.5) and (4.3))

5.1.3 Theorem

Take the assumptions as in Theorem 5.1.1. Then

$$\begin{aligned}
&\|(a+K_0+M_V)^{-1} - (a+K_0+M_W)^{-1}\| \\
&\leq \|(a+K_0)^{-1}|V-W|^{2\gamma}(a+K_0+2M_W)^{-1}\mathbf{1}\|_\infty^{1/2} \\
&\leq \|(a+K_0)^{-1}|V-W|^{2-2\gamma}(a+K_0+2M_V)^{-1}\mathbf{1}\|_\infty^{1/2}. \qquad (5.11)
\end{aligned}$$

The proof follows the same line as the proof of Theorem 5.1.1, which will be explained in detail in the lecture.

Now we estimate

$$\begin{aligned}
&((a+K_0+2M_V)^{-1}\mathbf{1})(x) \\
&= \int \mathrm{d}y \ (a+K_0+2M_V)^{-1}(x,y) \\
&= \int \mathrm{d}y \int_0^\infty \mathrm{d}s \ e^{-as} E_x^{y,s}\left\{ e^{-2\int_0^s V(X(u))\,\mathrm{d}u} \right\} \\
&= \int_0^\infty \mathrm{d}s \ e^{-as} E_x \left\{ e^{-2\int_0^s V(X(u))\,\mathrm{d}u} \right\} \\
&\leq \int_0^\infty \mathrm{d}s \ e^{-as} c e^{cs}, . \qquad (5.12)
\end{aligned}$$

which is finite if $a > c$. Taking $\gamma = \frac{1}{2}$ in the last theorem we get

5.1.4 Corollary

$$\|(a+K_0+M_V)^{-1} - (a+K_0+M_W)^{-1}\| \leq \frac{c_a}{1-e^{-a}} \|W-V\|_{\mathrm{KF}}. \qquad (5.13)$$

This is an interesting continuity result for the resolvents. Notice that the Kato-Feller norm depends naturally on K_0, too.

5.1.5 Theorem

Let φ, ψ be two weight functions, i.e. non-vanishing Borel functions (the reader should have in mind $\varphi = (1+|x|)^\alpha$, $\psi = (1+|x|)^\beta$ for some $\alpha, \beta \in \mathbb{R}$).
Let $\gamma \in [0,1]$. Then

$$\begin{aligned}
&\|\varphi \left[(a+K_0+M_V)^{-1} - (a+K_0+M_W)^{-1} \right] \psi \| \\
&\leq \ \| |\psi|^{2\beta} (a+K_0)^{-1} |V-W|^{2\gamma} (a+K_0+2M_V)^{-1} |\varphi|^{2\alpha} \|_\infty^{1/2} \\
&\cdot \ \| |\varphi|^{2-2\alpha} (a+K_0)^{-1} |V-W|^{2-2\gamma} (a+K_0+2M_V)^{-1} |\psi|^{2-2\beta} \|_\infty^{1/2}. \qquad (5.14)
\end{aligned}$$

5.2 Obstacle perturbations

In Section 4.2 we explained the obstacle perturbations. In particular we already considered Dynkin's formula (see (4.51)). Here we repeat the setting.

Let Γ be a (closed) obstacle region with $\Gamma = \Gamma^r$. Set $\Sigma = \mathbb{R}^d \setminus \Gamma$. The free Feller operator K_0 shall satisfy BASSA. Let $(K_0)_\Sigma$ be the singularly perturbed operator. Take $Jf = f|_\Sigma$ for all $f \in L^2(\mathbb{R}^d)$. J is the restriction operator, J^* is the extension operator from $L^2(\Sigma)$ to $L^2(\mathbb{R}^d)$. Then we have

$$(J^*Jf)(x) = \chi_\Sigma(x) f(x),$$

$f \in L^2(\mathbb{R}^d)$. On the other hand JJ^* is an operator from $L^2(\Sigma)$ to $L^2(\Sigma)$, such that

$$(JJ^*g)(x) = g(x),$$

$g \in L^2(\Sigma)$. That means JJ^* is the identity in $L^2(\Sigma)$. The harmonic extension operator was (see (4.50)) given by

$$(V_\Gamma^a f)(x) = E_x \left\{ e^{-a\tau_\Gamma}, \tau_\Gamma < \infty \right\}.$$

τ_Γ is the first hitting time of Γ. For the resolvent difference we get Dynkin's formula

$$(a + K_0)^{-1} - J^*(a + (K_0)_\Sigma)^{-1} J = V_\Gamma^a (a + K_0)^{-1}.$$

In particular we have

$$\begin{aligned}
&\left(\left[(1 + K_0)^{-1} - J^*(1 + (K_0)_\Sigma)^{-1} J \right] \mathbf{1} \right)(x) \\
&= \left[V_\Gamma (1 + K_0)^{-1} \mathbf{1} \right](x) \\
&= E_x \left\{ e^{-\tau_\Gamma} \left[(1 + K_0)^{-1} X(\tau_\Gamma) \right] \right\} \\
&\leq E_x \left\{ e^{-\tau_\Gamma}, \tau_\Gamma < \infty \right\} \\
&= v_\Gamma(x).
\end{aligned} \tag{5.15}$$

$v_\Gamma(.)$ is the equilibrium potential defined in (4.32). For (5.15) we have used

$$\begin{aligned} \left((1+K_0)^{-1}\mathbf{1}\right)(x) &= \int_{\mathbb{R}^d} (1+K_0)^{-1}(x,y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \mathrm{d}y \int_0^\infty \mathrm{d}\lambda\ e^{-\lambda} e^{-\lambda K_0}(x,y) \\ &\le \int_0^\infty \mathrm{d}\lambda\ e^{-\lambda} = 1. \end{aligned} \tag{5.16}$$

The kernel of the semigroup difference

$$S_{\Sigma,t} = e^{-tK_0} - J^* e^{-t(K_0)_\Sigma} J$$

is given by

$$\begin{aligned} S_{\Sigma,t}(x,y) &:= e^{-tK_0}(x,y) - E_x^{y,t}\{\tau_\Gamma > t\} \\ &= E_x^{y,t}\{\mathbf{1}\} - E_x^{y,t}\{\tau_\Gamma > t\} \\ &= E_x^{y,t}\{\tau_\Gamma \le t\}. \end{aligned} \tag{5.17}$$

That implies

$$\begin{aligned} (S_{\Sigma,t}\mathbf{1})(x) &= \int_{\mathbb{R}^d} S_{\Sigma,t}(x,y)\,\mathrm{d}y \\ &= E_x\{\tau \le \Gamma > t\} \\ &= E_x\{e^{\tau_\Gamma} e^{-\tau_\Gamma}, \tau_\Gamma < t\} \\ &\le e^t E_x\{e^{-\tau_\Gamma}, \tau_\Gamma < \infty\} \\ &= e^t v_\Gamma(x). \end{aligned} \tag{5.18}$$

Estimating the operator norm by

$$\|S_{\Sigma,t}\| \le \left[\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |S_{\Sigma,t}(x,y)|\,\mathrm{d}y\right]^{\frac{1}{2}} \cdot \left[\operatorname*{ess\,sup}_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} |S_{\Sigma,t}(x,y)|\,\mathrm{d}x\right]^{\frac{1}{2}}$$

we get only

$$\|S_{\Sigma,t}\| \le e^t \text{ess}\sup_{x\in\mathbb{R}^d} v_\Gamma(x) \le e^t$$

and

$$\|R_1^{\Sigma,t}\| \le \text{ess}\sup_{x\in\mathbb{R}^d} v_\Gamma(x) \le 1,$$

which is not interesting.
Another possibility is to use

$$\|S_{\Sigma,t}\|^2 \le \int\limits_{\mathbb{R}^d} \mathrm{d}x \int\limits_{\mathbb{R}^d} \mathrm{d}y \ \ |S_{\Sigma,t}|^2. \tag{5.19}$$

But this is a Hilbert-Schmidt estimate which will be studied in some more detail in the next chapter. Nevertheless, by using (5.19) we obtain

$$\begin{aligned}
\|S_{\Sigma,t}\|^2 &\le \int\limits_{\mathbb{R}^d} \mathrm{d}x \int\limits_{\mathbb{R}^d} \mathrm{d}y \ \ \left|E_x^{y,t}\{\tau_\Gamma \le t\}\right|^2 \\
&\le \int\limits_{\mathbb{R}^d} \mathrm{d}x \int\limits_{\mathbb{R}^d} \mathrm{d}y \ \ e^{-tK_0}(x,y) E_x^{y,t}\{\tau_\Gamma \le t\} \\
&\le ct^{-\rho} \int\limits_{\mathbb{R}^d} \mathrm{d}x \ \ E_x\{e^{\tau_\Gamma} e^{-\tau_\Gamma}, \tau_\Gamma \le t\} \\
&\le ct^{-\rho} e^t \int\limits_{\mathbb{R}^d} \mathrm{d}x \ \ v_\Gamma(x) \\
&= ct^{-\rho} e^t \ \ \mathrm{cap}(\Gamma).
\end{aligned} \tag{5.20}$$

Chapter 6

Resolvent and semigroup differences for Feller operators: Hilbert-Schmidt-norms

In the next two chapters we will study Hilbert-Schmidt and trace class properties of certain operator differences. These are special cases of Schatten classes $\mathcal{S}_p$.

6.1 Definition

Let T be a compact operator in $\mathcal{B}(\mathcal{H})$. Then T^*T is a nonnegative symmetric compact operator in $\mathcal{B}(\mathcal{H})$. Let α_k^2 be the eigenvalues of T^*T and φ_k the corresponding eigenvectors. Then T^*T has the spectral representation

$$T^*T = \sum_{k=1}^{\infty} \alpha_k^2 \langle ., \varphi_k \rangle \varphi_k$$

with $\alpha_1 \geq \alpha_2 \geq \cdots > 0$. α_k are called the (repeated) singular values.
The Schatten class $\mathcal{S}_p$ is a the set of compact operators where the p-norm

$$\|T\|_p = \left(\sum_{k=1}^{\infty} \alpha_k^p \right)^{1/p}, \tag{6.1}$$

$1 \leq p < \infty$ is finite. In this context $\|T\|_2$ is the Hilbert-Schmidt norm, which will be denoted by $\|T\|_{\mathrm{HS}}$. And $\|T\|_1$ is the trace norm of T denoted by $\|T\|_{\mathrm{tr}}$.

If K is an integral operator

$$(Kf)(x) = \int\limits_{\mathbb{R}^d} K(x,y) f(y)\,\mathrm{d}y,$$

then the Hilbert-Schmidt-norm, if it exists, is

$$\|K\|_{\mathrm{HS}}^2 = \int\limits_{\mathbb{R}^d} \mathrm{d}x \int\limits_{\mathbb{R}^d} \mathrm{d}y\ \ |K(x,y)|^2. \tag{6.2}$$

As long as the kernel has the property $K(x,y) = K(x-y)$, K can not be a Hilbert-Schmidt operator. However, this is the case for many kernels of the free Feller operators. On the other hand $M_{\mathbf{1}_\Gamma} e^{-t(-\Delta)}(x,y)$ is a Hilbert-Schmidt operator if $|\Gamma| < \infty$, because

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} \mathrm{d}x \int\limits_{\mathbb{R}^d} \mathrm{d}y\ \ |\mathbf{1}_\Gamma(x) e^{-t(-\Delta)}(x,y)|^2 \\
&= \int\limits_{\Gamma} \mathrm{d}x \int\limits_{\mathbb{R}^d} \mathrm{d}y\ \ e^{-t(-\Delta)}(x,y) e^{-t(-\Delta)}(x,y) \\
&= \int\limits_{\Gamma} \mathrm{d}x\ \ e^{-2t(-\Delta)}(x,x) \\
&\leq\ ct^{-\frac{d}{2}} |\Gamma|.
\end{aligned}$$

6.2 Regular potentials

For studying semigroup differences the basic tool is Duhamel's formula. It is based on the fundamental theorem of integral and differential calculus. Let F be a function in $C^1(\mathbb{R})$, then

$$F(t) - F(s) = \int_s^t F'(u)\,\mathrm{d}u.$$

The aim is to find an appropriate $F(.)$ such that $F(t) - F(0) = e^{-tA} - e^{-tB}$. (Here $\{e^{-tA}, t \geq 0\}, \{e^{-tB}, t \geq 0\}$ are supposed to be strongly continuous semigroups.)

There are several possibilities A good choice is

$$F(u) = e^{-uA} \cdot e^{-(t-u)B}.$$

Then $F(t) = e^{-tA}$, $F(0) = e^{-tB}$,
and

$$F'(u) = -Ae^{-uA}e^{-(t-u)B} + e^{-uA}Be^{-(t-u)B}.$$

If $\mathrm{dom}(B) \subseteq \mathrm{dom}(A)$ then Duhamel's formula is

$$\left(e^{-tA} - e^{-tB}\right) f = \int_0^t e^{-uA}(B - A)e^{-(t-u)B} f \, \mathrm{d}u, \tag{6.3}$$

for all f in the space considered.
The domain problem can be avoided. Let A and B be selfadjoint operators in $L^2(\mathbb{R}^d)$ and $-a \in \mathrm{res}(A) \cap \mathrm{res}(B)$, $a > 0$. Then we consider

$$\begin{aligned}
&(A+a)^{-1}(e^{-tA} - e^{-tB})(B+a)^{-1} f \\
&= \int_0^t \mathrm{d}u \ \frac{\mathrm{d}}{\mathrm{d}u} \left[(A+a)^{-1} e^{-uA} e^{-(t-u)B} (B+a)^{-1} f\right] \\
&= \int_0^t \mathrm{d}u \ e^{-uA} \left((A+a)^{-1} - (B+a)^{-1}\right) e^{-(t-u)B} f.
\end{aligned} \tag{6.4}$$

This version of Duhamel's formula can be used without any domain problem.
Now we take Feller operators

$$A = K_0, \qquad B = K_0 + M_V$$

in $L^2(\mathbb{R}^d)$. Furthermore we define

$$K(s) = K_0 + sM_V$$

such that $K(0) = K_0$ and $K(1) = K_0 + M_V$.

6.2.1 Theorem

Introducing

$$D_s(t) := -\frac{\mathrm{d}}{\mathrm{d}s} e^{-tK(s)}$$

then

$$D_s(t) = \int_0^t e^{-uK(s)} M_V e^{-(t-u)K(s)} \,\mathrm{d}u, \tag{6.5}$$

and

$$e^{-tK_0} - e^{-t(K_0+M_V)} = \int_0^1 D_s(t) \,\mathrm{d}s. \tag{6.6}$$

Proof:
By Duhamel's formula

$$e^{-tK(s)} - e^{-tK(s_0)} = \int_0^t e^{-uK(s)}(-sM_V + s_0 M_V) e^{-(t-u)K(s_0)} \,\mathrm{d}u.$$

That implies

$$\frac{e^{-tK(s)} - e^{-tK(s_0)}}{s - s_0} = -\int_0^t e^{-uK(s)} M_V e^{-(t-u)K(s_0)} \,\mathrm{d}u.$$

With $s \longrightarrow s_0$ we obtain (6.5). Then (6.6) follows from the definition of $D_s(t)$.

$$\begin{aligned}
\int_0^1 D_s(t) \,\mathrm{d}s &= -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} e^{-tK(s)} \,\mathrm{d}s \\
&= -\left(e^{-tK(1)} - e^{-tK(0)}\right) \\
&= e^{-tK_0} - e^{-t(K_0+M_V)}
\end{aligned}$$

□

Theorem 6.1.1 is the key for studying

$$\|e^{-tK_0} - e^{-t(K_0+M_V)}\|_{\mathrm{HS}}.$$

Set

$$S_{V,t} = e^{-tK_0} - e^{-t(K_0+M_V)}.$$

All the operators are integral operators. Thus

$$S_{V,t}(x,y) = \int_0^1 D_s(t,x,y) \,\mathrm{d}s,$$

with $S_{V,t}(x,y) = e^{-tK_0}(x,y) - e^{-t(K_0+M_V)}(x,y)$.

For the Hilbert-Schmidt-norm holds

$$\|S_{V,t}\|_{\mathrm{HS}} \leq \int_0^1 \|S_{V,t}(s)\|_{\mathrm{HS}} \ \mathrm{d}s$$

if the right hand side exists.

6.2.2 Lemma

The kernel of $D_t(s)$ is representable by

$$D_t(s,x,y) = E_x^{y,t}\left\{\int_0^t V(X(u))\,\mathrm{d}u \ \ e^{-s\int_0^t V(X(\rho))\,\mathrm{d}\rho}\right\}. \tag{6.7}$$

6.2.3 Lemma

It holds

$$\int\limits_{\mathbb{R}^d} |D_t(s,x,y)|^2\,\mathrm{d}x \leq E_y\left\{\left[\int_0^t V(X(u))\,\mathrm{d}u\right]^2\right\}\cdot \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} e^{-t(K_0+2sM_V)}(x,y). \tag{6.8}$$

The last two lemmata imply the following theorem

6.2.4 Theorem

$S_{V,t} = e^{-tK_0} - e^{-t(K_0+M_V)}$ is a Hilbert-Schmidt operator if V is a Kato-Feller potential and if $V \in L^1(\mathbb{R}^d)$ or $V \in L^2(\mathbb{R}^d)$.
The proof will be given in the lecture.

The last theorem implies that $S_{V,t}$ is a Hilbert-Schmidt operator if the potential has the form $V = V_1 + V_2$, where $V_1 \in L^1$ and $V_2 \in L^2$ and if V_1 and V_2 are Kato-Feller potentials.
By Weyl's theorem that implies

$$\sigma_{\mathrm{ess}}(K_0) = \sigma_{\mathrm{ess}}(K_0 + M_V).$$

A similar result holds for resolvent differences of the form

$$(K_0 + a)^{-p} - (K_0 + M_V + a)^{-p} = R_p^{V,a}$$

(see (5.1)). $R_p^{V,a}$ is a Hilbert-Schmidt operator if $V \in L^2$ and for p large enough so that

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \int_0^\infty t^{p-1} e^{-at} e^{-tK_0}(x,x)\,\mathrm{d}t < \infty.$$

Beacause $e^{-tK_0}(x,x) \le ct^{-\rho}$, $p > \rho$ is sufficient.

Of course, the Hilbert-Schmidt condition is stronger than necessary to imply the stability of the essential spectrum. It would be sufficient if $S_{V,t}$ or $R_p^{V,a}$ are compact.

6.2.5 Theorem

Let V be a Kato-Feller potential and suppose that $(K_0 + a)^{-1}|V| \in C_\infty(\mathbb{R}^d)$. Then

$$S_{V,t} = e^{-t(K_0 + M_V)} - e^{-tK_0}$$

is compact.

Scetch of a proof: We cut the potential by

$$V_{klm}(x) = V(x) \cdot \mathbf{1}_{B_m}(x) \cdot \mathbf{1}_{\{-k \le V(x) \le l\}}(x),$$

where B_m is a ball of radius m. Then

$$e^{-tK_0 + M_{V_{klm}}} - e^{-tK_0}$$

is a Hilbert-Schmidt operator. Hence it suffices to show that

$$\lim_{k,l,m \to \infty} \| e^{-tK_0 + M_{V_{klm}}} - e^{-tK_0} \|_{2,2} = 0. \tag{6.9}$$

Duhamel's formula, the Markov property and the Cauchy-Schwarz inequality give

$$\|e^{-tK_0+M_{V_{klm}}} - e^{-tK_0}\|_{2,2} \le ce^{ct} \ \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} E_x\left\{\int_0^t |V(X(u)) - V_{klm}(X(u))| \ \mathrm{d}u\right\}.$$

Then the Lemma of Dini can be used to obtain (6.9). □

6.2.6 Corollary

Let V be a Kato-Feller potential. Assume $V \in L^1 + L^2$ or $(K_0+a)^{-1}|V| \in C_\infty$, then $S_{V,t}$ is compact.

For $(K_0+a)^{-1}|V| \in C_\infty(\mathbb{R}^d)$ it is sufficient that $V \in C_\infty(\mathbb{R}^d)$.

6.3 Obstacle perturbations

As mentioned in the beginning of Chapter 5 we set again

$$S_{\Sigma,t} = e^{-tK_0} - J^* e^{-t(K_0)_\Sigma} J$$

and we have

$$S_{\Sigma,t}(x,y) = E_x^{y,t}\{\tau_\Gamma \le t\}.$$

A very rough estimate of the Hilbert-Schmidt norm is

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \ |S_{\Sigma,t}(x,y)|^2 \\
\le\ &\int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \ e^{-tK_0}(x,y) E_x^{y,t}\{\tau_\Gamma \le t\} \\
\le\ &ct^{-\rho} \int_{\mathbb{R}^d} \mathrm{d}x \ E_x\{\tau_\Gamma \le t\} \\
\le\ &ct^{-\rho} e^t \int_{\mathbb{R}^d} \mathrm{d}x \ E_x\{e^{-\tau_\Gamma}, \tau_\Gamma \le t\} \\
\le\ &ct^{-\rho} e^t \int_{\mathbb{R}^d} v_\Gamma(x)\,\mathrm{d}x \\
=\ &ct^{-\rho} e^t \mathrm{cap}(\Gamma).
\end{aligned}$$

A somewhat better estimate gives the following theorem

6.3.1 Theorem

$S_{\Sigma,t}$ is a Hilbert-Schmidt operator if the equilibrium potential v_Γ is in $L^2(\mathbb{R}^d)$.

The proof will be given in the lecture.
A similar result holds for the resolvent difference.

6.3.2 Theorem

The resolvent difference

$$R_p^{\Sigma,a} = (K_0 + a)^{-p} - J^*((K_0)_\Sigma + a)^{-p} J$$

is a Hilbert-Schmidt operator if $a > 2$, $p > \rho$ and $v_\Gamma \in L^2$.

Comparing with Theorem 6.1.3 we see that the equilibrium potential v_Γ plays essentially the same role as the regular potential in potential perturbations.

Chapter 7

Resolvent and semigroup differences for Feller operators: Trace class norms

7.1 Trace class conditions

Let A be a selfadjoint trace class operator in $L^2(\mathbb{R}^d)$. The set of trace class operators id denoted by $\mathcal{B}_{\text{tr}}$. If A is an integral operator in $L^2(\mathbb{R}^d)$, i.e.

$$(Af)(x) = \int_{\mathbb{R}^d} A(x,y)f(y)\,\mathrm{d}y,$$

and if $A(.,.)$ is continuous then

$$\operatorname{trace}(A) = \int_{\mathbb{R}^d} A(x,x)\,\mathrm{d}x. \tag{7.1}$$

Any trace class operator can be written as a product of two Hilbert-Schmidt operators. On the other hand the standard trick to prove a product AB to be trace class is

$$\|AB\|_{\text{tr}} \leq \|A\|_{\text{HS}} \cdot \|B\|_{\text{HS}} \tag{7.2}$$

where A, B are supposed to be Hilbert-Schmidt.

If $K = AB$ and if A and B are integral operators in $L^2(\mathbb{R}^d)$ and Hilbert-Schmidt then K is also an integral operator and

$$K(x,y) = \int_{\mathbb{R}^d} A(x,y)B(u,y)\,\mathrm{d}u.$$

If $K(.,.)$ is continuous we have again

$$\mathrm{trace}(K) = \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \;\; A(x,y)B(y,x).$$

Now let A, B be selfadjoint and semibounded. Then the standard way to show that

$$e^{-2tA} - e^{-2tB} \in \mathcal{B}_{\mathrm{tr}}$$

is the decomposition of this difference in the form

$$e^{-2tA} - e^{-2tB} = e^{-tA}\left(e^{-tA} - e^{-tB}\right) + \left(e^{-tA} - e^{-tB}\right)e^{-tB}. \tag{7.3}$$

But, of course, in many cases e^{-tA} is not Hilbert-Schmidt. Consider for instance

$$e^{-t(-\Delta)}(x,y) = p_w(t,x,y) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

One possible way is to introduce in the product a further operator

$$e^{-tA}\left(e^{-tA} - e^{-tB}\right) = e^{-tA} M_{\langle x\rangle^{-s}} M_{\langle x\rangle^{s}} \left(e^{-tA} - e^{-tB}\right) \tag{7.4}$$

with $\langle x\rangle = \sqrt{1+|x|^2}$. Then one has to choose s large enough, such that

$$\int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \;\; \left|\left(e^{-tA}M_{\langle x\rangle^{-s}}\right)(x,y)\right|^2 < \infty. \tag{7.5}$$

For instance if $A = -\Delta$, $s > \frac{d}{2}$ is sufficient. However the final condition for the perturbation becomes very strong, because the rest has to satisfy

$$\int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \;\; \langle x\rangle^{2s} \left|e^{-tA}(x,y) - e^{-tB}(x,y)\right|^2 < \infty. \tag{7.6}$$

That means the kernel of the semigroup difference has to decrease fast at infinity.

A better condition will be given in the next theorem.

7.1.1 Theorem (Demuth, Stollmann, Stolz, van Casteren)

(See also Weidmann [15], Vol. I, p. 158.)
Let A and B be integral operators in $L^2(\mathbb{R}^d)$ with measurable kernels. Let

$$A(.,x) \in L^2(\mathbb{R}^d),$$

i.e. $\int_{\mathbb{R}^d} |A(y,x)|^2 \,\mathrm{d}y < \infty$, and

$$B(x,.) \in L^2(\mathbb{R}^d).$$

Assume that

$$\int_{\mathbb{R}^d} \|A(.,x)\|_{L^2} \ \|B(x,.)\|_{L^2} \,\mathrm{d}x < \infty.$$

Then AB is a trace class operator and

$$\|AB\|_{\mathrm{tr}} \leq \int_{\mathbb{R}^d} \|A(.,x)\|_{L^2} \ \|B(x,.)\|_{L^2} \,\mathrm{d}x. \tag{7.7}$$

7.1.2 Remark

- The estimate (7.7) is better than the usual Hilbert-Schmidt estimate (see 7.4) if A and B are Hilbert-Schmidt operators. This follows from

$$\begin{aligned}
&\int_{\mathbb{R}^d} \|A(.,x)\|_{L^2} \ \|B(x,.)\|_{L^2} \,\mathrm{d}x \\
\leq\ & \left(\int_{\mathbb{R}^d} \|A(.,x)\|_{L^2}^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \|B(x,.)\|_{L^2}^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\
=\ & \left(\int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \,|A(y,x)|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \,|B(x,y)|^2 \right)^{\frac{1}{2}} \\
=\ & \|A\|_{\mathrm{HS}} \|B\|_{\mathrm{HS}}.
\end{aligned} \tag{7.8}$$

- For special examples we have the equality in (7.7).
  Take $A(x,y) = a_1(x)a_2(y)$ and $B(x,y) = b_1(x)b_2(y)$, where a_i, b_i are positive

functions. Assume $a_1, b_2 \in L^2(R^d)$. Then

$$\begin{aligned}\|AB\|_{\mathrm{tr}} &= \int_{\mathbb{R}^d} \|A(.,x)\|_{L^2}\|B(x,.)\|_{L^2}\,\mathrm{d}x \\ &= \|a_1\|_{L^2}\|b_2\|_{L^2}\int_{\mathbb{R}^d} a_2(x)b_1(x)\,\mathrm{d}x. \qquad (7.9)\end{aligned}$$

- In (7.9) one sees that neither A nor B has to be Hilbert-Schmidt. For instance $a_2 \in L^\infty$ and $b_1 \in L^1$ is allowed.
  (7.9) can be estimated like in (7.8):

$$\begin{aligned}\|AB\|_{\mathrm{tr}} &= \|a_1\|_{L^2}\|b_2\|_{L^2}\langle a_2, b_1\rangle \\ &\leq \|a_1\|_{L^2}\|b_2\|_{L^2}\|a_2\|_{L^2}\|b_1\|_{L^2} \\ &= \|A\|_{\mathrm{HS}}\|B\|_{\mathrm{HS}}.\end{aligned}$$

Proof of Theorem 7.1.1

The proof uses an optimized Hilbert-Schmidt estimate. We have

$$\begin{aligned}\|AB\|^2_{\mathrm{tr}} &= \|AM_\varphi M_{1/\varphi}B\|^2_{\mathrm{tr}} \\ &\leq \|AM_\varphi\|^2_{\mathrm{HS}}\|M_{1/\varphi}B\|^2_{\mathrm{HS}} \\ &= \left(\int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}^d}\mathrm{d}u\ |A(x,u)|^2|\varphi(u)|^2\right)\left(\int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}^d}\mathrm{d}v\ |B(x,v)|^2\left|\frac{1}{\varphi(x)}\right|^2\right) \\ &= \left(\int_{\mathbb{R}^d}\mathrm{d}u\ |\varphi(u)|^2\|A(.,u)\|^2_{L^2}\right)\left(\int_{\mathbb{R}^d}\mathrm{d}x\ \frac{1}{|\varphi(u)|^2}\|B(x,.)\|^2_{L^2}\right).\end{aligned}$$

Now we choose φ in such a way that

$$|\varphi(u)|^2\|A(.,u)\|^2_{L^2} = \frac{1}{|\varphi(u)|^2}\|B(u,.)\|^2_{L^2},$$

i.e. we choose

$$|\varphi(u)|^2 = \frac{\|B(u,.)\|^2_{L^2}}{\|A(.,u)\|^2_{L^2}}.$$

Then

$$\|AB\|_{\mathrm{tr}}^2 \le \left(\int_{\mathbb{R}^d} \|A(.,x)\|_{L^2}^2 \|B(x,.)\|_{L^2}^2 \right)^2 . \qquad \square$$

7.1.3 Remark

Let us apply Theorem 7.1.1 to semigroup differences and compare the result with (7.5) and (7.6). Let

$$A = e^{-tK_0} \text{ and } e^{-tK_0}(x,y) = e^{-tK_0}(x-y).$$

Then

$$\begin{aligned} \|e^{-tK_0}(.,x)\|_{L^2}^2 &= \int_{\mathbb{R}^d} \mathrm{d}y \ |e^{-tK_0}(y,x)|^2 \\ &= e^{-2tK_0}(x,x) = e^{-2tK_0}(0,0) \\ &\le ct^{-\rho}. \end{aligned} \tag{7.10}$$

In order that the semigroup difference is trace class the final condition for the perturbation is here

$$\int_{\mathbb{R}^d} \mathrm{d}x \sqrt{\int_{\mathbb{R}^d} \mathrm{d}y \ |e^{-tK_0}(x,y) - e^{-tB}(x,y)|^2} < \infty, \tag{7.11}$$

which is much better than (7.6).

7.2 Regular perturbations

Theorem 7.1.1 is now applied to the standard situation of Feller operators.

7.2.1 Theorem

Let K_0 be a free Feller operator and V a Kato-Feller potential. Assume that

$$\int_{\mathbb{R}^d} \mathrm{d}x \ \sqrt{\int_{\mathbb{R}^d} \mathrm{d}y \ \left(e^{-tK_0}(x,y)\right)^2 |V(y)|^2} < \infty, \tag{7.12}$$

and

$$\int_{\mathbb{R}^d} \mathrm{d}x \, \sqrt{\int_{\mathbb{R}^d} \mathrm{d}y \, \left(e^{-t(K_0+M_V)}(x,y)\right)^2 |V(y)|^2} < \infty, \tag{7.13}$$

then $S_{V,2t} = e^{-2t(K_0+M_V)} - e^{-2tK_0}$ is a trace class operator.

Proof:
Decompose $M_V e^{-2tK_0} = M_V e^{-tK_0} e^{-tK_0}$. Then (7.12) implies $V e^{-2tK_0}$ is trace class. Using Theorem 7.1.1 (7.13) gives that $V e^{-t(K_0+M_V)}$ is trace class. We decompose $S_{V,2t}$ by

$$S_{V,2t} = e^{-t(K_0+M_V)} S_{V,t} + S_{V,t} e^{-t(K_0)}.$$

Here we consider only $e^{-t(K_0+M_V)} S_{V,t}$.

$$\begin{aligned}
& \|e^{-t(K_0+M_V)} S_{V,t}\|_{\mathrm{tr}} \\
\leq \; & \|\int_0^t e^{-t(K_0+M_V)} e^{-(t-\rho)(K_0+M_V)} V e^{-\rho K_0} \, \mathrm{d}\rho\|_{\mathrm{tr}} \\
\leq \; & \int_0^t \|V e^{-t(K_0+M_V)}\|_{\mathrm{tr}} \|e^{-\rho K_0}\|_{2,2} \|e^{(t-\rho)(K_0+M_V)}\|_{2,2} \, \mathrm{d}\rho \\
\leq \; & c \int_0^t e^{c(t-\rho)} \; \|V e^{-t(K_0+M_V)}\|_{\mathrm{tr}} \, \mathrm{d}\rho.
\end{aligned}$$

The same argument can be used for $S_{V,t} e^{-tK_0}$. □

The condition

$$\int_{\mathbb{R}^d} \mathrm{d}x \, \sqrt{\int_{\mathbb{R}^d} \mathrm{d}y \, \left(e^{-t(K_0+M_V)}(x,y)\right)^2 |V(y)|^2} < \infty,$$

is not very elegant. For the stability of the absolutely continuous spectrum one expects an L^1-condition for the potential V. For that one has to generalize the usual trace class condition, which says that $\sigma_{ac}(K_0) = \sigma_{ac}(K_0 + M_V)$ if $S_{V,t}$ is trace class. This possibility is considered in Section 7.4.

7.3 Obstacle perturbations

In the same way as in Section 7.2 we can handle

$$\begin{aligned} S_{\Sigma,2t} &= e^{-2tK_0} - J^* e^{-2t(K_0)_\Sigma} J \\ &= e^{-tK_0} S_{\Sigma,t} - S_{\Sigma,t} J^* e^{-t(K_0)_\Sigma} J. \end{aligned}$$

Here

$$\begin{aligned} &\int_{\mathbb{R}^d} \left| \left(J^* e^{-t(K_0)_\Sigma} J \right)(x,y) \right|^2 \mathrm{d}y \\ &= \int_{\mathbb{R}^d} \left| E_x^{y,t}\{\tau_\Gamma > t\} \right|^2 \mathrm{d}y \\ &\leq \int_{\mathbb{R}^d} \left| e^{-tK_0}(x,y) \right|^2 \mathrm{d}y \\ &= e^{-2tK_0}(x,x) \leq ct^{-\rho}. \end{aligned}$$

Thus it remains to study

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathrm{d}x \ \| S_{\Sigma,t}(.,x) \|_{L^2} \\ &= \int_{\mathbb{R}^d} \mathrm{d}x \sqrt{\int_{\mathbb{R}^d} \mathrm{d}y \ |S_{\Sigma,t}(y,x)|^2} \\ &= \int_{\mathbb{R}^d} \mathrm{d}x \sqrt{\int_{\mathbb{R}^d} \mathrm{d}y \ \left| E_x^{y,t}\{\tau_\Gamma \leq t\} \right|^2} \\ &\leq \operatorname*{ess\,sup}_{x,y} \sqrt{e^{-tK_0}(x,y)} \int_{\mathbb{R}^d} \sqrt{E_x\{\tau_\Gamma \leq t\}}\, \mathrm{d}x \\ &\leq ct^{-\rho/2} \int_{\mathbb{R}^d} e^{t/2} \sqrt{E_x\{e^{-\tau_\Gamma}, \tau\Gamma < \infty\}}\, \mathrm{d}x \\ &= ct^{-\rho/2} e^{t/2} \int_{\mathbb{R}^d} \sqrt{v_\Gamma(x)}\, \mathrm{d}x. \end{aligned}$$

Here one sees again that the equilibrium potential $v_\Gamma(x)$ plays the role of the potential function V in Section 7.2.
Thus we proved

7.3.1 Theorem

$S_{\Sigma,2t}$ is a trace class operator if the equilibrium potential satisfies

$$\int_{\mathbb{R}^d} \sqrt{v_\Gamma(x)}\,\mathrm{d}x < \infty.$$

In a similar way one obtains

7.3.2 Proposition

Let $S_{\Sigma,2t}$ be a trace class operator. Then its trace can be estimated by

$$\begin{aligned} \operatorname{trace}(S_{\Sigma,2t}) &\leq c(t) \int_{\mathbb{R}^d} v_\Gamma(x)\,\mathrm{d}x \\ &\leq c(t)\ \operatorname{cap}(\Gamma). \end{aligned} \tag{7.14}$$

7.4 Generalized stability condition for the stability of the absolutely continuous spectrum

Let A, B be two selfadjoint operators in a Hilbert space $\mathcal{H}$. Assume A and B to be bounded from below. For the stability of the absolutely continuous spectrum, i.e. for

$$\sigma_{ac}(A) = \sigma_{ac}(B),$$

it is sufficient if the wave operators

$$\Omega_\pm(B,A) = \text{s-}\lim_{t\to\pm\infty} e^{itB}e^{-itA}P_{ac}(A)$$

exist and are complete. Complete means

$$\operatorname{ran}(\Omega_\pm(B,A)) = P_{ac}(B)\mathcal{H}.$$

The wave operators exist and are complete if $B - A$ is a trace class operator. However, in potential scattering $B - A$ is not trace class. Fortunately, in scattering theory we have the invariance principle which enables us to restrict the investigation to bounded functions of A and B.

7.4.1 Definition

Let I_n, $n \in \mathbb{N}$ be mutually disjoint open intervals in $\mathbb{R}$ such that $\bigcup_n \overline{I_n} = \mathbb{R}$.
A function $\alpha : \mathbb{R} \to \mathbb{R}$ is called admissible if it has the following properties on every I_n:

1) α is continuous differentiable

2) $\alpha' > 0$ or $\alpha' < 0$ on any I_n

3) α' is locally of bounded variation.

For admissible functions the wave operators coincide if $\alpha(A) - \alpha(B) \in \mathcal{B}_{\text{tr}}$.
If $\alpha' > 0$ on $\mathbb{R}$ we have

$$\Omega_{\pm}(\alpha(A), \alpha(B)) = \Omega_{\pm}(B, A). \tag{7.15}$$

For the stability of σ_{ac} it is sufficient if

$$e^{-2B} - e^{-2A} \in \mathcal{B}_{\text{tr}}$$

or if

$$\begin{aligned} & e^{-B}\left(e^{-B} - e^{-A}\right) \in \mathcal{B}_{\text{tr}} \\ \text{and} \qquad & \left(e^{-B} - e^{-A}\right) e^{-A} \in \mathcal{B}_{\text{tr}}. \end{aligned} \tag{7.16}$$

That we have used in Sections 7.2 and 7.3. The sufficient conditions there were

$$\begin{aligned} \int_{\mathbb{R}^d} \sqrt{V(x)}\, \mathrm{d}x < \infty, \\ \int_{\mathbb{R}^d} \sqrt{v_\Gamma(x)}\, \mathrm{d}x < \infty. \end{aligned}$$

On the other hand in Chapter 6 the sufficient Hilbert-Schmidt conditions were

$$V \in L^2 \quad \text{or} \quad v_\Gamma \in L^2.$$

In this case σ_{ess} was stable.
Hence it is natural to find a condition on the semigroups such that $V \in L^1$ or

$v_\Gamma \in L^1$ is sufficient.
For that we need a generalized criterion for the existence and completeness of wave operators.

7.4.2 Theorem

Assume two selfadjoint, bounded below operators, A and B, in a Hilbert space $\mathcal{H}$. Suppose

$$e^{-B}\left(e^{-B} - e^{-A}\right)e^{-A} \in \mathcal{B}_{\text{tr}} \tag{7.17}$$

and

$$e^{-B} - e^{-A} \in \mathcal{B}_{\text{compact}}. \tag{7.18}$$

Then $\sigma_{ac}(A) = \sigma_{ac}(B)$.
The proof will be given in the lecture.

As we will see the condition in (7.17) is more general than that in (7.16) because the semigroup difference is sandwiched from both sides.
Now we will apply Theorem 7.4.2 for Feller operators, i.e. we set $A = K_0$ and $B = K_0 + M_V$, $\mathcal{H} \in L^2(\mathbb{R}^d)$.

7.4.3 Theorem

Let K_0 e the free Feller operator and V a Kato-Feller potential Then

$$\sigma_{ac}(K_0 + M_V) = \sigma_{ac}(K_0)$$

if V is additionally in $L^1(\mathbb{R}^d)$.

Proof idea
The difference to the proof of Theorem 7.2.1 is the new decomposition of the operator

$$e^{-2(K_0+M_V)}\left(e^{-2(K_0+M_V)} - e^{-2K_0}\right)e^{-2K_0}$$

by

$$A = e^{-t(K_0+M_V)}$$

and

$$B = e^{-t(K_0+M_V)} \left(e^{-2(K_0+M_V)} - e^{-2K_0}\right) e^{-2K_0}.$$

Then

$$\|A(.,x)\|^2 = e^{-2(K_0+M_V)}(x,x) < \text{const.}$$

And

$$\begin{aligned} & \int_{\mathbb{R}^d} \mathrm{d}x \ \|B(x,.)\| \\ \leq \ & c \int_{\mathbb{R}^d} \mathrm{d}u \int_{\mathbb{R}^d} \mathrm{d}v \ \left| e^{-2(K_0+M_V)}(u,v) - e^{-2(K_0)}(u,v) \right| \\ \leq \ & c \int_{\mathbb{R}^d} |V(u)| \ \mathrm{d}u. \qquad \Box \end{aligned}$$

Analogous considerations are possible for the obstacle perturbations. We know already that $S_{\Sigma,t}$ is a Hilbert-Schmidt operator if

$$\int_{\mathbb{R}^d} v_\Gamma(x)^2 \, \mathrm{d}x < \infty.$$

Using Theorem 7.4.2 again we can prove that

$$e^{-K_0} S_{\Sigma,1} e^{-(K_0)_\Sigma} \in \mathcal{B}_{\text{tr}}$$

if $v_\Gamma \in L^1$. Thus follows

7.4.4 Corollary

The absolutely continuous spectra of K_0 and $(K_0)_\Sigma$ coincide if $v_\Gamma \in L^1(\mathbb{R}^d)$ or, what is the same, if the capacity of Γ is finite, i.e.

$$\text{cap}(\Gamma) < \infty. \tag{7.19}$$

7.4.5 Remark

The condition in (7.19) that obstacles of finite capacity are sufficient for the stability of the absolutely continuous spectra does not mean that Γ has to be bounded. There are unbounded Γ with finite capacity.

For instance, take $K_0 = -\Delta$ and $\Gamma = \bigcup_n B_n$, a union of balls with radius r_n and center in $(n, 0, \ldots, 0) \in \mathbb{R}^d$, $d \geq 3$. Then

$$\operatorname{cap}(\Gamma) \leq \sum_n \operatorname{cap}(B_n)$$

and

$$v_{B_n}(x) \leq \frac{r_n^{d-2}}{|x - (n, 0, \ldots, 0)|^{d-2}}, \quad x \in B_n$$

Hence,

$$\operatorname{cap}(B_n) \leq c\ r_n^{d-2}$$

for $r_n < 1$, such that

$$\operatorname{cap}(\Gamma) \leq \sum_n r_n^{d-2}.$$

We can choose $r_n = \frac{1}{2}\left(\frac{1}{n}\right)^{2/d}$. Then $\operatorname{cap}(\Gamma) < \infty$ for $d > 4$ but $\sum\limits_{n=1}^{\infty} r_n = \infty$.

7.4.6 Perturbations by negative measures

The Feynman-Kac formula is

$$\left(e^{-t(K_0+M_V)}f\right)(x) = E_x\left\{e^{-\int_0^t V(X(u))\,\mathrm{d}u} f(X(t))\right\}.$$

If, for instance, $V(x) = \mathbf{1}_\Gamma(x)$, the functional $e^{-\int_0^t \mathbf{1}_\Gamma(X(u))\,\mathrm{d}u}$ makes sense as long as $|\Gamma| > 0$. If $|\Gamma| = 0$ we obtain $\int_0^t \mathbf{1}_\Gamma(X(u))\,\mathrm{d}u = 0$. On sets of measure zero we could change the values of $\mathbf{1}_\Gamma(x)$ arbitrarily. For instance it could be equal to $-\infty$.

On the other hand perturbations on sets of measure zero can play an essential role in several models. For instance how can we realize operators of the form

$$H_0 \text{ "} + \text{" } \delta(x) \tag{7.20}$$

or even

$$H_0 \text{ "} - \text{" } \delta(x), \tag{7.21}$$

where $\delta(x)$ are assumed to be δ-like perturbations? Of course, the operators in

(7.20) and (7.21) have to be defined correctly.
Here we develop one method to model negative, δ-like perturbations. We approach the δ-function by

$$\frac{1}{\epsilon}\mathbf{1}_{[\partial\Gamma-\epsilon,\partial\Gamma+\epsilon]}(x),$$

where $\partial\Gamma$ is a hyperplane in $\mathbb{R}^d$, e.g. for $d=3$ a plane and for $d=2$ a line. As long as $\epsilon > 0$ this is a bounded function, so the Feynman-Kac formula holds. Hence one has to find a condition such that the limit $\epsilon \to 0$ becomes possible. For that purpose the following general theorem is useful.

7.4.7 Theorem

Let $\{K_n\}$ be a sequence of trace class operators in a separable Hilbert space $\mathcal{H}$. Assume that the weak limit of K_n exists, i.e.

$$\text{w-}\lim_{n\to\infty} K_n = K.$$

Moreover assume a uniform bound of the trace norms,

$$\|K_n\|_{\text{tr}} < M,$$

where M is independent of n.
Then K is also a trace class operator.

Proof: In the Hilbert space of Hilbert-Schmidt operators the inner product is defined by

$$\langle A^*, B\rangle = \sum_{k=1}^{\infty}\langle A^*\varphi_k, B\varphi_k\rangle = \text{trace}(AB),$$

where $\{\varphi_k\}$ is any orthogonal basis in $\mathcal{H}$.
Due to the theorem of Alaoglu there is a w^*-convergent subsequence $\{K_{n_j}\}$ and

$$\lim_{j\to\infty} \text{trace}(K_{n_j}C) = \text{trace}(K'C)$$

for all $C \in \mathcal{B}_{\text{compact}}$. Moreover, K' is then also trace class.

For a special $C_0 = \langle f, .\rangle g$ with fixed $f, g \in \mathcal{H}$ and $\|f\| = \|g\| = 1$ we obtain

$$\begin{aligned} \lim_{j\to\infty} \operatorname{trace}(K_{n_j} C_0) = & \langle f, K'g\rangle \\ = \quad \lim_{j\to\infty} \langle f, K_{n_j} g\rangle = & \langle f, Kg\rangle. \end{aligned}$$

Hence $K' = K$, such that K is trace class. □

7.4.8 Corollary

Let A be a selfadjoint semibounded operator in $\mathcal{H}$. Let $\{B_n\}$ be a sequence of selfadjoint semibounded operators and B the strong resolvent limit of $\{B_n\}$, i.e.

$$\text{s-}\lim_{n\to\infty} (B_n + a)^{-1} = (B + a)^{-1},$$

$a \in \bigcap_n \operatorname{res}(B_n)$.
Assume now

(i) $e^{-B_n} \left(e^{-B_n} - e^{-A}\right) e^{-A} \in \mathcal{B}_{\text{tr}}, \quad \forall n \in \mathbb{N}$,

(ii) $\|e^{-B_n} \left(e^{-B_n} - e^{-A}\right) e^{-A}\|_{\text{tr}} < M$,

(iii) $e^{-B_n} - e^{-A} \in \mathcal{B}_{\text{compact}}, \quad \forall n \in \mathbb{N}$,

(iv) $\|e^{-B} - e^{-A}\| \le K$ uniformly in n.

Then

$$e^{-B} \left(e^{-B} - e^{-A}\right) e^{-A} \in \mathcal{B}_{\text{tr}}$$

and

$$e^{-B} - e^{-A} \in \mathcal{B}_{\text{compact}}.$$

Corollary 7.5.2 implies that $\sigma_{ac}(A) = \sigma_{ac}(B)$, although one does not have an expression for B.

7.4.9 Example

Take $K_0 = (-\Delta)^\alpha$, $0 < \alpha \le 1$ and $d = 1$, i.e. K_0 acts in $L^2(\mathbb{R})$. Then assume that the limit

$$\text{s-}\lim_{\epsilon\to 0}\left(K_0 - \frac{1}{\epsilon}\mathbf{1}_{[x_0-\epsilon,x_0+\epsilon]} + a\right)^{-1} =: (K_\delta + a)^{-1}$$

exists. Then K_δ is a good model for the formal operator $K_0 - \delta(x_0)$, $x_0 \in \mathbb{R}$.

Using Corollary 7.4.8 we can prove that $\sigma_{ac}(K_0) = \sigma_{ac}(K_\delta)$. According to Corollary 7.5.2 one has to verify

$$\|e^{-K_\epsilon}\left(e^{-K_\epsilon} - e^{-K_0}\right)e^{-K_0}\|_{\mathrm{tr}} \le M_1$$

for $K_\epsilon = K_0 - \frac{1}{\epsilon}\mathbf{1}_{[x_0-\epsilon,x_0+\epsilon]}$. The operator is trace class for $V \in L^1$, i.e. here for

$$\frac{1}{\epsilon}\int_{\mathbb{R}} \chi_{[x_0-\epsilon,x_0+\epsilon]}(x)\,\mathrm{d}x < \infty.$$

M_1 is independent of ϵ if

$$\sup_x E_x\left\{\exp\left(\frac{1}{\epsilon}\int_0^1 \chi_{[x_0-\epsilon,x_0+\epsilon]}(X(u))\,\mathrm{d}u\right)\right\} < M_2$$

and if M_2 is independent of ϵ. Using the Lemma of Kashminskii M_2 is independent of ϵ if

$$\sup_x E_x\left\{\frac{1}{\epsilon}\int_0^{t_0} \chi_{[x_0-\epsilon,x_0+\epsilon]}(X(u))\,\mathrm{d}u\right\} \le \beta < 1$$

for small t_0 and β independent of ϵ. That follows from

$$\begin{aligned}
&\sup_x E_x\left\{\frac{1}{\epsilon}\int_0^{t_0} \chi_{[x_0-\epsilon,x_0+\epsilon]}(X(u))\,\mathrm{d}u\right\}\\
&= \frac{1}{\epsilon}\sup_x\int_0^{t_0}\mathrm{d}s\int_{[x_0-\epsilon,x_0+\epsilon]}\mathrm{d}y\;e^{-sK_0}(x,y)\\
&\le c\int_0^{t_0} s^{-\rho}\,\mathrm{d}s = \beta.
\end{aligned}$$

Here $\rho = \frac{1}{2\alpha}$. Then β becomes smaller than one for small t_0 if $\frac{1}{2} < \alpha \leq 1$.

Chapter 8

Further possible spectral results

¿From here the lecture can be extended in several directions.

8.1 Scattering theory

We can study more details in scattering theory, for instance we can estimate the wave operators, the scattering matrices or the spectral measures in case of regular or obstacle pertubations. However for that we should have a "detour" to scattering theory. In the case of $(-\Delta)^{\alpha}$, $0 < \alpha \leq 1$, we get criteria for the absence of singularly continuous spectra.

8.2 Domain perturbations

Till now we studied only obstacles Γ which are "small" in comparison with Σ where the operators $(K_0)_\Sigma$ are defined. We can also study the contrary situation. Let the obstacle be large such that Σ becomes small, for instance a ball in $\mathbb{R}^d$. To avoid confusions we denote the domain where the perturbed operator is defined with $U \subseteq \mathbb{R}^d$. For the Laplacian the perturbation is given by Dirichlet boundary conditions on ∂U. If $\overline{U}$ is compact the operator $(K_0)_U$ defined in $L^2(U)$ has only eigenvalues. Then we can investigate the behaviour of the eigenvalues if we modify U.

Let U_s be a (smaller) set contained in U_l (large). The corresponding operators are denoted by $(K_0)_s$,$(K_0)_l$, respectively. Let f be a nonnegative function. Then

$$\left(e^{-t(K_0)_l} f\right)(x) \geq \left(e^{-t(K_0)_s} f\right)(x). \tag{8.1}$$

Let λ_s and λ_l be the lowest eigenvalues of $(K_0)_s$ and $(K_0)_l$, respectively. Then $\lambda_l \leq \lambda_s$. Hence the question arises when $\lambda_l = \lambda_s$. This was studied by Gesztesy, Zhao[8], Demuth, van Casteren [5], Arendt, Monniaux [1] . The result is:
If $\text{cap}(U_l \setminus U_s) = 0$ then $\lambda_s = \lambda_l$. And we can also show the inverse direction, i.e. if $\text{cap}(U_l \setminus U_s) > 0$ then $\lambda_s > \lambda_l$.

8.3 Eigenvalue estimates

In a similar situation as in Section 8.2 we assume that $(K_0)_U$ is a free Feller operator considered in a set $U \subset \mathbb{R}^d$. $(K_0)_U$ are assumed to have only eigenvalues and the lowest eigenvalue $\lambda_0 > 0$ is simple. Now we take a compact set Ω inside U. We define

$$K_\beta = (K_0)_U + \beta \mathbf{1}_\Omega.$$

This family tends in strong resolvent sense to an operator denoted by $(K_0)_{U \setminus \Omega}$. Let λ_Ω be the lowest eigenvalue of $(K_0)_{U \setminus \Omega}$. It is well-known that $\lambda_\Omega \geq \lambda_0$. But the difference $\lambda_\Omega - \lambda_0$ can be estimated more accurately.

8.3.1 Theorem

Let $(K_0)_U$, $(K_0)_{U \setminus \Omega}$ be defined as described above. Let Ω be a set of small capacity. Then there are nonnegative constants c_1, c_2 such that

$$c_1 \ \text{cap}(\Omega) \leq \lambda_\Omega - \lambda_0 \leq c_2 \ \text{cap}(\Omega). \tag{8.2}$$

This is of interest for instance for the crushed ice problem.
Consider the Laplacian and assume that Ω consist of a union of Balls B_i with radii r_i. Set

$$\begin{aligned} & U_n &= U \setminus \bigcup_{i=1}^{n} B_i \\ \text{and} \qquad & U_0 &= U. \end{aligned}$$

Let $(-\Delta)_{U_n}$ be the Laplacian with Dirichlet boundary condition on ∂U and ∂B_i. The norm of the semigroup $e^{-t(-\Delta)_{U_n}}$ determines the rate of cooling. We have

$$\|e^{-t(-\Delta)_{U_n}}\| = e^{-t\inf\sigma((-\Delta)_{U_n})}.$$

Let $\lambda_{U_n} = \inf\sigma((-\Delta)_{U_n})$. By 8.2 we have

$$c_1 \ \operatorname{cap}(\bigcup_{i=1}^{n} B_i) \le \lambda_{U_n} - \lambda_{U_0} \le c_2 \ \operatorname{cap}(\bigcup_{i=1}^{n} B_i). \tag{8.3}$$

Now the capacity of the ball in $\mathbb{R}^3$ is $\operatorname{cap}(B_i) = cr_i$, so that

$$\|e^{-t(-\Delta)_{U_n}}\| = e^{-t\lambda_{U_n}} \le e^{-t\lambda_{U_0}} e^{-ctnr_n}.$$

That means the cooling rate is improved by the factor e^{-ctnr_n}.
If $nr_n \to \infty$ we have a fast cooling, if $nr_n \to 0$ we obtain the same rate as without the balls B_i.

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