Combinatorics and Convexity

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Connections between Euclidean convex geometry and combinatorics go back to Euler, Cauchy, Minkowski and Steinitz. The theory was advanced greatly since the '50's and was influenced by the discovery of the simplex algorithm, the connections with extremal combinatorics, the introduction of methods from commutative algebra and the relations with complexity theory.

The first part of this paper deals with convexity in general and the second part deals with the combinatorics of convex polytopes. There are many excellent surveys [20, 9] and collections of open problems [13, 29]. I try to discuss several specific topics and to zoom in on issues which I am more familiar with.

1 Convex sets in general

1.1 Covering, Packing and Tiling

Borsuk conjectured (1933) that every bounded set in $\mathbf{R}^d$ can be covered by $d+1$ sets of smaller diameter. Kahn and Kalai [22] showed that Borsuk's conjecture is very false in high dimensions.

Here is the disproof of Borsuk's conjecture. Let $f(d)$ be the smallest integer such that every bounded set in $\mathbf{R}^d$ can be covered by $f(d)$ sets of smaller diameter. For a bounded metric space X, let $b(X)$ be the minimum number of sets of smaller diameter needed to cover X. Consider $\mathbf{P}^{d-1}$ the space of lines through the origin in $\mathbf{R}^d$ where the metric is given by the angle between two lines. The diameter of $\mathbf{P}^{d-1}$ is $\pi/2$ and the distance between two lines is $\pi/2$ iff they are orthogonal. Let $d = 4p$, p a prime. Frankl and Wilson [17] , see also [39, 16] proved that there are at most 1.8^d vectors in $\{-1,+1\}^d$ such that no two are orthogonal. This yields $b(\mathbf{P}^{d-1}) > 1.1^d$, since if $\mathbf{P}^{d-1}$ is covered by t sets of smaller diameter, each such set contains at most 1.8^d of the lines spanned by the vectors in $\{-1,+1\}^d$. But there are 2^{d-1} such lines and therefore $t \geq (2/1.8)^d$. Now, embed $\mathbf{P}^{d-1}$ into $\mathbf{R}^{d^2}$ by the map $x \to x \otimes x$, where x is a vector of norm 1 in $\mathbf{R}^d$. Note [1] that $< x \otimes x, y \otimes y >=< x, y >^2$. Therefore, the order relation between distances is preserved, and the image of $\mathbf{P}^{d-1}$ is the required counterexample. This example gives $f(d) > 1.2^{\sqrt{d}}$, for sufficiently large d.

Betke, Henk and Wills [7] proved for sufficiently high dimensions Fejes Toth's sausage conjecture. They showed that the minimum volume of the convex hull of n nonoverlapping congruent balls in $\mathbf{R}^d$ is attained when the centers are on a line.

Keller conjectured (1930) that in every tiling of $\mathbf{R}^d$ by cubes there are two cubes which share a complete facet. Lagarias and Shor [30] showed this to be false for $d \geq 10$. They used a reduction to a purely combinatorial problem which was found by Corŕadi and Szabó.

[1] If $x = (x_1, x_2 \ldots, x_d)$ and $y = (y_1, y_2 \ldots, y_k)$, you can regard $x \otimes y$ as the $d \times k$ matrix whose (i,j)-entry is $x_i \cdot y_j$.

Some problems

There are many problems on packing, covering and tiling and the most famous are perhaps the sphere packing problem in $\mathbf{R}^3$ and the (asymptotic) sphere packing problem in $\mathbf{R}^d$. There are several open problems around Borsuk's problem. What is the asymptotic behavior of $f(d)$? What is the situation in low dimensions? What is the behavior of $b(\mathbf{P}^n)$? Witsenhausen conjectured (see, [16]) that if A is a subset of the unit sphere without two orthogonal vectors, then $vol(A) \leq 2v_{\pi/4}$, where $v_{\pi/4}$ is the volume of a spherical cap of radius $\pi/4$. This would imply that $b(\mathbf{P}^n) \leq (\sqrt{2}+o(1))^n$. Perhaps the algebraic methods used for the Frankl-Wilson theorem can be of help.

Schramm [41] proved an upper bound $f(d) \leq s(d) = (\sqrt{3/2}+o(1))^d$. He showed that every set of constant width can be covered by $s(d)$ smaller homothets. Bourgain and Lindenstrauss [12] proved the same bound by covering every boundede set by $s(d)$ balls of the same diameter. (Danzer already showed that exponential number of balls is sometimes necessary.) In his proof Schramm related the value of $f(d)$ with another classical problem in convexity, that of finding or estimating the minimal volume of Euclidean (and more generally spherical) sets of constant width.

It is not known if there are sets of constant width 1 in $\mathbf{R}^d$ whose volume is *exponentially* smaller than the volume of a ball of radius 1/2. Perhaps the following series of examples (suggested by Schramm) $K_d \subset \mathbf{R}^d$ will do, but we do not know to compute or estimate their volumes. $K_0 = 0$ and K_{d+1} is obtained as follows. Consider K_d as sitting in the hyperplane given by $x_{d+1} = 0$ in $\mathbf{R}^{d+1}$. Now, take $K_{d+1} = A_{d+1} \cup B_{d+1}$ where A_{d+1} is the set of all points z with $x_{d+1} \geq 0$ such that the ball of radius 1 around z contains K_d and B_{d+1} is the set of all points z with $x_{d+1} \leq 0$ which belong to every ball of radius 1 which contains K_d. Schramm also conjectured that the minimal volume of a spherical set of constant width $\pi/4$ is obtained for an orthant.

Finally, what is the minimal diameter d_n such that the unit n-ball can be covered by $n+1$ sets of diameter d_n? It is known that $2 - O(\log n/n) \leq d_n \leq 2 - O(1/n)$, see [31]. Hadwiger conjectured that the upper bound (which corresponds to the standard symmetric decomposition of the ball to $n+1$ regions,) is the truth. Perhaps also here the natural conjecture is false?

1.2 Helly-type theorems

Tverberg's theorem

Sarkaria [40] found a striking simple proof of the following theorem of Tverberg: [49]

Every $(d+1)(r-1)+1$ points in $\mathbf{R}^d$ can be partitioned into r parts such that the convex hulls of these parts have nonempty intersection.

He used the following result of Barany [2]. *Let $A_1, A_2, \ldots, A_{d+1}$ be sets in $\mathbf{R}^d$ such that $x \in conv(A_i)$ for every i. Then it is possible to choose $a_i \in A_i$ such that $x \in conv(a_1, a_2, \ldots, a_{d+1})$.* (To prove this consider the minimal distance t between x and such $conv(a_1, a_2, \ldots, a_{d+1})$ and show that if $t > 0$ one of the a_i's can be replaced to decrease t.)

Now consider $m = (d+1)(r-1)+1$ points $a_1, a_2, \ldots a_m$ in $\mathbf{R}^d$ and regard them as points in $V = \mathbf{R}^{d+1}$ whose sum of coordinates is 1. Sarkaria's idea was to consider the tensor product $V \otimes W$ where W is a $(r-1)$-dimensional space spanned by r vectors $w_1, w_2, \ldots, w_r$ whose sum is zero. Next define m $(= \; dim V \otimes U - 1)$ sets in $V \otimes U$ as follows:

$$A_i = \{a_i \otimes w_1, a_i \otimes w_2, \ldots a_i \otimes w_r\}.$$

Note that 0 is in the convex hull of each A_i and by Barany's theorem $0 \in conv\{a_1 \otimes w_{i_1}, a_2 \otimes w_{i_2}, \ldots, a_m \otimes w_{i_m}\}$, for some choices of $i_1, i_2, \ldots, i_m$. The required partition of the points is given by $\Omega_j = \{a_k : i_k = j\}$, $j = 1, 2, \ldots, r$. To see this write $0 = \sum \lambda_k a_k \otimes w_{i_k}$, where the coefficients λ_k are nonnegative and sum to 1. Deduce that the vectors $v_j = \sum_{k \in \Omega_j} \lambda_k a_k$, $1 \leq j \leq r$, are all equal and so are the scalars $\alpha_j = \sum_{k \in \Omega_j} \lambda_k$.

There are many beautiful problems and results concerning Tverberg's theorem, see [15]. Topological versions were found for the case where r is a prime [3] and were extended to derive colored versions of Tverberg's theorem [52]. Sierksma conjectured, see [50], that the number of Tverberg partitions is at least $(r-1)!^d$. For a finite set A in $\mathbf{R}^d$ let $f(A, r) = \max\{\dim \cap_{i=1}^{r} conv(\Omega_i)\}$, where the maximum is taken over all partitions $(\Omega_1, \Omega_2, \ldots, \Omega_r)$ of A.

Conjecture: $\sum_{r=1}^{|A|} f(A, r) \geq 0$. (Note: $\dim \emptyset = -1$.)

This extension of Tverberg's theorem was proved by Kadari for planar sets.

The Hadwiger-Debrunner Piercing Conjecture

Alon and Kleitman [1] proved the Hadwiger-Debrunner Piercing Conjecture.

For every d and every $p \geq d+1$ there is a $c = c(p, d) < \infty$ such that the following holds. For every family H of compact, convex sets in $\mathbf{R}^d$ in which any set of p members of the family contains a subset of cardinality $d+1$ with a nonempty intersection there is a set of c points in $\mathbf{R}^d$ that intersects each member of H.

Helly's theorem asserts that $c(d+1, d+1) = 1$ and it is not difficult to see that $c(p, 1) = p - 1$. We describe the proof for the first (typical) case $d = 2, p = 4$ We are given a family of n planar convex sets and out of every four sets in the family we can nail three with a point. We want to nail the entire family with a fixed number of points. The first step is to show that there is a way to nail a constant fraction (independent from n) of the sets with one point. This follows from a "fractional Helly theorem" of Katcalski and Liu. A more sophisticated use of the Katcalski Liu theorem shows that for every assignment of nonnegative weights to the sets in the family we can nail with one point sets representing a constant proportion of the entire weight. Using linear programming duality Alon and Kleitman proceeded to show that there is a collection Y of points (their number may depend on n) such that every set in the family is nailed by a constant fraction of the points in Y. The final step, replacing Y with a set of bounded cardinality which meets all the sets in the family is done using the theorems of Barany and Tverberg mentioned above.

2 Convex polytopes

2.1 Polytopes, spheres and Steinitz theorem

Convex polytopes are among the most ancient mathematical objects of study. The combinatorial theory of polytopes is the study of their face-structure and in particular their face numbers. There is also a developed metric theory of polytopes (problems concerning volume, width, sections, projections etc.) and arithmetic theory (lattice points in polytopes). These three aspects of convex polytopes are related and some of the algebraic tools mentioned below are relevant to all of them.

A convex d-dimensional polytope (briefly, a d-polytope) is the convex hull of a finite set of points which affinely span $\mathbf{R}^d$. A (nontrivial) face of a d-polytope P is the intersection of P with a supporting hyperplane. The empty set and P itself are regarded as trivial faces. 0-faces are called *vertices*, 1-faces are called *edges* and $(d-1)$-faces are called *facets*. The set of faces of a polytope is a graded lattice. Two polytopes P and Q are *combinatorially isomorphic* if there is an order preserving bijection between their face lattices. P and Q are *dual* if there is an order reversing bijection between their face lattices.

Simplicial polytopes are polytopes all whose proper faces are simplices. Duals of simplicial polytopes are called *simple* polytopes. A d-polytope P is simple iff every vertex of P belongs to d edges. Denote by $f_i(P)$ the number of i-faces of P. The vector $(f_0(P), f_1(P), \ldots, f_d(P))$ is called the f-vector of P. Euler's famous formula $V - E + F = 2$ is the beginning of a rich theory on face-numbers of convex polytopes and related combinatorial structures.

The wild behavior for $d \geq 4$

The boundary of every simplicial d-polytope is a triangulation of a $(d-1)$-sphere, but there are triangulations of $(d-1)$-spheres which cannot be realized as boundary complexes of simplicial polytopes (for $d \geq 4$). Goodman and Pollack [19] proved that the number of combinatorial types of polytopes is surprisingly small. The number of d-polytopes with 1,000,000 vertices (in any dimension) is bounded above by $2^{2^{70}}$ while the number of triangulations of spheres with 1,000,000 vertices is between $2^{2^{692,225\pm 25}}$. (This is achieved for $d \sim 552,786$.) There are combinatorial types of convex polytopes that cannot be realized by points with rational coordinates ([21, 51]) and there are polytopes which have a combinatorial automorphism which cannot be realized geometrically and whose realization space is not connected. Mnev [38] showed that for every simplicial complex C, there is a polytope whose realization space is homotopy equivalent to C. Recently, Richter announced that all these phenomena occur already in dimension 4, that all algebraic numbers are needed to coordinatize all 4-polytopes and that there is a non-rational 4-polytope with 34 vertices.

The tame behavior for $d = 3$

All these "pathologies" do not occur for 3-polytopes by a deep theorem of Steinitz asserting that every 3-connected planar graph is the graph of a polytope and related theorems. Relatives of Koebe-Andreev-Thurston circle packing theorem provide new approach to Steinitz theorem, see [42]. Andreev and Thurston proved that there is a realization of every 3-polytope P such that all its edges are tangent to the unit ball, and this realization is unique up to projective transformations preserving the unit sphere. Schramm observed that by choosing the realization so that the hyperbolic center of the tangency points of edges with the unit sphere is at the origin, you get the following result: (Which answers a question of Grünbaum, and extends a result of Mani.)

Let P be a 3-polytope, and let ? be the group of combinatorial isomorphisms of the pair (P, P^*), where P^* is the dual of P. (In other words, each element of ? is either a combinatorial automorphism of P or an isomorphism from P to P^*.) Then there is a realization of the polyhedron so that every element of ? is induced by a congruence.

An open problem of Perles is whether every combinatorial automorphism ϕ of a centrally symmetric d-polytope (P is centrally symmetric if $x \in P$ implies $-x \in P$) satisfies $\phi(-v) = -\phi(v)$.

2.2 Face numbers and h-numbers of simplicial polytopes

The upper bound theorem and the lower bound theorem

Motzkin conjectured in 1957 and McMullen proved in 1970 [37] the *upper bound theorem*: Among all d-polytopes with n vertices the *cyclic* polytope has the maximal number of k-faces for every k. The cyclic d-polytope with n vertices is the convex hull of n points on the *moment curve* $x(t) = (t, t^2, \ldots, t^d)$. Cyclic d-polytopes have the remarkable property that every set of k vertices determines a $(k-1)$-face for $1 \leq k \leq [d/2]$.

Klee proved in 1964 the assertion of the upper bound theorem when n is large w.r.t. d for arbitrary *Eulerian complexes*, namely $(d-1)$-dimensional simplicial complexes such that the link of every r-face has the same Euler characteristics as a $(d-r-1)$-sphere. The assertion of the upper bound theorem for arbitrary Eulerian complexes (even manifolds) is still open.

Brückner conjectured in 1909 and Barnette [4] proved in 1970 the *lower bound theorem*: The minimal number of k-faces for simplicial d-polytopes with n vertices is attained for *stacked* polytopes. Stacked polytopes are those polytopes built by gluing simplices along facets.

The g-conjecture

Let $d > 0$ be a fixed integer. Given a sequence $f = (f_0, f_1, \ldots, f_{d-1})$ of nonnegative integers, put $f_{-1} = 1$ and define $h[f] = (h_0, h_1, \ldots, h_d)$ by the relation

$$\textstyle\sum_{k=0}^{d} h_k x^{d-k} = \sum_{k=0}^{d} f_{k-1}(x-1)^{d-k}.$$

If $f = f(K)$ is the f-vector of a $(d-1)$-dimensional simplicial complex K then $h[f] = h(K)$ is called the *h-vector* of K. The h-vectors are of great importance in the combinatorial theory of simplicial polytopes. The upper bound theorem and the lower bound theorem have simple forms in terms of the h-numbers. The upper bound theorem follows from the inequality $h_k \leq \binom{n-d+k-1}{k}$. The lower bound theorem amounts to the relation $h_1 \leq h_2$. The Dehn-Sommerville relations for the face numbers of simplicial polytope assert that $h_k = h_{d-k}$.

In 1970 McMullen proposed a complete characterization of f-vectors of boundary complexes of simplicial d-polytopes. McMullen's conjecture was settled in 1980. Billera and Lee [8] proved the sufficiency part of the conjecture and Stanley [44] proved the necessity part. Recently, McMullen [35, 36] found an elementary proof of the necessity part of the g-theorem.

McMullen conjecture, now called the *g-theorem* asserts that $(h_0, h_1, \ldots, h_d)$ *is the h-vector of a simplicial d-polytope if and only if the following conditions hold: (a)* $h_i = h_{d-i}$, *(b) there is a graded standard algebra* $M = \oplus_{i=0}^{d/2} M_i$ *such that* $dim M_i = h_i - h_{i-1}$, *for* $0 \leq i \leq [d/2]$. (A graded algebra is standard if it is generated as an algebra by elements of degree 1.)

The second condition was originally given in purely combinatorial terms which is equivalent to the formulation given here by an old theorem of Macaulay. In the rest of this section we will describe methods used to attack the upper and lower bound theorems and the g-conjecture.

It is conjectured that the assertion of the g-theorem applies to *arbitrary simplicial spheres.*

Shellability and the h-vector

A *shelling* of a simplicial sphere is a way to introduce the facets (maximal faces) one by one so that at each stage you have a topological ball until the last facet is introduced and you get the entire sphere. Let P be a simplicial polytope and let P^* be its polar (which is a simple polytope). A shelling order for the facets of P is obtained simply by ordering the vertices of P^* according to some linear objective function ϕ . The number h_k has a simple interpretation as the number of vertices v of P^* of degree k where the degree of a vertex is the number of its neighboring vertices with lower value of the objective function. Switching from ϕ to $-\phi$ we get the Dehn-Sommerville relations $h_k = h_{d-k}$. (Including the Euler relation for $k = 0$.)

We are ready to describe McMullen's proof of the upper bound theorem (in a dual form). Consider a linear objective function ϕ which gives higher values to vertices in a facet F than to all other vertices to obtain that (*) $h_{k-1}(F) \le h_k(P)$. Next,

$$(**) \sum h_k(F) = (k+1)h_{k+1}(P) + (d-k)h_k(P),$$

where the sum is over all facets F of P. To see this note that every vertex of degree k in P has degree $k-1$ in k facets containing v and degree k in the remaining $d-k$ facets. (*) and (**) gives the upper bound relations $h_k \le \binom{n-d+k-1}{k}$ by induction on k.

Cohen-Macaulay rings

Stanley, see [46], proved the upper bound theorem for arbitrary simplicial spheres using the theory of Cohen-Macaulay rings. Let K be a $(d-1)$-dimensional simplicial complex on n vertices $x_1, \dots x_n$. The *face-ring* $R(K)$ of K is the quotient $\mathbf{R}[x_1, x_2 \dots x_n]/I$ were I is the ideal generated by non-faces of K. (Namely, I is generated by monomials of the form $x_{i_1} \cdot x_{i_2} \cdots x_{i_m}$ where $[x_{i_1}, x_{i_2}, \dots, x_{i_m}]$ is *not* a face of K.) $R(K)$ is a *Cohen-Macaulay* ring if it decomposes into direct sum of (translation of) polynomial rings as follows: there are elements of $R(K)$, $\theta_1, \theta_2, \dots, \theta_d$ and $\eta_1, \eta_2 \dots, \eta_t$ such that

$$R(K) = \oplus_{i=1}^{t} \eta_i \mathbf{R}[\theta_1, \theta_2, \dots \theta_d].$$

It turns out that the θ's can be chosen as linear combinations of the variables and when this is the case the number of η's of degree i is precisely h_i. Reisner found topological conditions for the Cohen-Macaulayness of $R(K)$ which imply that $R(K)$ is Cohen-Macaulay when K is a simplicial sphere. All this implies the upper bound inequalities for the h numbers since (roughly) after moding out by d linear forms the dimension of the space of homogeneous polynomials of degree k (from which the $\eta's$ are taken) is $\binom{n-d+k-1}{k}$.

Toric varieties

For every rational d-polytope P one associates an algebraic variety $T(P)$ of dimension $2d$. If P has n vertices $v_1, v_2, \dots, v_n$ then consider n complex variables $z_1, \dots z_n$ and replace each affine relation with integer coefficients $\sum n_i v_i = 0$, where $\sum n_i = 0$, by the polynomial relation $\prod z_i^{n_i} = 1$. When P is simplicial Danilov proved that the $2i$-th Betti number of $T(P)$ is h_i. This enabled Stanley [44] to prove the necessity part of the g-conjecture via the Hard-Lefschetz theorem for $T(P)$.

Rigidity and stresses

Let P be a simplicial d-polytope, $d \ge 3$. then P is *rigid*. Namely, every small perturbation of the vertices of P which does not change the length of the edges of P is induced by an affine rigid motion of $\mathbf{R}^d$. The rigidity of simplicial 3-polytopes follows from Cauchy's rigidity theorem which asserts that if two combinatorially isomorphic convex polytopes have pairwise congruent 2-faces then they are congruent. (It follows also from Dehn's infinitesimal rigidity theorem for simplicial 3-polytope.) There is a simple inductive argument on the dimension to prove rigidity of simplicial d-polytopes starting with the case $d = 3$. If P is a simplicial d-polytope with n vertices, there are dn degrees of freedom to move the vertices and the dimension of the group of rigid motions of $\mathbf{R}^d$ is $\binom{d+1}{2}$. Therefore the rigidity of P implies the lower bound inequality $f_1(P) \geq dn - \binom{d+1}{2}$ This observation gives also various extensions of the lower bound theorem, see Kalai [24]. Lee [33] extended this idea to higher h-numbers and found relations to the face-ring.

The algebra of weights

A remarkable recent development is McMullen's elementary proof of the necessity part of the g-conjecture [35, 36]. McMullen proved in fact the assertion of the the Hard-Lefschetz theorem and his proof applies to non-rational simplicial polytopes. (There, the toric varieties do not exist but the assertion of the Hard-Lefschetz theorem in terms of the face-ring still makes sense.) McMullen defines r-weights of simple d-polytopes to be an assignment of weights $w(F)$ to each r-face F such that in each $(r+1)$-face G, $\sum w(F)u_{F,G} = 0$, where the sum is taken over all r-faces F of G and $u_{F,G}$ is the outer normal of F in G. Let $\Omega_r(P)$ denote the space of r-weights of the polytope P. A well known theorem of Minkowski asserts that assigning to an r-face its r-dimensional volume is an r-weight. These special weights have a central role in the proof.

McMullen's proof proceeds in the following steps: 1. He defines an algebra structure on weights and show that this algebra is generated by 1-weights. 2. He proves that $dim\Omega_r(P) = h_r(P)$. 3. He considers the special 1-weight ω which assigns to each edge its length and proves that $\omega^{d-2r} : \Omega_r \to \Omega_{d-r}$ is an isomorphism. To show this McMullen computes the signature of the quadratic form $\omega^{d-2r}x^2$ on $\Omega_r(P)$. This is achieved via new geometric inequalities of Brunn-Minkowski type.

Algebraic shifting

Algebraic shifting, introduced by Kalai in [23], is a way to assign to every simplicial complex K an auxiliary simplicial complex $\Delta(K)$ of a special type. The vertices of $\Delta(K)$ are $v_1, v_2, v_3, \ldots$ and the r-faces of $\Delta(K)$ respect a certain partial order. Namely, if $S = (v_{i_0}, v_{i_1}, \ldots, v_{i_r})$ form an r-face of $\Delta(K)$ then if one of the vertices v_j of S is replaced with a vertex v_i with $i < j$ this results also with a face of $\Delta(K)$. (For example, if (v_3, v_7) is a 1-face of $\Delta(K)$ then so is (v_3, v_5).) The definition of $\Delta(K)$ is given by a certain generic change of basis for the cochain groups of K, see [10].

Various combinatorial and topological properties of simplicial complexes are preserved by the operation $K \to \Delta(K)$. $\Delta(K)$ has the same f-vector as K. $\Delta(K)$ also have the same Betti numbers as K but other homotopical information is eliminated as $\Delta(K)$ has the homotopy type of a wedge of spheres. K has the Cohen-Macaulay property (its face-ring is Cohen-Macaulay) iff $\Delta(K)$ has.

What is still missing is the relation of algebraic shifting with embeddability in $\mathbf{R}^n$. It is a well known fact that K_5, the complete graph with five vertices, cannot be embedded in the plane. More generally, van-Kampen and Flores proved that σ_r^{2r+2}, the r-skeleton of the $(2r+2)$-simplex, cannot be embedded in $\mathbf{R}^{2r}$. Kalai and Sarkaria propose

Conjecture: σ_r^{2r+2} *is not contained in* $\Delta(K)$ *whenever* K *is embeddable in* $\mathbf{R}^{2r}$.

This conjecture would imply the assertion of the g-theorem for arbitrary simplicial spheres.

2.3 Other topics

Flag numbers and and other invariants of general polytopes

Flag numbers of polytopes count chains of faces of prescribed dimensions. There are 2^d flag numbers but Bayer and Billera [5] showed that the affine space of flag numbers of d-polytopes has dimension $c_d - 1$ where c_d is the d-th Fibonacci number. Toric varieties supply interesting invariants of general polytopes. It turns out that the dimensions of their (middle perversity) intersection homology groups are linear combinations of flag numbers. See, [45, 25]. There are mysterious connections between these invariants of a polytope P and its dual P^* (See [24] Sec. 12, [6, 47].) Another remarkable invariant of general polytopes (and Eulerian posets) which was defined by Fine is the CD-index, see [6, 48].

The following very simple problem is open: *Show that a centrally symmetric polytope P in $\mathbf{R}^d$ must have at least 3^d nonempty faces.*

Reconstruction theorems

Whitney proved that the graph of a 3-polytope determines its face structure. The 2-faces of the polytope are given by the induced cycles which do not separate the graph. This can be extended to show that the $(d-2)$-skeleton of a d-polytope determines the face structure and for general polytopes this cannot be improved. (See [21] Ch. 12.) Perles proved that the $[d/2]$-skeleton of a *simplicial* d-polytope determines the face structure, and Dancis [14] extended this result to arbitrary simplicial spheres. Perles conjectured and Blind and Mani [11] proved that the face structure of every simple d-polytope is determined by the graph (1-skeleton) of the polytope. For a simple proof see, Kalai [26]. Consider a simplicial $(d-1)$-dimensional sphere and a *puzzle* in which the pieces are the facets and for each piece there is a list of the d neighboring pieces. The Blind-Mani theorem asserts that for boundary complexes of simplicial polytopes (and for a certain class of shellable spheres) the puzzle has only one solution. Conjecture: *For an arbitrary simplicial sphere the puzzle has a unique solution.* Perhaps the machinery of Cohen-Macaulay rings can be of help.

Polytopes of triangulations

Lee [32] and Haiman proved that the set of triangulations of the regular n-gon with non-crossing diagonals corresponds to the vertices of an $(n-3)$-dimensional polytope. The r-faces of this polytope correspond to all triangulations containing a given set of $n-3-r$ diagonals. Independently, (as part of a theory of generalized hypergeometric functions) Gelfand, Kapranov and Zelevinsky [18] defined a much more general objects called "secondary polytopes", which correspond to certain triangulations of arbitrary polytopes, and further extensions were given by several authors, including Billera and Sturmfelds "Fiber polytopes". It looks now that these constructions are quite fundamental in convex polytope theory and the reader is referred to Zeigler's book [51]. In another independent development Slater, Tarjan and Thurston [43] proved a sharp lower bound on the (combinatorial) diameter of the associahedron using volume estimates of hyperbolic polytopes.

2.4 The simplex algorithm and the diameter of graphs of polytopes

The simplex algorithm solves linear programming problems by moving from vertex to vertex of a polytope (the set of feasible solutions) along its edges. Let $\Delta(d, n)$ be the maximum diameter of the graphs of d-polytopes P with n facets. It is not known if $\Delta(d,n)$ is bounded above by a linear function of d and n, or even by a polynomial function of d and n. In 1970 Larman proved that $\Delta(d,n) \leq 2^{d-3}n$. Recently, quasi-polynomial bounds where found, see Kalai and Kleitman [28] for a simple proof for $\Delta(d,n) \leq n^{\log d+1}$. All the known upper bounds use only the facts that the intersection of faces of a polytope is a face and that the graph of every face is connected.

Consider a linear programming problem with d variables and n constraints. Given the fact that the diameter of the feasible polytope is relatively small, the next step would be to find a pivot rule for linear programming which requires for *every* linear programming problem a subexponential number of pivot steps. Here, we assume, that each individual pivot step should be performed by a polynomial number of arithmetic operations in d and n. However, no such pivot rule is known. Recently, Kalai [27] and independently Matousek, Sharir and Welzl [34] found a *randomized* pivot rule such that the *expected* number of pivot steps needed is at most $exp(c\sqrt{d}\log n)$.

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