A Topological Colorful Helly Theorem

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August 18, 2003

Abstract

Let $\mathcal{F}_1,\ldots,\mathcal{F}_{d+1}$ be $d+1$ families of convex sets in $\mathbb{R}^d$. The Colorful Helly Theorem (see [3]) asserts that if $\bigcap_{i=1}^{d+1} F_i \neq \emptyset$ for all choices of $F_1 \in \mathcal{F}_1,\ldots,F_{d+1} \in \mathcal{F}_{d+1}$ then there exists an $1 \leq i \leq d+1$ such that $\bigcap_{F\in\mathcal{F}_i} F \neq \emptyset$.

Our main result is both a topological and a matroidal extension of the colorful Helly theorem. A simplicial complex X is d-*Leray* if $H_i(Y;\mathbb{Q}) = 0$ for all induced subcomplexes $Y \subset X$ and $i \geq d$.

Theorem: Let X be a d-Leray complex on the vertex set V. Suppose M is a matroidal complex on the same vertex set V with rank function ρ. If $M \subset X$ then there exists a simplex $\tau \in X$ such that $\rho(V-\tau) \leq d$.

1 Introduction

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. Helly's theorem (see e.g. [6]) asserts that if $\bigcap_{F\in\mathcal{G}} F \neq \emptyset$ for all $\mathcal{G} \subset \mathcal{F}$ such that $|\mathcal{G}| \leq d+1$, then $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$. Dual to Helly's theorem is Carathéodory's theorem: If A is a set of points in $\mathbb{R}^d$ and $x \in convA$ then there exists a subset $A' \subset A$ such that $x \in convA'$ and $|A'| \leq d+1$. In [3] Imre Bárány described remarkable extensions of these classical theorems.

Theorem 1.1 (Colorful Helly, Lovász) *Let $\mathcal{F}_1,\ldots,\mathcal{F}_{d+1}$ be $d+1$ finite families of convex sets in $\mathbb{R}^d$. If $\bigcap_{i=1}^{d+1} F_i \neq \emptyset$ for all choices of $F_1 \in \mathcal{F}_1,\ldots,F_{d+1} \in \mathcal{F}_{d+1}$ then $\bigcap_{F\in\mathcal{F}_i} F \neq \emptyset$ for some $1 \leq i \leq d+1$.*

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Theorem 1.2 (Colorful Carathéodory, Bárány) *Let $A_1, A_2, \ldots A_{d+1}$ be finite sets of points in $\mathbb{R}^d$. If $x \in \cap_{i=1}^{d+1} conv(A_i)$ then there exist $a_1 \in A_1, \ldots, a_{d+1} \in A_{d+1}$ such that $x \in conv\{a_1, a_2, \ldots, a_{d+1}\}$.*

These results reduce to the theorems of Helly and Carathéodory when all families $\mathcal{F}_i$'s and respectively all sets A_i's coincide. As in the classical case, Theorems 1.1 and 1.2 are related by linear programming duality (see [4]). Bárány's Theorem 1.2 has several important applications in discrete geometry. For example, it plays key roles in Sarkaria's proof of Tverberg's Theorem [9, 4] and in the proof of existence of weak ϵ-nets for the family of convex sets in $\mathbb{R}^d$ [2]. The deeper nature of the colorful versions of Carathéodory's and Helly's theorems is demonstrated by the fact that while there is a simple polynomial time algorithm that finds the points $a_1, a_2, \ldots a_{d+1}$ guaranteed by Carathéodory's theorem, no such algorithm is known in the more general setting of Bárány's theorem.

In this paper we discuss some geometric and topological extensions of the colorful Helly Theorem. Helly himself realized (see [6]) that in his theorem, convex sets can be replaced by topological cells if you impose the additional requirement that all non-empty intersections of these cells are again topological cells. (This requirement is automatically satisfied in the original geometric version since the intersection of convex sets is also convex). Helly's topological version of his theorem also follows from the later nerve theorems of Leray and others (see below). Our main result implies a similar topological version for the colorful Helly theorem.

Let $\mathcal{F}$ be a finite family of sets. The *nerve* $N(\mathcal{F})$ of $\mathcal{F}$ is the simplicial complex whose vertex set is $\mathcal{F}$ and whose simplices are all $\mathcal{G} \subset \mathcal{F}$ such that $\bigcap_{F \in \mathcal{G}} F \neq \emptyset$. A simplicial complex X is *d-representable* if it is isomorphic to a nerve of a finite family of convex sets in $\mathbb{R}^d$. The class of d-representable complexes is denoted by $\mathcal{K}^d$.

Let M be a matroid on the vertex set V with rank function ρ. We identify M with its *matroidal complex*, namely the simplicial complex on V whose simplices are the independent sets of the matroid. For a simplicial complex X on V and a subset $S \subset V$, let $X[S] = \{\sigma \in X : \sigma \subset S\}$ denote the induced subcomplex on S. Matroidal complexes are characterized by the property that every induced subcomplex $M[S]$ is pure, i.e. all maximal faces of $M[S]$ have the same dimension. The matroid rank function ρ satisfies $\rho(S) = \dim M[S]+1$ for all $S \subset V$, in particular $\mathrm{rk}(M) = \rho(V) = \dim M+1$. In section 2 we prove

Theorem 1.3 *Let X be a d-representable complex on V and let M be a matroidal complex on V such that $M \subset X$. Then there exists a simplex $\tau \in X$ such that $\rho(\tau) = \mathrm{rk}(M)$ and $\rho(V - \tau) \le d$.*

The proof is geometric and relies on a certain collapsibility property of d-representable complexes due to Wegner [11].

Theorem 1.3 implies by duality a matroidal version of Bárány's result. Let $\{v_1, \ldots, v_m\}$ be a multiset of points in $\mathbb{R}^d$, and let M be a matroid on $[m] = \{1, \ldots, m\}$ with rank function ρ.

Corollary 1.4 *Suppose $v \in conv\{v_i : i \in S\}$ for all $S \subset [m]$ such that $\rho(S) = \mathrm{rk}(M)$ and $\rho([m] - S) \le d$. Then there exists an independent set $T \in M$ such that $v \in conv\{v_i : i \in T\}$.*

$\square$

Suppose $\bigcup_{i=1}^{d+1} V_i$ is a partition of V into $d+1$ sets. Let M denote the corresponding partition matroid: $\sigma \in M$ iff $|\sigma \cap V_i| \le 1$ for all $1 \le i \le d+1$. Applying Theorem 1.3 with this M we obtain the following stronger version of Lovász Theorem.

Corollary 1.5 *Let X be a d-representable complex on V such that $\{v_i\}_{i=1}^{d+1} \in X$ for all choices of $v_1 \in V_1, \ldots, v_{d+1} \in V_{d+1}$. Then there exist an $1 \le i \le d+1$ and $v_j \in V_j$ for all $j \neq i$ such that $V_i \bigcup \{v_j : j \neq i\} \in X$.*

$\square$

For a simplicial complex X, let $H_j(X)$ denote the j-th homology group of X with rational coefficients and let $\beta_j(X) = \dim H_j(X)$ be its j-th Betti number. The reduced rational homology and Betti numbers are denoted by $\tilde{H}_j(X)$ and $\tilde{\beta}_j(X)$. The *link* of a simplex $\sigma \in X$ is defined by

$$\mathrm{lk}(X, \sigma) = \{\tau \in X \;:\; \tau \cap \sigma = \emptyset \;,\; \tau \cup \sigma \in X \,\} \;.$$

X is *d-Leray* if $\tilde{H}_i(Y) = 0$ for all induced subcomplexes $Y \subset X$ and $i \ge d$. Equivalently X is d-Leray if $\tilde{H}_i(\mathrm{lk}(X, \sigma)) = 0$ for all $\sigma \in X$ and $i \ge d$. The class of d-Leray complexes is denoted by $\mathcal{L}^d$. Let $\mathcal{F}$ be a family of topological (or homological) cells in $\mathbb{R}^d$ with the property that every non-empty intersection of members in $\mathcal{F}$ is again a topological (or homological) cell. The nerve theorem (see e.g. Theorem 3.2 below) implies that $N(\mathcal{F})$ is d-Leray. In particular, $\mathcal{K}^d \subset \mathcal{L}^d$. Examples (see [11]) show that the containment is strict for all $d \ge 1$.

Our main result is an extension of Theorem 1.1 to d-Leray complexes.

Theorem 1.6 *Let X be a d-Leray complex on V and let M be a matroidal complex on V such that $M \subset X$. Then there exists a simplex $\tau \in X$ such that $\rho(V - \tau) \leq d$.*

The proof of Theorem 1.6 applies a homological criterion (see [1, 7]) for the existence of colorful simplices in colored complexes. For completeness, we include a short proof of this result in section 3. The proof of Theorem 1.6 is given in section 4.

2 Colorful Helly for d-Collapsible Complexes

Let σ be a face of dimension at most $d-1$ of a simplicial complex X which is contained in a *unique* maximal face τ of X, and let $[\sigma, \tau] = \{\eta \, : \, \sigma \subset \eta \subset \tau\}$. The operation $X \to Y = X - [\sigma, \tau]$ is called an *elementary d-collapse*. X is *d-collapsible* if there exists a sequence of elementary d-collapses

$$X = X_1 \to X_2 \to \cdots \to X_m = \{\emptyset\} \quad .$$

The class of d-collapsible complexes is denoted by $\mathcal{C}^d$. A fundamental result of Wegner [11] asserts that the nerve of every finite family of convex sets in $\mathbb{R}^d$ is d-collapsible, thus $\mathcal{K}^d \subset \mathcal{C}^d$. (Wegner's proof is based on sliding a hyperplane from infinity towards the family and studying the first time this hyperplane passes the intersection of some d members of the family.) Since an elementary d-collapse does not effect the homology in dimensions at least d it follows that $\mathcal{C}^d \subset \mathcal{L}^d$. Examples (see [11]) show that the first inclusion is strict for $d \geq 1$ while the second is strict for $d \geq 2$. We prove the following extension of Theorem 1.3.

Theorem 2.1 *Let X be a d-collapsible complex on V and let M be a matroidal complex on V. If $M \subset X$ then there exists a simplex $\tau \in X$ such that $\rho(\tau) = \mathrm{rk}(M)$ and $\rho(V - \tau) \leq d$.*

Proof: Let $X = X_1 \to X_2 \to \cdots \to X_m = \{\emptyset\}$ be a d-collapsing of X, where $X_i \to X_{i+1} = X_i - [\sigma_i, \tau_i]$ is an elementary d-collapse. Let $1 \leq k \leq m$ be the maximal index such that $X_k \supset M$. If $k = m$ then M is void and the theorem holds trivially. Suppose $k < m$. We show that the theorem holds with $\tau = \tau_k$. By maximality of k there exists an $\sigma_k \subset S \subset \tau_k$ such that $S \in M$, thus $\sigma_k \in M$. Let T be any basis of M that contains σ_k. Since $M \subset X_k$ it follows that $T \subset \tau_k$, hence $\rho(\tau_k) = \mathrm{rk}(M)$. If $v \notin \tau_k$ then $\sigma_k \cup \{v\} \notin M$. Therefore $V - \tau_k \subset$ *Span* σ_k and $\rho(V - \tau_k) \leq |\sigma_k| \leq d$.

□

3 Colorful Simplices via Homology

For a simplicial complex X on a vertex set V let

$$\eta(X) = \min\{j : \tilde{\beta}_j(X) \neq 0\} + 1 \ .$$

Let Y be a simplicial complex on a vertex set V' such that $V \cap V' = \emptyset$. The *join* $X * Y$ is a simplicial complex with vertices $V \cup V'$ and simplices $\sigma \cup \tau$ where $\sigma \in X$, $\tau \in Y$. The Künneth formula

$$\tilde{H}_k(X * Y) = \bigoplus_{i+j=k-1} \tilde{H}_i(X) \otimes \tilde{H}_j(Y)$$

implies that $\eta(X * Y) = \eta(X) + \eta(Y)$.
Let Z be a simplicial complex on the vertex set W and let $\bigcup_{i=1}^m W_i$ be a partition of W. A simplex $\tau \in Z$ is *colorful* if $|\tau \cap W_i| = 1$ for all $1 \leq i \leq m$. The following Hall's type condition for the existence of colorful simplices appears in [1] and in [7].

Proposition 3.1 *If for all $I \subset [m]$*

$$\eta(Z[\bigcup_{i \in I} W_i]) \geq |I|$$

then Z contains a colorful simplex.

For completeness we reproduce a short proof from [7]. Suppose $\mathcal{U} = \{U_i\}_{i=1}^m$ is a family of subcomplexes of Z such that $\bigcup_{i=1}^m U_i = Z$. For $\sigma \subset [m]$ let $U_\sigma = \bigcap_{i \in \sigma} U_i$. Identify the nerve of $\mathcal{U}$ with the simplicial complex N on the vertex set $[m]$ whose simplices are all $\sigma \subset [m]$ such that $U_\sigma \neq \emptyset$. Let $N^{(k)}$ denote the k-dimensional skeleton of N. We shall use the following homology variant of Leray's Nerve Theorem (see [5, 7]).

Theorem 3.2 *If $\tilde{H}_j(U_\sigma) = 0$ for all $\sigma \in N^{(k)}$ and $0 \leq j \leq k - \dim \sigma$, then*
(i) $\tilde{H}_j(Z) \cong \tilde{H}_j(N)$ for $0 \leq j \leq k$.
(ii) If $\tilde{H}_{k+1}(N) \neq 0$ then $\tilde{H}_{k+1}(Z) \neq 0$.

$\square$

Proof of Proposition 3.1: Suppose Z does not contain a colorful simplex. Let $U_i = Z[\bigcup_{j \neq i} W_j]$, then $\mathcal{U} = \{U_i\}_{i=1}^m$ is a cover of Z. Let $\sigma \subset [m]$ then $U_\sigma = Z[\bigcup_{j \notin \sigma} W_j]$ hence by assumption $\tilde{H}_j(U_\sigma) = 0$ for $-1 \leq j \leq$

$m-|\sigma|-2 = m-\dim\sigma-3$. Therefore $\mathcal{U}$ meets the conditions of Theorem 3.2 with $k = m-3$. Since $\tilde{H}_{m-2}(Z) = 0$ it follows by 3.2(ii) that $\tilde{H}_{m-2}(N) = 0$. But N is clearly the $(m-2)$-skeleton of the $(m-1)$-simplex on the vertex set $[m]$, hence $\tilde{H}_{m-2}(N) \neq 0$, a contradiction.

$\square$

Remark: The first result on the existence of colorful simplices in simplicial complexes is Sperner's lemma. The relevance of Sperner's lemma and (implicitly) Proposition 3.1 to matching theory of hypergarphs was discovered in a recent remarkable paper by Aharoni and Haxell [1]. Further applications may be found in [7, 8].

4 Colorful Helly for Leray Complexes

We recall the combinatorial version of the Alexander Duality Theorem (see e.g. [10]). Let X be a simplicial complex on the vertex set V. The *Alexander Dual* of X is the simplicial complex on V given by

$$\mathrm{D}(X) = \{\tau \subset V \ : \ V - \tau \not\in X \} \ .$$

The homology groups of X and $\mathrm{D}(X)$ are related by the following

Theorem 4.1 *(Alexander Duality)*
If $V \not\in X$ then for all $-1 \leq i \leq |V|-2$

$$\tilde{H}_i(\mathrm{D}(X)) \simeq \tilde{H}_{|V|-i-3}(X) \quad .$$

Suppose $S \not\in \mathrm{D}(X)$ then $\overline{S} = V - S \in X$. Applying Alexander Duality to $\mathrm{D}(X)[S]$ we obtain

Corollary 4.2 *If $V \not\in X$ and $S \not\in \mathrm{D}(X)$ then for all $-1 \leq i \leq |S|-2$*

$$\tilde{H}_i(\mathrm{D}(X)[S]) \cong \tilde{H}_{|S|-i-3}(\mathrm{lk}(X, \overline{S})) \quad .$$

$\square$

Proof of Theorem 1.6: Suppose for contradiction that for all $\sigma \in X$

$$\rho(V - \sigma) \geq d+1 \ . \tag{1}$$

Let $V = \{v_1, \ldots, v_m\}$ be the vertex set of X and let $Y = \mathrm{D}(X)$. Let $V' = \{v_1', \ldots, v_m'\}$ be a disjoint copy of V and let M' be an isomorphic copy

of M on V'. Consider the join $Z = Y * M'$ with the vertex set $W = V \cup V'$. For $1 \leq i \leq m$ let $W_i = \{v_i, v_i'\}$. We now use Proposition 3.1 to show that Z contains a colorful simplex with respect to the partition $\{W_i\}_{i=1}^m$.
Let $I \subset [m]$ and let $S = \{v_i : i \in I\}$, $S' = \{v_i' : i \in I\}$. Then

$$\eta(Z[\bigcup_{i \in I} W_i]) = \eta(Y[\{v_i : i \in I\}] * M'[\{v_i' : i \in I\}]) =$$

$$\eta(Y[S]) + \eta(M'[S']) = \eta(Y[S]) + \eta(M[S]) \ . \qquad (2)$$

If $S \in Y$ then $Y[S]$ is contractible and $\eta(Y[S]) = \infty$. We thus assume that $S \notin Y$ hence $\sigma = V - S \in X$. By Alexander duality (Corollary 4.2)

$$\tilde{H}_i(Y[S]) \simeq \tilde{H}_{|S|-i-3}(\mathrm{lk}(X,\sigma)) \ .$$

Since $\tilde{H}_j((\mathrm{lk}(X,\sigma)) = 0$ for all $j \geq d$ it follows that $\tilde{H}_i(Y[S]) = 0$ for all $i \leq |S| - d - 3$. Thus

$$\eta(Y[S]) \geq |S| - d - 1 \ . \qquad (3)$$

On the other hand it is well known (see e.g. [5]) that for matroid complexes

$$\eta(M[S]) \geq \rho(S) \ . \qquad (4)$$

Combining (2),(3),(4) and (1) it follows that

$$\eta(Z[\bigcup_{i \in I} W_i]) = \eta(Y[S]) + \eta(M[S]) \ \geq$$

$$|S| - d - 1 + \rho(S) = |S| - d - 1 + \rho(V - \sigma) \geq |S| = |I| \ .$$

Proposition 3.1 now implies that there exists a simplex $\tau \in Z = Y * M'$ such that $|\tau \cap W_i| = 1$ for all $1 \leq i \leq m$. Therefore $A = \tau \cap V \in Y$ and $B = V - A \in M$. But $A \in Y$ implies $B = V - A \notin X$, in contradiction with $M \subset X$.

$\square$

Remark: It would be interesting to decide whether the stronger conclusion of Theorem 1.3 holds in the topological setting as well.

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