Flag Numbers and FLAGTOOL

Gil Kalai	Peter Kleinschmidt and Günter Meisinger
Institute of Mathematics	Wirtschaftswissenschaftliche Fakultät
The Hebrew University	Universität Passau
91904 Jerusalem	94032 Passau
ISRAEL	GERMANY
`kalai@math.huji.ac.il`	`kleinsch@winf.uni-passau.de`

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Abstract

FLAGTOOL is a computer program for proving automatically theorems about the combinatorial structure of polytopes of dimensions at most 10. Its starting point is the known linear relations (equalities and inequalities) for flag number of polytopes. After describing the state of the art concerning such linear relations we describe various applications of FLAGTOOLS and we conclude by indicating several direction for future research and automation. As an appendix we describe FLAGTOOL's main tools and demonstrate one working session with the program.

Contents

1 Face numbers, flag numbers, g-numbers and convolutions

1.1 Face numbers

For a d-polytope P the number of its k-faces is denoted by $f_k(P)$. The vector

$$(f_0(P), f_1(P), \dots f_{d-1}(P))$$

is called the f-vector of P.

A basic problem (perhaps hopeless for $d > 3$) is:

Problem 1.1 *Characterize all the f-vectors of d-polytopes.*

A more realistic subproblem is:

Problem 1.2 *Find all the linear relations (linear equalities and linear inequalities) among face numbers of d-polytopes*

It is well known [Gr67, Zie95] that the only linear equality that holds among face numbers of d-polytopes is Euler's relation:

$$\sum_{i=0}^{d-1} f_i(P) = 1 + (-1)^d.$$

As for inequalities for $d > 5$ the only known linear inequalities are quite trivial:

$$f_r(P) \geq \binom{d+1}{r+1},$$

$$2f_1(P) \geq df_0(P),$$
$$2f_{d-2}(P) \geq df_{d-1}(P).$$

1.2 Flag numbers

For a d-polytope P, and a subset $S = \{i_1, \ldots, i_k\} \subset \{0, 1, \ldots, d-1\}$ the flag number $f_S^d(P)$ is the number of chains of faces of P $F_1 \subset F_2 \subset \cdots \subset F_k$ such that $dimF_j = i_j$. We will omit the superscript d if its value is clear from the context. (The same definition applies to ranked lattices.) The vector of flag numbers $f_S(P)$ (where the indices are ordered according to some fixed ordering) is called the flag vector of P. For simplicial polytopes the flag numbers are determined by the face numbers, but for general polytopes flag numbers seems to be the "correct" invariants.

Problem 1.3 *Characterize flag vectors of d-polytopes.*

Again this may be hopeless and a more realistic task is:

Problem 1.4 *Find all the linear relations among flag numbers of d-polytopes*

We will denote by $\mathcal{A}_d$ the affine space spanned by flag vectors of d-polytopes and by $\mathcal{P}_d \subset \mathcal{A}_d$ the cone spanned by flag vectors of d-polytopes.

Remark: While these problems on flag numbers are more general than the corresponding problems for face-numbers, deriving conclusions for the face numbers from information on flag numbers may also be a non-trivial task.

1.3 The theorem of Bayer and Billera

A remarkable theorem of Bayer and Billera [BayB85] asserts that the affine dimension of the space $\mathcal{A}_d$ of flag vectors of d-polytopes is $c_d - 1$, were c_d is the d-th Fibonacci number.

Bayer and Billera used Euler's formula to deduce the following relation commonly referred to as the "generalized Dehn-Sommerville relations":

Let $S \subset \{0, 1, \ldots d-1\}$, $k \in S \cup \{-1, d\}$ and $i \leq k-2$. If S contains no integer between i and k then

$$\sum_{j=i+1}^{k-1} f^d_{S\cup\{j\}}(P) = (1-(-1)^{k-i-1})f^d_S(P).$$

Bayer and Billera also showed that these relations span all the affine relations among flag numbers of d-polytopes.

1.4 Bases for flag vectors and bases for polytopes

A d-form will denote a linear combination of flag numbers f^d_S. A *basis* for the space of flag numbers of d-polytopes is a collection of d-forms which affinely span the space $\mathcal{A}_d$. The *special* flag numbers are those flag numbers f^d_S such that $S \subset \{1, 2, \dots d-2\}$ and S does not contain two consecutive integers. It follows from the generalized Dehn-Sommerville relation that every flag number can be represented as an affine combination of special flag numbers. Other bases of flag numbers of d-polytopes were found in [Kal88, BilL].

A *basis* of polytopes is a collection of c_d polytopes whose flag vectors are affinely independent. See [BayB85, Kal88] for two such constructions.

1.5 h- and g-numbers for simplicial polytopes

Let $d > 0$ be a fixed integer. Given a sequence $f = (f_0, f_1, \dots, f_{d-1})$ of nonnegative integers, put $f_{-1} = 1$ and define $h[f] = (h_0, h_1, \dots, h_d)$ by the relation

$$\sum_{k=0}^{d} h_k x^{d-k} = \sum_{k=0}^{d} f_{k-1}(x-1)^{d-k}. \tag{1}$$

If $f = f(P)$ is the f-vector of a simplicial d-polytope P then $h[f] = h(P)$ is called the *h-vector* of P. The g-vector $g(K) = (g_0, g_1, \dots, g_{[d/2]})$ of P is defined by $g_i = h_i - h_{i-1}$. Thus, $g_0 = 1$, $g_1 = f_0 - (d+1)$, $g_2 = f_1 - df_0 + \binom{d+1}{2}$ and $g_3 = f_2 - (d-1)f_1 + \binom{d}{2}f_0 + \binom{d+1}{3}$ and so on.

In 1970 McMullen [McM71] proposed a complete characterization of f-vectors of boundary complexes of simplicial d-dimensional polytopes. McMullen's conjecture was settled in 1980. Billera and Lee [BiLe81] proved the sufficiency part of the conjecture and Stanley [Sta80] proved the necessity part. Stanley's proof relies on deep algebraic machinery including the hard Lefschetz theorem for toric varieties. Recently, McMullen [McM93] found a self-contained proof of the necessity part of the g-theorem. It is conjectured that the g-theorem applies to arbitrary simplicial spheres.

For positive integers $n \geq k > 0$ there is a unique expression of n of the form

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \dots + \binom{a_i}{i}, \tag{2}$$

where $a_k > a_{k-1} > \dots > a_i \geq i > 0$. This given, define

$$\partial^k(n) = \binom{a_{k-1}-1}{k-1} + \binom{a_{k-1}-1}{k-2} + \dots + \binom{a_i-1}{i-1}. \tag{3}$$

Theorem 1.1 (The g-theorem) *For a vector $h = (h_0, h_1, \dots, h_d)$ of nonnegative integers the following conditions are equivalent:*
(i) *h is the h-vector of some simplicial d-polytope.*
(ii) *h satisfies the following conditions*
(a) $h_k = h_{d-k}$ *for* $k = 0, 1, \dots, [\frac{d}{2}]$
Put $g_k = h_k - h_{k-1}$.
(b) g_0=*1 and* $g_k \geq 0$, $k = 1, 2, , \dots, [\frac{d}{2}]$.
(c) $\partial^k(g_{k+1}) \leq g_k, k < [\frac{d}{2}]$

1.6 h- and g-numbers for general polytopes

Intersection homology theory has led to deep and mysterious extensions of h- and g-numbers from simplicial polytopes to general polytopes. The definition [Sta94″] goes as follows. For a polytope P denote by P_k the set of k-faces of P. Define by induction two polynomials

$$h_P(x) = \sum_{k=0}^{d} h_k^d x^{d-k}, \quad g_P(x) = \sum_{k=0}^{[d/2]} g_k^d x^{d-k},$$

by the rules: (a) $g_k^d = h_k^d - h_{k-1}^d$, (b) If P is the empty polytope or a 0-polytope P, $h_P = g_P = 1$, and

$$h_P(x) = \sum_{k=0}^{d} (x-1)^{d-k} \sum \{g_F(x) : x \in P_k\}.$$

Thus $g_1^d(P) = f_0(P) - d - 1$ and

$$g_2^d(P) = f_1(P) + \sum \{f_0(F) - 3 : F \in P_2\} - df_0(P) + \binom{d+1}{2}.$$

The value of g_2^d for general polytopes has also a rigidity theoretic meaning and is nonnegative for every polytope. The nonnegativity of g_2^d is still open for more general objects like polyhedral spheres and manifolds. It follows from intersection homology theory for toric varieties that g_k^d is nonnegative for every rational polytope. This is still open for general polytopes.

Problem 1.5 *Characterize g-vectors of d-polytopes.*

It is conjectured that g-vectors of arbitrary d-polytopes satisfy (and therefore are characterized by) the same non linear relations which were proved for simplicial polytopes.

Problem 1.6 *What is the significance of the g-numbers for the combinatorial theory of d-polytopes?*

1.7 Duality of polytopes

For a d-polytope P we denote by P^* the dual of P. There is an order-reversing bijection between faces of P and faces of P^*. Put $\bar{g}_k^d(P) = g_k^d(P^*)$. Clearly $\bar{g}_k^d(P)$ is nonnegative for every rational polytope P.

There are various connections between flag numbers of polytopes and their dual which are quite mysterious and are related to mirror symmetry. See [Sta92, BaBo96, Kal88].

1.8 Intervals in face-lattices of polytopes

The set of faces of a d-polytope form a ranked poset of rank $d+1$ with the lattice property. We will use the dimension as the grading and thus the empty face will have grade -1. Intervals in the face lattices of polytopes are themselves face lattices of polytopes, see [Gr67, Zie95, Kal88]. Intervals of type $[a, b]$ are intervals $[F, G]$ where $\dim F = a$ and $\dim G = b$. If $a = -1$ then F is the empty face and $[F, G]$ is simply the face lattice of G.

1.9 Convolutions

Let m^d, m^e be linear combinations of flag numbers of d- and e-polytopes respectively. For a polytope P of dimension $d + e + 1$ define the convolution of m^d and m^e by

$$m^d * m^e(P) = \sum\{m^d(F) \cdot m^e(P/F) : F \text{ a } d\text{-face of } P\}.$$

For a Hopf-algebraic treatment of flag numbers and their convolutions, see [BilL]. The following lemma [Kal88] is immediate.

Lemma 1.2
(1) *$m^d * m^e(P)$ is a linear combination of flag numbers of $(d + e + 1)$-polytopes.*

(2) *If $m^d(P) = 0$ for every d-polytope P or $m^e(Q) = 0$ for every e-polytope Q then $m^d * m^e(R) = 0$ for every $d + e + 1$-polytope R.*

(3) *If $m^d(P) \geq 0$ for every d-polytope P and $m^e(Q) \geq 0$ for every e-polytope Q then $m^d * m^e(R) \geq 0$ for every $d + e + 1$-polytope R.*

Convolutions of the g_i's and $\bar{g}_i$'s yield a large collection of linear inequalities for flag numbers of d-polytopes. We will denote by $\mathcal{Q}_d$ the cone of flag numbers described by all these inequalities. As it turns out the simple inequalities $f_i^d(P) \geq \binom{d+1}{i+1}$ do not follow from convolutions of the g_i's. We denote by $\mathcal{Q}'_d$ the cone of flag numbers obtained by adding these inequalities, their polars and the derived inequalities by convolutions.

1.10 The cd-index

Remarkable classes of invariants for Eulerian posets are given by Fine's cd-index. See [BayK91, Sta94′]. The cd-index for d-dimensional polytopes is a polynomial of degree d in two non-commuting variables of degrees 1 and 2, respectively. This polynomial has c_d

coefficients, each one of which is a d-form. (Namely, a linear combination of flag numbers of d-polytopes) and together they consist of a basis of such forms. It was proved by Stanley [Sta94′] that these coefficients are nonnegative for every d-polytope, and Billera and Ehrenborg [BiEh] proved that the values of these forms are at least as large as their value for the d-simplex. In low dimensions these inequalities already follows from those in $\mathcal{Q}'_d$.

2 FLAGTOOL

The previous section shows that the Generalized Dehn Sommerville Equations and the nonnegativity of convolutions of the numbers g_i^k and $\overline{g_i}^k$ ($0 \leq k \leq d$, $0 \leq i \leq \lfloor k/2 \rfloor$) yield a lot of linear relations between the flag numbers of general d-polytopes for a fixed dimension d. It is very hard to compute them or to derive new results from them without using a computer because the number of those relations is large already for small dimensions. Therefore, Meisinger [Mei94] developed a program called FLAGTOOL. The main purpose of this program is to

- compute all (known) linear relations between the flag numbers of general d-polytopes for small dimensions, say $3 \leq d \leq 10$,
- extract and automatically prove new results from those relations.

The aim of the following section is to present the main features of the program.

2.1 Basic ideas and motivation

In 1990 Kalai [Kal90] proved that every d-polytope ($d \geq 5$) has a 2-face with less than 5 vertices. This implies that there does not exist a 5-polytope all 2-faces of which are pentagons. The proof was obtained by taking for dimension five all known linear inequalities for flag numbers and all possible convolutions of the inequality corresponding to the negation of the theorem's claim ($f_0^2 - 5 \geq 0$ for intervals of type $[-1, 2]$). The resulting set of linear inequalities, as an input of a Linear Programming Problem, had no feasible solution and therefore the correctness of the theorem followed.

This suggests that more results can be proved in a similar way by using this idea systematically. The main aspect for the development of FLAGTOOL is based on this idea and can be summarized as follows. We include examples for a better understanding. See also Figure 1 for the general scheme of supporting theorem proving with FLAGTOOL.

1. Consider a fixed dimension d. FLAGTOOL works with dimensions $3 \leq d \leq 10$. This value can be increased if it does not exceed the computer's memory capacity or causes runtime problems.
2. For a fixed dimension d we derive all known nonnegative d-forms from the numbers g_i^k and $\overline{g_i}^k$ ($0 \leq k \leq d$, $0 \leq i \leq \lfloor k/2 \rfloor$) and their convolutions. (It turned out that a

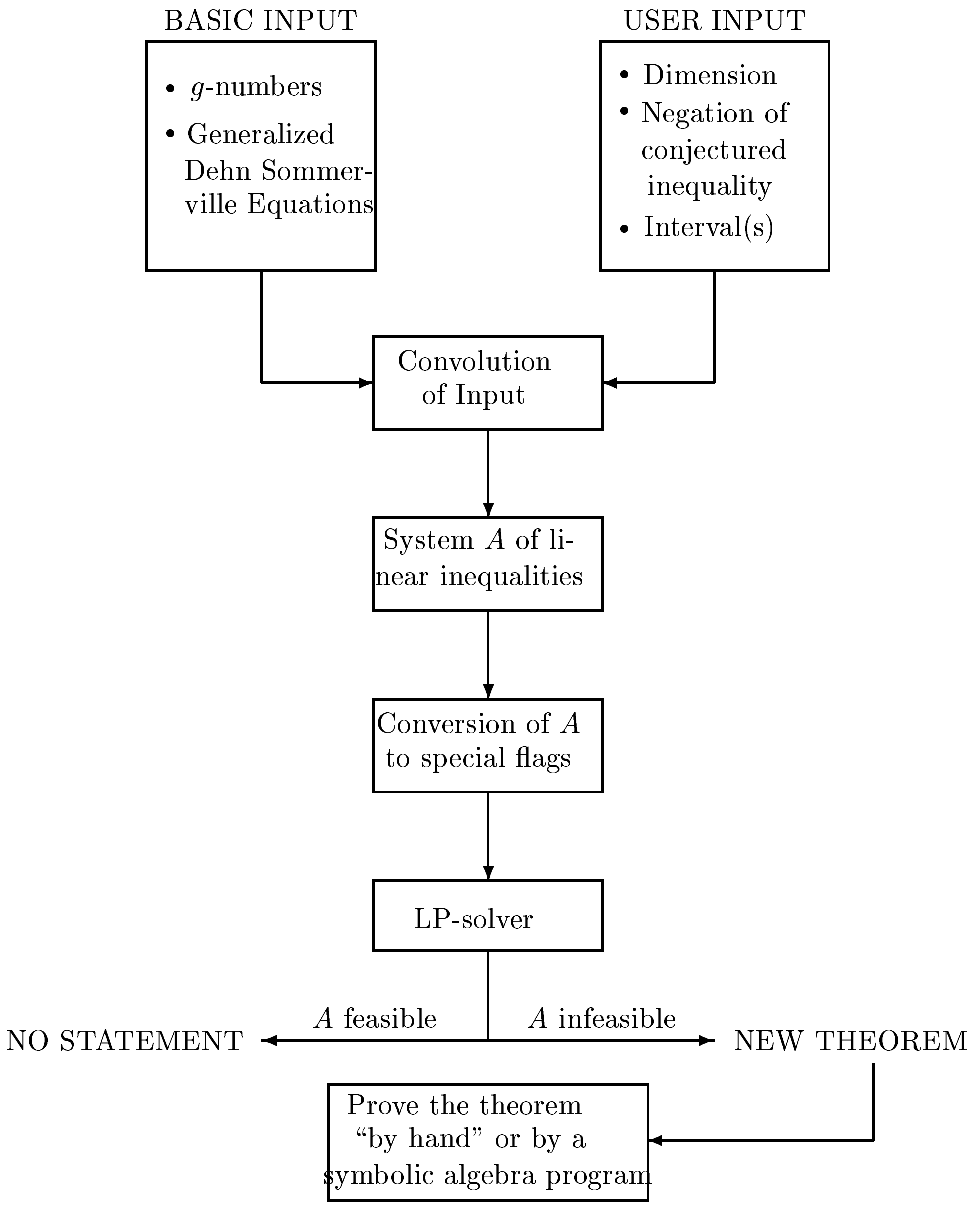

Figure 1: Automated theorem proving by FLAGTOOL

lot of the resulting inequalities are redundant. FLAGTOOL omits those redundant inequalities and computes a system B of linear inequalities in terms of flag numbers for each dimension d.)

3. (a) In order to prove a linear inequality for flag or face numbers in this fixed dimension d, we have to add the negation of this inequality to the system B. To prove $-2f_1+3f_2-2f_3 \geq 0$ which holds for all 5-polytopes, we add the negation $2f_1 - 3f_2 + 2f_3 - 1 \geq 0$.

   (b) To prove facts about low dimensional faces or quotients, we have to add inequalities (negation of conjectured inequalities) to intervals (in the face-lattices) of a d-polytope and to convolve them with the numbers g_i^k and $\overline{g_i}^k$ ($0 \leq k \leq d$, $0 \leq i \leq \lfloor k/2 \rfloor$) to create a system of inequalities which contains the system B. To prove the result that 5-polytopes always have a 2-face with less than 5 vertices, we have to add the 2-dimensional inequality $f_0^2 - 5 \geq 0$ to the bottom interval $[-1, 2]$.

4. Express all the new linear inequalities in terms of special flag numbers. This results in a set A (that contains B) of linear inequalities.

5. If the set A of linear inequalities has no feasible solution, the negation of at least one of the added inequalities is true for all rational polytopes. If only g_0^d, g_1^d and g_2^d are used in convolutions, the result holds for all polytopes. The infeasibility can be proved for example by using phase I of an LP-solver or symbolic mathematical programs.

The program itself consists of a set of subtools which are described briefly in the Appendix. For more details see the user manual of FLAGTOOL [Mei94].

3 The linear cone of flag vectors and f-vectors

3.1 What are all the linear inequalities for flag numbers?

It was conjectured in [Kal88] that $\mathcal{P}_d = \mathcal{Q}_d$, i. e., that every linear inequality on the flag numbers of general d-polytopes is equivalent to the nonnegativity of some nonnegative combination of convolutions of the g_i^k and $\overline{g_i}^k$ ($0 \leq k \leq d$, $0 \leq i \leq \lfloor k/2 \rfloor$). It turned out that this conjecture is false.

Proposition 3.1 *The linear inequality $(*) f_2 - 35 \geq 0$ which holds for all 6-polytopes is not a nonnegative combination of convolutions of the numbers g_i^k and $\overline{g_i}^k$ $(0 \leq k \leq 6, 0 \leq i \leq \lfloor k/2 \rfloor)$.*

Proof: We determined using FLAGTOOL all the linear inequalities for the cone $\mathcal{Q}_6$. The point

$$(f_0, f_1, f_2, f_3, f_4, f_{02}, f_{03}, f_{04}, f_{13}, f_{14}, f_{24}, f_{024}) \quad = \quad (7, 21, 0, 0, 21, 105, 350, 315, 630, 840, 630, 2520)$$

satisfies all these inequalities but violates (*).

3.2 Flag numbers of 4-polytopes

Euler's theorem easily implies a complete description of face numbers (hence flag numbers) of 3-polytopes. In the Appendix we give a list of the known nonredundant inequalities for 4-, 5- and (rational) 6-polytopes.

The situation for 4-dimensional polytopes was considered by Barnette [Ba74] and Bayer [Bay87]. Bayer described the cone $\mathcal{Q}_4$ and in particular identified its seven extreme rays. It is not hard to show that four of these rays are indeed extreme rays of $\mathcal{P}_4$ the cone of flag numbers of 4-polytopes. In order to show that $\mathcal{Q}_4$ describes all linear inequalities among 4-polytopes certain constructions of 4-polytopes are needed. For examples an infinite class of self-dual 2-simplicial polytopes with vanishing g_2 will take care of one such ray. An infinite family of self-dual 2-simplicial polytopes so that the ratio $\frac{f_2}{f_1}$ is unbounded will take care of a second ray.

On the other hand, Billera and Ehrenborg conjectured that for 4-polytopes

$$10f_0 - f_1 + 9f_3 - 2f_{03} \;\geq\; 45.$$

They showed that if true this relation added to $\mathcal{Q}_4$ would characterize the linear cone of flag numbers of 4-polytopes.

Note the interesting consequences of this inequality for 2-simplicial 2-simple 4-polytopes. For all such polytopes $f_0 = f_3$ and $f_1 = f_2$ and $f_{03} = f_0 + 2f_1$. The inequality of Billera and Ehrenborg would be for such polytopes $f_1 \leq 9f_0 - 45$, namely it will give a linear upper bound for the number of edges in terms of the number of vertices.

3.3 What are all the linear inequalities for face numbers?

When studying the linear inequalities for the flag numbers of arbitrary d-polytopes the question arises what information on the ordinary f-vector can be derived from those inequalities. For a fixed dimension $d \geq 4$ we have a set of n linear inequalities $A_1 \geq 0, \ldots, A_n \geq 0$ obtained by convolution of the numbers g_i^k and $\overline{g_i}^k$ ($0 \leq k \leq d$, $0 \leq i \leq \lfloor k/2 \rfloor$). Each A_j ($1 \leq j \leq n$) is a d-form in $c_d - 1$ variables (the special flag numbers without $f_\emptyset$) and $c_d - d$ variables among them are not face numbers.

Information about the ordinary f-vector can be obtained by projecting the cone of flag-vectors onto the space of f-vector. We obtained the image of $\mathcal{Q'}_d$ under this projection by a successive elimination of the $c_d - d$ variables which are not face numbers. Elimination of a variable f_S (f_S a special flag number) here means to generate all possible nonnegative combinations of the A_i where f_S does no longer appear.

For dimensions 4 and 5 projecting $\mathcal{Q'}_d$ into the space of face-numbers gave, beside the inequalities mentioned in the introduction, two nontrivial inequalities which were found earlier by Bayer and Kalai. Bayer [Bay87] showed that for 4-polytopes

$$g_2^4 + g_0^0 * g_1^2 * g_0^0$$

$$
\begin{aligned}
&= (f_{02} - 3f_2 + f_1 - 4f_0 + 10) + (6f_1 - 6f_0 - f_{02}) \\
&= -3f_2 + 7f_1 - 10f_0 + 10 \geq 0.
\end{aligned}
$$

and Kalai [Kal88] showed that for 5-polytopes

$$
\begin{aligned}
& g_1^2 * g_1^2 + g_0^0 * g_1^2 * g_0^1 + g_0^1 * g_1^2 * g_0^0 \\
= \;& (-6f_3 + 3f_{03} - f_{13} - 3f_{02} + 9f_2) \\
& +(2f_{13} - 3f_{03}) \\
& +(-6f_1 - f_{03} - f_{13} + 3f_{02}) \\
= \;& 6f_1 - 9f_2 + 6f_3 \geq 0 \\
\Rightarrow \;& 2f_1 - 3f_2 + 2f_3 \geq 0
\end{aligned}
$$

No further inequalities for 4- and 5-polytopes are obtained by projecting $\mathcal{Q}'_4$ and $\mathcal{Q}'_5$.

In dimension $d = 6$ we have 28 nonredundant linear inequalities $A_1 \geq 0, \ldots, A_{28} \geq 0$ obtained by FLAGTOOL. When we eliminated successively the variables f_{02}, f_{03}, f_{04}, f_{13}, f_{14} and f_{024} and remove redundancy the result is that no new linear inequality for the ordinary f-vector is obtained.

If no further inequalities for face numbers exist this would imply, for example, that there is a sequence P_n of 6-polytopes so that for every $k \neq 3$, $f_3(P) = o(f_k(P_n))$ as n tends to infinity. This seems very unlikely. Barany conjectured that for every d-polytope $f_k(P) \geq min\{f_0(P), f_{d-1}(P)\}$ and this seems very likely albeit beyond our reach.

4 Low dimensional faces and quotients of high dimensional polytopes

4.1 Some basic conjectures

Much of the rest of the paper is related to the following three conjectures:

Conjecture 4.1 *For every integer $k > 0$ there exist integers $n(k)$ and $d(k)$ so that every d-polytope $d \geq d(k)$ has a k-dimensional face with at most $n(k)$ vertices.*

It can be conjectured that $n(k)$ can be chosen to be 2^k and that the following stronger conjecture holds

Conjecture 4.2 *For every integer $k > 0$ there exists $d'(k)$ so that every d-polytope, $d \geq d'(k)$ has a k-dimensional face which is either a simplex or combinatorially isomorphic to a cube.*

Conjecture 4.3 (Perles) *For every integer $k > 0$ there exists $f(k)$ so that every d-polytope $d \geq f(k)$ has a k-dimensional quotient which is a simplex.*

For simple polytopes the first conjecture follows from a fundamental result of Nikulin, see [Nik86, Kal90]. The second conjecture is open even for simple polytopes.

And finally,

Problem 4.4 *Is it true that for every k there is $g(k)$ such that for every d-polytope, $d > g(k)$ either P or its polar P^* has a k-dimensional face which is a simplex.*

4.2 Small low dimensional faces of high dimensional polytopes

Theorem 4.1 *Every rational d-polytope ($d \geq 9$) has a 3-face with less than 78 vertices or 78 facets. Using the simple relation $f_0 \leq 2f_2 - 4$, which holds for all 3-polytopes, this implies that there exists always a 3-face with less than or equal to 150 vertices.*

Proof: Assume that every 3-face of an arbitrary 9-polytope has 78 or more vertices or facets. This assumption can be expressed by the inequalities $f_0^3 - 78 \geq 0$ and $f_2^3 - 78 \geq 0$. A system of 53 linear 9-forms obtained by convolutions of the g-numbers and of these two added inequalities (in the bottom interval $[-1, 3]$) has no nonnegative feasible solution and therefore Theorem 4.1 is proved. See [MKK]. The proof was obtained as follows. FLAGTOOL creates 227 linear inequalities which contain the 53 inequalities above. The infeasibility was first checked using phase I of the LP-solver CPLEX and then proved using the symbolic mathematical program MAPLE V. It seems impossible to prove the infeasibility "by hand." The details of this proof (which for a computer generated proof is quite short) do not seem to contribute to our (human) insight for understanding why the theorem is true.

4.3 Small $(k+1)$-faces for high dimensional k-simplicial polytopes

Conjecture 4.2 would imply that for high enough dimension d every 2-simplicial polytope contain a k-dimensional face which is a simplex. In this section we show that every k-simplicial ($2 \leq k \leq 3$) d-polytope P (all k-dimensional faces of P are simplices) has small $(k+1)$-dimensional faces if the dimension d is high enough.

Theorem 4.2 *Every 2-simplicial d-polytope ($d \geq 5$) has a 3-face with less than 8 vertices.*

Proof: It suffices to show that Theorem 4.2 holds for 5-polytopes, because every 5-dimensional face of every 2-simplicial d-polytope ($d > 5$) is again 2-simplicial and therefore has a small 3-face.

Assume that every 3-face of a 2-simplicial 5-polytope has 8 or more vertices. This assumption is expressed by the inequality $f_0^3 - 8 \geq 0$ in the bottom interval $[-1, 3]$. Consider the following five inequalities for 5-polytopes obtained by convolutions of the g-numbers and the added inequality.

$$
\begin{array}{lcll}
[1] & g_0^1 * g_1^2 * g_0^0 & = & -6f_1 - f_{13} + 3f_{02} \geq 0 \\
[2] & (f_0 - 8) * g_0^1 & = & f_{03} - 8f_3 \geq 0 \\
[3] & g_1^2 * g_1^2 & = & -6f_3 + 3f_{03} - f_{13} - 3f_{02} + 9f_2 \geq 0 \\
[4] & g_0^0 * g_2^4 & = & -8f_1 + 2f_{13} + f_{02} - 3f_{03} + 10f_0 \geq 0 \\
[5] & \overline{g_1}^5 & = & -f_0 + f_1 - f_2 + f_3 - 4 \geq 0
\end{array}
$$

By using the fact that for 2-simplicial polytopes the inequality $f_{02} = 3f_2$ holds we can prove the infeasibility of these five inequalities. The following nonnegative combination of the inequalities shows the infeasibility. This was obtained by Fourier-Motzkin Elimination.

$$3 * [1] + 3 * [2] + [3] + 2 * [4] + 24 * [5] = -4f_0 - 10f_1 - 6f_3 - 96 < 0$$

□

Theorem 4.3 *Every 2-simplicial d-polytope ($d \geq 7$) has a 3-face with less than 7 vertices.*

Proof: Again it suffices to prove the theorem for 7-polytopes. Assume that every 3-face of a 2-simplicial 7-polytope has 7 or more vertices (inequality $f_0^3 - 7 \geq 0$ in the bottom interval $[-1, 3]$) and that every 2-face is triangular (inequality $3 - f_0^2 \geq 0$ in the interval $[-1, 2]$). Note that $g_1^2 = f_0^2 - 3 \geq 0$ and therefore $f_0^2 = 3$. Consider the following 15 inequalities for 7-polytopes obtained by convolutions of the g-numbers, their duals and the added inequalities. The theorem follows again from the infeasibility of this system of linear inequalities.

$$
\begin{array}{lcll}
[1] & (3 - f_0) * g_0^0 * g_1^2 * g_0^0 & = & 18f_{03} - 36f_3 + 18f_{24} + 3f_{035} - 6f_{35} - 6f_{13} \\
 & & & -6f_{024} - f_{135} \geq 0 \\
[2] & (f_0 - 7) * g_1^2 * g_0^0 & = & -6f_{03} - f_{035} + 3f_{024} + 42f_3 + 7f_{35} - 21f_{24} \\
 & & & +15f_{14} - 15f_{04} \geq 0 \\
[3] & g_0^1 * g_1^4 * g_0^0 & = & -f_{024} + f_{025} - 10f_1 - 3f_{13} + 5f_{14} - 5f_{15} \\
 & & & +5f_{02} \geq 0 \\
[4] & g_0^1 * g_2^4 * g_0^0 & = & 3f_{024} - f_{025} + 20f_1 + 4f_{13} - 10f_{14} + 4f_{15} \\
 & & & -10f_{02} \geq 0 \\
[5] & g_1^6 * g_0^0 & = & -14 + 7f_0 + 7f_2 - 7f_3 + 7f_4 - 7f_5 - f_{02} \\
 & & & +f_{03} - f_{04} + f_{05} - 5f_1 \geq 0 \\
[6] & (3 - f_0) * g_1^2 * g_0^1 & = & 6f_{35} - 3f_{035} + f_{135} - 9f_{25} + 3f_{025} \geq 0 \\
[7] & (f_0 - 7) * g_0^0 * g_1^2 & = & -3f_{024} + 2f_{035} + 15f_{04} - 15f_{14} + 21f_{24} \\
 & & & -14f_{35} \geq 0 \\
[8] & (3 - f_0) * g_0^1 * g_1^2 & = & f_{135} - 3f_{035} + 6f_{35} - 9f_{24} + 3f_{024} \geq 0 \\
[9] & g_0^1 * g_1^2 * g_1^2 & = & -3f_{024} - 6f_{15} - f_{135} + 3f_{025} + 9f_{14} \geq 0 \\
[10] & g_0^4 * g_1^2 & = & 2f_5 - f_{05} + f_{15} - f_{25} + f_{35} - 3f_4 \geq 0 \\
[11] & (f_0 - 7) * g_0^3 & = & f_{03} - 7f_3 \geq 0 \\
[12] & g_0^2 * g_2^4 & = & -8f_3 + 4f_{03} - 4f_{13} + 2f_{35} - f_{035} + f_{135} \\
 & & & +f_{24} - 3f_{25} + 10f_2 \geq 0 \\
[13] & g_1^2 * g_2^4 & = & 24f_3 - 12f_{03} - 6f_{35} + 3f_{035} + 4f_{13} - f_{135} \\
 & & & +f_{024} - 3f_{025} + 10f_{02} - 3f_{24} + 9f_{25} \\
 & & & -30f_2 \geq 0 \\
[14] & (3 - f_0) * g_0^4 & = & -f_{02} + 3f_2 \geq 0 \\
[15] & g_0^0 * \overline{g_1}^6 & = & -f_{02} + f_{03} - f_{04} + f_{05} + 2f_1 - 7f_0 \geq 0
\end{array}
$$

The following positive combination of the 15 inequalities obtained by FLAGTOOL proves the infeasibility of the system.

$112*[1]+84*[2]+84*[3]+105*[4]+540*[5]+112*[6]+168*[7]+336*[8]+210*[9]+1260*[10]+252*[11]+189*[12]+315*[13]+1260*[14]+720*[15] = -1260f_5 - 1260f_0 - 7560 < 0$

□

Theorems 4.2 and 4.3 were obtained by direct support of FLAGTOOL. The added inequalities, as part of the user input for our program, are $f_0 - 3 \leq 0$ in the bottom interval $[-1, 2]$ and $f_0 - x_d \geq 0$ in the bottom interval $[-1, 3]$. The quantity x_d is a conjectured lower bound for the number of vertices of a 3-face. Several tests with different values for the dimension d and the quantity x_d lead to the following lower bounds for x_d, which are sufficient to show the infeasibility of the system of n inequalities. Note that in all cases the numbers g_0, g_1, g_2 are sufficient to prove the theorems and so they are not restricted to rational polytopes. (The first and third rows of the table are just Theorems 5 and 6 above.)

d	x_d	use of g_i	n
5	8	g_2	16
6	8	g_2	32
7	7	g_2	65
8	7	g_2	122
9	7	g_2	227
10	7	g_2	424

Similar work with FLAGTOOL shows that every 3-simplicial d-polytope $(d \geq 7)$ has a small 4-face.

The added inequalities are $f_0 - 4 \leq 0$ in the bottom interval $[-1, 3]$ and $f_0 - x_d \geq 0$ in the bottom interval $[-1, 4]$. The following lower bounds for the vertex number x_d of a 4-face for which the infeasibility holds, the use of the g-numbers and the number n of generated inequalities were obtained.

d	x_d	use of g_i	n
7	10	g_2	60
8	10	g_2	111
9	9	g_2	206
10	9	g_2	382

4.4 Low dimensional quotients with few vertices

It is a classical result that every 3-polytope or its dual has a triangular 2-face. It follows that every d-polytope $(d \geq 3)$ has a 2-quotient Q which is a triangle. Indeed if a 3-polytope P and his dual both has no triangular face then we will obtain that

$$\begin{array}{llcl} [1] & (f_0-4)*g_0^0 & = & -8+4f_0-2f_1 \geq 0 \\ [2] & g_0^0*(f_0-4) & = & 2f_1-4f_0 \geq 0 \end{array}$$

$$[1]+[2] = -8 < 0$$

In what follows we will prove that higher dimensional analogous of this fact hold as well. We show that high dimensional polytopes always have small 3-dimensional quotients in some interval of the face lattice. The most important result is that every d-polytope ($d \geq 9$) has a 3-quotient which is a simplex. For lower dimensions we show that every d-polytope ($5 \leq d \leq 8$) has a 3-quotient with less than x_d vertices. These results were obtained by adding the 3-dimensional inequality $f_0 - x_d \geq 0$ to all intervals $[-1,3]\ldots[d-4,d]$. The lower bounds for the quantity x_d, the use of the g-numbers and the number n of nonnegative convolutions (inequalities) produced by FLAGTOOL are as follows:

d	x_d	use of g_i	n
5	8	g_1	14
6	7	g_1	28
7	6	g_1	55
8	6	g_1	103
9	5	g_2	243

In particular,

Theorem 4.4 *Every d-polytope ($d \geq 9$) has a 3-quotient which is a simplex.*

In a similar way we prove that every d-polytope ($7 \leq d \leq 9$) has a 4-quotient with less than x_d vertices. The values for x_d, the use of the g-numbers and the number n of nonnegative d-forms produced by FLAGTOOL are as follows:

d	x_d	use of g_i	n
7	16	g_2	61
8	13	g_2	112
9	10	g_2	210

4.5 Low dimensional "small" quotients in prescribed locations

Note that Conjecture 4.2 would imply that if d is large enough we can always find a k-dimensional quotient of the form G/F which is a simplex, where the dimension e of F is specified and $e \neq -1, d-k-1$. In this section we show that 2- and 3-dimensional quotients with small number of vertices appear in certain specified intervals of a high dimensional polytope.

A consequence of the fact that every 3-polytope or its dual has a triangular 2-face and of the fact that the union of two adjacent 2-dimensional intervals are representing a 3-dimensional interval is the following corollary.

Corollary 4.5 *Every d-polytope ($d \geq 3$) has a triangle as a 2-quotient either in the interval $[e, e+3]$ or in the interval $[e+1, e+4]$ $(-1 \leq e \leq d-4)$.*

In what follows we present similar, but new results concerning triangular 2-quotients in certain locations. Again all results were obtained by direct support of FLAGTOOL.

Theorem 4.6 *Every d-polytope ($d \geq 6$) has a triangular 2-quotient either in the interval $[0, 3]$ or in the interval $[2, 5]$. In particular, every 6-polytope or its dual has a 3-face with a 3-valent vertex.*

Theorem 4.7 *Every 7-polytope has a triangle as a 2-quotient of a 1-face in a 4-face, i. e., it has a 4-face with an edge that is contained in three 3-faces.*

Moreover we prove that there always exist small 3-quotients in certain interesting locations. For example we show that every d-polytope ($7 \leq d \leq 9$) has a small 3-face or its dual has a small 3-face with less than x_d vertices. The added inequalities for FLAGTOOL are $f_0 - x_d \leq 0$ in the bottom interval $[-1, 3]$ and in the top interval $[d-4, d]$. The quantities x_d, the use of the g-numbers and the number n of nonnegative convolutions (inequalities) produced by FLAGTOOL are as follows:

d	x_d	use of g_i	n
7	17	g_2	62
8	55	g_2	116
9	21	g_2	194

Next we show that from a certain dimension there exist small 3-quotients in the intervals $[0, 4]$ and $[1, 5]$. Several tests with FLAGTOOL by adding the inequality $f_0 - x_d \leq 0$ in the interval $[0, 4]$ lead to the following lower bounds x_d for which the infeasibility of the n inequalities obtained by convolutions of the g-numbers and the added inequality holds.

d	x_d	use of g_i	n
5	12	g_2	14
6	12	g_2	28
7	10	g_2	57
8	10	g_2	104
9	10	g_2	172

For the interval $[1, 5]$ the result is as follows.

d	x_d	use of g_i	n
6	12	g_2	28
7	8	g_2	56
8	8	g_2	103
9	8	g_2	168

4.6 Low dimensional quotients with small g_2

In the previous sections low dimensional quotients with small g_1 were considered. We will now consider quotients with small g_2.

Theorem 4.8 *Every d-polytope ($d \geq 7$) has a 4-quotient Q such that $g_2(Q) = 0$.*

5 Other possible applications and extensions

5.1 Non-linear inequalities and linear consequences

The nonlinear relations among g-numbers which are known to hold for simplicial polytopes are conjectured for general polytopes. Other nonlinear inequalities were recently proved. Convolutions still apply, what can be derived from these inequalities? Do they imply *linear* inequalities for the flag numbers?

For simplicial d-polytopes (and more generally, for subcomplexes of their boundary complexes) one can derive quite sharp upper bounds for f_i in terms of f_{i-1}. (These bounds called the generalized upper bound inequalities are attianed for cyclic polytopes). See [Kal91]. For $i \geq [d/2] + 1$ these bounds are linear. Are these bounds continue to apply for the non-simplicial case? This is conjectured to be true in [Kal91]. This conjecture would imply a positive answer to Barany's question (Section 3.3) as well as the following remarkable extension of Björner's partial unimodality results for simplicial polytopes [Bj94]:

Conjecture 5.1 *The face numbers f_i of d-polytopes are non-decreasing for $i \leq [(d+3)]/4$ and nonincreasing for $i \geq [3(d-1)/4]$.*

The result of Braden and MacPherson which we are going now to discuss may be relevant or even the key for proving such a conjecture.

There are various relations involving flag numbers of a polytope P and those of a specific face F and the quotient P/F. Braden and MacPherson [BrMP] proved for rational polytopes that

$$g_P(x) \geq g_F(x) \times g_{P/F}(x).$$

(Here the inequality means that all coefficients of the polynomial on the left hand side are at least as large as those in the right hand side.) FLAGTOOL does not involve such relations. Can they be added to the picture? For simplicial complexes relations between face numbers of a complex and its links are fundamental in proving nonlinear relations [McM70, BjK91]. The Braden-MacPherson inequalities already have various interesting applications [Kal88, Bay98, BiEh]. Bayer found sharp form of the upper bound theorem for general polytopes and Billera and Ehrenborg used the Braden-MacPhersob result and their own monotonicity theorem for the cd-index to derive various nonlinear inequalities. It seems that there is much yet to be explored.

5.2 Special classes of polytopes

Simplicial and cubical polytopes

The face numbers of simplicial polytopes are completely characterized. (Although finding interesting combinatorial consequences from this characterization is still a challenge, see e. g. [Bj94].) Perhaps it is the right time to introduce and study more delicate numerical invariants for them. (For some ideas see [Gr70].) E. g., for simplicial 4-polytopes we can consider the number of pairs of facets which has an edge in common. (Or, similarly, we can study the f-vector of the deleted join of the polytope.)

The knowledge of cubical polytopes is much less complete (see [Ad96, BBC97, JZ]). For both simplicial and cubical polytopes flag numbers are determined by face numbers and the affine space spanned by face numbers is of dimension $[d/2]$.

In the cubical case there are analogs of the g_i's introduced by Adin [Ad96] who conjectured them to be non-negative. It would be interesting to find combinatorial consequences from the nonnegativity of the g_i for the links (and possibly also Adin's conjecture) e. g., finding an analog of Nikulin's theorem [Nik86, Theorem C] or showing that dual-to-cubical d- polytopes always have a e-dimensional face which is a simplex (where e tends to infinity with d).

Quasi simplicial polytopes

Quasi simplicial polytopes are polytopes all whose faces are simplicial. For these polytopes the face numbers determine all flag numbers [Kal88] and the affine space of face numbers is $d-1$. It seems that finding the linear inequalities for face-numbers for such polytopes which can be derived from the nonnegativity of the g_i's can be done automatically for much higher dimensions ($d \leq 100$ seems realistic).

As pointed out by Anders Björner such a study can contribute to the classification of hyperbolic reflection groups, see [Nik86, Kho86]. The results obtained so far heavily use Nikulin's theorem [Nik86, Thm. C] as well as its extension by Khovanskii to the quasi-simplicial case. For this application we need to consider only the restricted class of polytopes whose facets are Cartesian products of simplices.

k-simplicial $(d-k)$-simple d-polytopes

Recall that a polytope P is called k-simplicial if all its k-faces are simplices. P is k-simple if P^* is k-simplicial. If P is a k-simplicial, r-simple d-polytope $(k, r, \leq d)$ and $k + r > d$ then P is a simplex. Finding k-simplicial $(d-k)$-simple d-polytopes (for $k, d-k > 1$), apart from the simplex itself, is of great interest. No such example is known lfor $k, d-k \geq 4$. The convex hulls of middle points of the edges of the simplex (the 2-hypersimplex) are 2-simplicial $(d-2)$-simple polytopes. The convex hull of the even vertices of the d-cube are 3-simplicial $(d-3)$-simple. (These examples and others were worked out by Perles who may have had also a single example for $k = d - k = 4$ that was forgotten.) It would be interesting to understand the linear relations among flag vectors of such polytopes.

Facet-forming polytopes

A d-polytope P is a facet forming polytope if there is a $(d+1)$-polytope all whose facets are isomorphic to P. By eliminating variables we can, in principle, determine the average behavior of flag numbers of facets of a d-polytope (and more generally the behavior of the averages of flag numbers for e-dimensional faces of d-polytope). Clearly every such inequality will apply to facet-forming polytopes. It turns out that for $d = 4, 5$ no new inequalities are obtained. More sophisticated applications of the inequalities in $\mathcal{Q}_5$ can be used to show that certain 4-polytopes are not facet-forming, see [Kal90]. It would be interesting to extend these results to higher dimension.

Barycentric subdivisions

Another interesting direction is to try to determine all linear relations between face numbers of barycentric subdivisions of d-polytopes. Recall that the number $f_k(BP)$ of the barycentric subdivision BP of P is simply the sum of $f_S(P)$ for all sets S of cardinality $k+1$. Again, this is a question about the projection of $\mathcal{Q}'_d$ to some small dimensional subspace. This direction was not worked out yet.

Zonotopes

Understanding flag numbers of zonotopes is of great interest. There are several famous open problems and some advances seem to be related to the kind of arguments used here. Billera, Ehrenborg and Readdy [BER97] showed that the affine space spanned by flag number of d-dimensional zonotopes is the entire space spanned by flag numbers of d-polytopes. (In other words, they proved the existence of a basis of polytopes all whose members are zonotopes. They did not construct explicitly such a basis.)

Other posets

There is much activity concerning the combinatorics and the linear relation of flag numbers of various classes of ranked posets [BayH, BilH, BilH′]. The generalized Dehn-Sommerville inequalities apply to all Eulerian posets while the inequality $g_1 \geq 0$ aplly to all relatively complemented lattices. Thus, both these properties apply to general Eulerian lattices and FLAGTOOL can thus be useful to their study. (We do not know, for example, if the conjectures of Section 4.1 may apply to arbitrary Eulerian lattices.)

5.3 Extensions of FLAGTOOL

High dimensions

It may be useful to try to extend FLAGTOOL to higher dimensions. Since the number of variables and inequalities grows rapidly and the resulting LP problems are not sparse we cannot expect too much but extending the program up to 20 dimensions may be feasible. As we already mentioned, for restricted classes of polytopes (quasisimplicial, for example) we may hope for a similar program applicable in much larger dimensions.

Further automation

FLAGTOOL is used to automate certain combinatorial arguments which were done "by hand" in a few papers and with some computer support in another [Ba74, Bay87, Kal90].

Let $\mathcal{G}$ be a specific system of linear inequalities for flag numbers of k-polytopes. Most of the theorems proved using FLAGTOOL are of the following form:

Every d-polytope contain a k-dimensional quotient (possibly in a prescribed location) which satisfy the inequalities in $\mathcal{G}$.

The current modus operandi (see Figure 1) was that we tested various conjectures of this kind using FLAGTOOL, but here also further automation seems possible.

Problem 5.2 *Find automatically (all the) theorems of this form that are derived from the known linear inequalities.*

Note that this problem is not identical to finding all flag-number inequalities which holds on average for k-dimensional quotients in prescribed locations.

Using other types of reasoning

There are arguments which are similar to those we use but use additional ingredients. For example: studying the behavior of certain face number relations along a shelling process of the polytope. See for example [Ba80, BlBl90]. Can these arguments be automated (and thus systematically extended) too?

Blind and Blind [BlBl90] proved that every d-polytope with no triangular 2-faces must contain at least as many k-faces as the d-dimensional cube. This is a type of theorem that could have been derived using FLAGTOOL but it does not follow from the known flag number inequalities. The proof of Blind and Blind contains further ingredients.

Challenge 5.3 *Automate (and extend) the theorem of Blind and Blind.*

Challenge 5.4 *Automate (and extend) the proofs of [BBMM90].*

Seymour's arguments [Sey82] in connection with the points-lines-planes conjecture have some similar flavor to the arguments used here.

Challenge 5.5 *Automate (and extend) Seymour's theorems.*

6 Conclusion

FLAGTOOL can serve as a useful tool for proving theorems concerning the combinatorial structure of polytopes of dimension $d \leq 10$ and for testing and making conjectures for arbitrary polytopes. At present the proofs obtained from FLAGTOOL do not seem to give much insight (to humans) about the theorems and, in particular, do not supply a recipe for extensions to higher dimensions.

7 Appendix

FLAGTOOL is a computer program implementing the ideas described in this paper; see also [Mei94]. The code may be obtained from the second author on request.

7.1 Description of the available tools

After starting FLAGTOOL, a menu whose topics correspond to the available tools appears on the screen. After executing a tool the program returns to the menu. Most of the tools are simple I/O-programs, i. e. they transfer data from or to a file or screen. Other tools like (5), (8), (13) and (18) require nontrivial data structures and algorithms. Here is a complete enumeration of the available tools.

(1) DIMENSION

This simple tool is used to change the current working dimension d and provides the new basic input for d. FLAGTOOL accepts values from dimension 3 up to dimension 10 (this can easily be raised to higher dimensions).

(2) GMAX

This changes the use of the g-numbers. The maximum possible value depends on the current working dimension. If this value is less than three, i. e. only g_0, g_1 and g_2 are used in convolutions, results hold for all polytopes, not only for rational polytopes (see section 1.6).

(3) ADD

This tool is used for adding a new inequality for proper faces or quotients of d-polytopes to the system (maybe the negation of a conjectured inequality) and for specifying the dimension of this inequality and the intervals in which this inequality appears.

(4) DELETE

Delete is used for removing added inequalities (with ADD) and specified intervals.

(5) MAKE

This tool computes a system A of linear inequalities by convolution of the g-inequalities and the added inequalities in specified intervals. Every inequality is expressed in terms of special flag numbers. See section 4.4 of [Mei94] for data structures and algorithms.

(6) INADD
This adds a new inequality in the present working dimension to a current system A of linear inequalities, which was created by the MAKE-tool by convolutions of the g-numbers and added inequalities in specified intervals and changed by tool (6), (7), (8) or (9).

(7) INDEL
This serves for deleting an inequality from a current system A of linear inequalities.

(8) ELIM
This tool eliminates a special flag number from the current system A of linear inequalities.

(9) DSELIM
This eliminates a special flag number from the current system A of linear inequalities using only Generalized Dehn Sommerville Equations.

(10) MPS
This tool creates an mps-file (LP input format) which corresponds either to the current system of linear inequalities (an objective function in terms of special flag numbers can be specified) or to the dual problem.

(11) SAVE
A current system of linear inequalities is saved on a file.

(12) FETCH
This reads a system of linear inequalities from a file created by the SAVE-tool.

(13) DISPLAY
This tool is used for displaying the current input or more information about flag numbers. The following options are available.

(a) STATUS
Display the current input of FLAGTOOL (working dimension, use of the g-numbers and added inequalities).

(b) SYSTEM
Display the current system A of inequalities.

(c) G-NUM
Display the g-numbers and their duals up to the present working dimension.

(c) DEHN
Display the expression of the flag numbers by the special flag numbers as a solution of the Generalized Dehn Sommerville Equations.

(14) READ
This reads input for FLAGTOOL from a file.

(15) WRITE
This writes the current FLAGTOOL input to a file.

(16) SOLVE
This tool is available only if FLAGTOOL is linked with the linear programming solver $CPLEX^{TM}$ [1]. It computes the objective function value, if it is specified, or simply tests the infeasibility of an mps-file created by FLAGTOOL.

(17) RED
This tool removes redundant inequalities from a given system A of linear inequalities produced, for example, by the ELIM-tool. Only if FLAGTOOL is linked with the LP-solver $CPLEX^{TM}$ RED-tool is available.

(18) CD
This tool computes the coefficients in the *cd index* (see section 1.10) for the present working dimension in terms of special flag numbers.

(19) DUAL
For a current system of linear inequalities the system of the corresponding dual inequalities is computed.

(19) HELP
This provides the user with online information about the program, the tools and the interpretation of the output.

7.2 Working with FLAGTOOL

A short demonstration into how FLAGTOOL works is given by explaining a typical FLAGTOOL-session that proves Kalai's result about the existence of small 2-faces in d-polytopes ($d \geq 5$).

FLAGTOOL starts with the following menu. Note, that the following conventions are used to distinguish computer output from user input. All output produced by the computer will appear in `typewriter-like` font. Text entered by the user will appear in *italic* font.

```
Welcome to FLAGTOOL!

Here is a list of available commands.
Type 'help' followed by a command name for more
information on commands, for example 'helpadd'.
```

[1] $CPLEX^{TM}$ is a registered trademark of CPLEX OPTIMIZATION INC.

```
(dim)ension   set or change the working dimension
(gm)ax        set or change the use of the $g$-numbers
(ad)d         add an inequality for proper faces or quotients
(de)lete      delete one or more added inequalities
(ma)ke        compute a system of inequalities by convolution
(ina)dd       add an inequality to a current system
(ind)el       delete inequalities from a current system
(el)im        eliminate a flag from a current system
(ds)elim      eliminate a flag by using Dehn Sommerville
(mp)s         create an mps-file
(sa)ve        write a current system to a file
(fe)tch       get a current system from a file
(dis)play     display the input, the current system or
              more information about flag numbers
(re)ad        read input from a file
(wr)ite       write input to a file
(so)lve       solve an mps-file with CPLEX
(red)undant   remove redundancy from a current system
(cd)index     compute the coefficients in the cd index
(du)al        compute the dual system of inequalities
(com)mands    list the FLAGTOOL commands
(qu)it        leave FLAGTOOL

Commands may be executed by entering the command name (or
at least the letters in the bracket) and FLAGTOOL will
prompt you for additional required information.
```

First we have to set the working dimension and the use of the g-numbers. Note that g_2 is needed for the proof, i. e. FLAGTOOL cannot prove Kalai's theorem only with g_1.

`FLAGTOOL>` *dim*

`Present value for the working dimension:  3`
`New value for the working dimension:` *5*

`Okay, new value for the working dimension:  5`

`FLAGTOOL>` *gm*

`Present value for the use of the g-numbers:  1`
`New value for the use of the g-numbers:` *2*

`New value for the use of the g-numbers:  2`

In order to prove that every 5-polytope has a 2-face with less than 5 vertices we have to add the inequality $f_0^2 - 5 \geq 0$ to the bottom interval $[-1, 2]$ ('add' command), convolve this inequality with the numbers g_i^k and $\overline{g_i}^k$ $(0 \leq k \leq 5, 0 \leq i \leq \lfloor k/2 \rfloor)$ ('make' command), create an mps-file ('mps' command) and solve the corresponding linear program ('solve' command).

`FLAGTOOL>` *add*

`There are 0 added inequalities.`

`Enter the dimension of the new inequality:` *2*

`Enter the new inequality:` *f0-5*

`Enter the interval(s) in which it appears:` *[-1,2]*

`Inequality added!`

`FLAGTOOL>` *make*

```
Note, that you have added inequalities for proper
faces or quotients!
```

`Flagtool computes a system of inequalities, please wait ...`

`Current system A with 15 inequalities created!`

`FLAGTOOL>` *mps*

`Enter '1' (dual) or '0' (primal):` *0*

`Enter '1' (min) or '0' (max):` *1*

`Enter the objective function:` *0*

`Enter a name for the new mps-file :` *kalai5.mps*

`File kalai5.mps created!`

`FLAGTOOL>` *solve*

`Enter the name for the mps-file :` *kalai5.mps*

`problem is infeasible!`

Before the session ends we have a look at the current system of linear inequalities by using the '(dis)play' command.

`FLAGTOOL>` *dis*

```
Display options :

(sta)tus      display the current input
(sys)tem      display the currrent system of inequalities
              and their meanings
(g_n)um       display the g-numbers up to the present
              working dimension
(deh)n        display the Dehn Sommerville Equations
              for the specified working-dimension
```

```
Display what: sys

Working dimension: 5

   1  -6f1-f13+3f02 = (G_0_1)*(G_1_2)*(G_0_0)
   2  -10+5f0+5f2-5f3-f02+f03-3f1 = (G_1_4)*(G_0_0)
   3  -10+5f0-5f1+5f2-3f3 = (G^_1_4)*(G_0_0)
   4  20-10f0-10f2-f03+4f3+4f1+3f02 = (G_2_4)*(G_0_0)
   5  2f13-3f03 = (G_0_0)*(G_1_2)*(G_0_1)
   6  2f3-f03+f13-3f2 = (G_0_2)*(G_1_2)
   7  f02-3f2 = (G_1_2)*(G_0_2)
   8  -6f3+3f03-f13-3f02+9f2 = (G_1_2)*(G_1_2)
   9  f02-5f2 = (-5+f0)*(G_0_2)
  10  -10f3+5f03-3f13-3f02+15f2 = (-5+f0)*(G_1_2)
  11  2f1-5f0 = (G_0_0)*(G_1_4)
  12  -f02+f03+2f1-5f0 = (G_0_0)*(G^_1_4)
  13  -8f1+2f13+f02-3f03+10f0 = (G_0_0)*(G_2_4)
  14  -6+f0 = (G_1_5)
  15  -f0+f1-f2+f3-4 = (G^_1_5)

FLAGTOOL> quit
```

The infeasibility of the system of 15 inequalities and thus the correctness of the theorem can be proved by hand. Consider the six nonnegative 5-forms 1, 5, 9, 10, 13 and 15. The following nonnegative combination of these six inequalities results in an inequality which is strictly less than zero and therefore the infeasibility is proved.

$$
\begin{array}{rl}
 & (-6f_1 - f_{13} + 3f_{02}) \\
+ & 4 * (2f_{13} - 3f_{03}) \\
+ & 8 * (f_{02} - 5f_2) \\
+ & 3 * (-10f_3 + 5f_{03} - 3f_{13} - 3f_{02} + 15f_2) \\
+ & (-8f_1 + 2f_{13} - 3f_{03} + f_{02} + 10f_0) \\
+ & 10 * (-f_0 + f_1 - f_2 + f_3 - 4) \\
= & -20f_0 - 5f_1 - 4f_2 - 40 < 0
\end{array}
$$

7.3 Known flag number inequalities for d-polytopes, $d \leq 6$

We describe now (nonredundant) d-forms (linear combinations of flag numbers of d-polytopes) which are known to be nonnegative, for $d = 3, 4, 5, 6$.

Dimension 3

1	$g_1^2 * g_0^0$	$= -6 + 3f_0 - f_1$
2	$g_0^0 * g_1^2$	$= 2f_1 - 3f_0$

Dimension 4

1	$g_0^0 * g_1^2 * g_0^0$	$= -6f_0 - f_{02} + 6f_1$
2	$g_1^2 * g_0^1$	$= f_{02} - 3f_2$
3	$g_0^1 * g_1^2$	$= f_{02} - 3f_1$
4	g_1^4	$= -5 + f_0$
5	$\overline{g_1}^4$	$= f_0 - f_1 + f_2 - 5$
6	g_2^4	$= 10 - 4f_0 + f_{02} - 3f_2 + f_1$

Dimension 5

1	$g_0^1 * g_1^2 * g_0^0$	$= -6f_1 - f_{13} + 3f_{02}$
2	$g_1^4 * g_0^0$	$= -10 + 5f_0 + 5f_2 - 5f_3 - f_{02} + f_{03} - 3f_1$
3	$\overline{g_1}^4 * g_0^0$	$= -10 + 5f_0 - 5f_1 + 5f_2 - 3f_3$
4	$g_2^4 * g_0^0$	$= 20 - 10f_0 - 10f_2 - f_{03} + 4f_3 + 4f_1 + 3f_{02}$
5	$g_0^0 * g_1^2 * g_0^1$	$= 2f_{13} - 3f_{03}$
6	$g_0^2 * g_1^2$	$= 2f_3 - f_{03} + f_{13} - 3f_2$
7	$g_1^2 * g_0^2$	$= f_{02} - 3f_2$
8	$g_1^2 * g_1^2$	$= -6f_3 + 3f_{03} - f_{13} - 3f_{02} + 9f_2$
9	$g_0^0 * g_1^4$	$= 2f_1 - 5f_0$
10	$g_0^0 * \overline{g_1}^4$	$= -f_{02} + f_{03} + 2f_1 - 5f_0$
11	$g_0^0 * g_2^4$	$= -8f_1 + 2f_{13} + f_{02} - 3f_{03} + 10f_0$
12	g_1^5	$= -6 + f_0$
13	$\overline{g_1}^5$	$= -f_0 + f_1 - f_2 + f_3 - 4$

Dimension 6

1	$g_0^2 * g_1^2 * g_0^0$	$= -6f_2 - f_{24} + 3f_{13} - 3f_{03} + 6f_3$
2	$g_1^2 * g_1^2 * g_0^0$	$= -6f_{02} - f_{024} + 18f_2 + 3f_{24} - 3f_{13} + 9f_{03} - 18f_3$
3	$g_0^0 * g_1^4 * g_0^0$	$= -2f_{13} + 2f_{14} - 10f_0 - 3f_{02} + 5f_{03} - 5f_{04} + 10f_1$
4	$g_0^0 * \overline{g_1}^4 * g_0^0$	$= -10f_0 - 5f_{02} + 5f_{03} - 3f_{04} + 10f_1$

5	$g_0^0 * g_2^4 * g_0^0$	$= 6f_{13} - 2f_{14} + 20f_0 + 4f_{02} - 10f_{03} + 4f_{04} - 20f_1$
6	$g_1^5 * g_0^0$	$= -6f_2 + 6f_3 - 6f_4 - 4f_0 + f_{02} - f_{03} + f_{04} + 4f_1$
7	$\overline{g_1}^5 * g_0^0$	$= -6f_0 + 6f_1 - 6f_2 + 6f_3 - 4f_4$
8	$g_0^1 * g_1^2 * g_0^1$	$= f_{024} - 3f_{14}$
9	$g_1^4 * g_0^1$	$= f_{04} - 5f_4$
10	$\overline{g_1}^4 * g_0^1$	$= f_{04} - f_{14} + f_{24} - 5f_4$
11	$g_2^4 * g_0^1$	$= f_{14} - 3f_{24} + f_{024} - 4f_{04} + 10f_4$
12	$g_1^2 * g_0^0 * g_1^2$	$= 2f_{024} + 18f_3 - 9f_{03} + 3f_{13} - 6f_{24}$
13	$g_0^0 * g_1^2 * g_0^2$	$= 2f_{13} - 3f_{03}$
14	$g_0^0 * g_1^2 * g_1^2$	$= -6f_{13} - 6f_{04} - f_{024} + 6f_{14} + 9f_{03}$
15	$g_0^3 * g_1^2$	$= f_{04} - f_{14} + f_{24} - 3f_3$
16	$g_1^2 * g_0^3$	$= f_{02} - 3f_2$
17	$g_0^1 * g_1^4$	$= f_{02} - 5f_1$
18	$g_0^1 * \overline{g_1}^4$	$= -f_{13} + f_{14} + f_{02} - 5f_1$
19	$g_0^1 * g_2^4$	$= -4f_{02} + f_{024} + f_{13} - 3f_{14} + 10f_1$
20	$g_0^0 * g_1^5$	$= 2f_1 - 6f_0$
21	$g_0^0 * \overline{g_1}^5$	$= f_{02} - f_{03} + f_{04} - 2f_1 - 4f_0$
22	g_1^6	$= -7 + f_0$
23	$\overline{g_1}^6$	$= f_0 - f_1 + f_2 - f_3 + f_4 - 7$
24	g_2^6	$= 21 - 6f_0 + f_{02} - 3f_2 + f_1$
25	$\overline{g_2}^6$	$= f_{24} - f_{14} + f_{04} - 6f_0 + 6f_1 - 6f_2 + 21 + 3f_3 - 5f_4$
26	g_3^6	$= -35 + 15f_0 - 4f_{02} - 5f_1 + f_{14} - 3f_{24} + f_{024} - 4f_{04} + 10f_4$ $+f_{03} - 4f_3 + 13f_2$ (nonnegativity is known only for rational polytopes)
27	*Added*	$= f_2 - 35$ (see Section 1.9)
28	*Added*	$= f_3 - 35$

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