INFLUENCES OF VARIABLES AND THRESHOLD INTERVALS UNDER GROUP SYMMETRIES

J. Bourgain[1]   G. Kalai[2]

0. Introduction.

A subset A of $\{0,1\}^n$ is called monotone provided if $x \in A, x' \in \{0,1\}^n$, $x_i \le x'_i$ for $i = 1, \ldots, n$ then $x' \in A$. For $0 \le p \le 1$, define μ_p the product measure on $\{0,1\}^n$ with weights $1-p$ at 0 and p at 1. Thus

$$\mu_p(\{x\}) = (1-p)^{n-j}p^j \quad \text{where} \quad j = \#\{i = 1, \ldots, n | x_i = 1\}. \tag{0.1}$$

If A is monotone, then $\mu_p(A)$ is clearly an increasing function of p. Considering A as a "property", one observes in many cases a threshold phenomenon, in the sense that $\mu_p(A)$ jumps from near 0 to near 1 in a short interval when $n \to \infty$. Well known examples of these phase transitions appear for instance in the theory of random graphs. A general understanding of such threshold effects has been pursued by various authors (see for instance Margulis [M] and Russo [R]). It turns out that this phenomenon occurs as soon as A depends little on each individual coordinate (Russo's zero-one law). A precise statement was given by Talagrand [T] in the form of the following inequality.

Define for $i = 1, \ldots, n$

$$A_i = \{x \in \{0,1\}^n | x \in A, \, U_i x \notin A\} \tag{0.2}$$

where $U_i(x)$ is obtained by replacement of the i^{th} coordinate x_i by $1 - x_i$ and leaving the other coordinates unchanged. Let

$$\gamma = \sup_{i=1,\ldots,n} \mu_p(A_i). \tag{0.3}$$

[1]Institute for Advanced Study, Princeton, NJ 08540, USA
[2]Hebrew University of Jerusalem, Jerusalem, Israel

Typeset by $\mathcal{AMS}$-TEX

Then
$$\frac{d\mu_p(A)}{dp} \geq c \frac{\log(1/\gamma)}{p(1-p)\log[2/p(1-p)]}\, \mu_p(A)\ [1-\mu_p(A)] \tag{0.4}$$
where $c > 0$ is some constant.

Defining for $i = 1, \ldots, n$ the functions
$$\varepsilon_i(x) = 2x_i - 1 \tag{0.5}$$
one gets
$$\frac{d\,\mu_p}{d\mu_{1/2}} = \otimes_{i=1}^n\ [1 + (2p-1)\varepsilon_i] \tag{0.6}$$

$$\begin{aligned} \frac{d\mu_p(A)}{dp} &= 2/p \sum_{i=1}^n \int \chi_A(x)\varepsilon_i(x)\ \ \otimes_{j\neq i}\ [1+(2p-1)\varepsilon_j]\mu_{\frac{1}{2}}(dx) \\ &= 2/p \sum_{i=1}^n \mu_p(A_i). \end{aligned} \tag{0.7}$$

The number $\mu_p(A_i)$ is the influence of the i^{th} coordinate (with respect to μ_p) and the right side of (0.7) represents thus the sum of the influences. Hence a small threshold interval corresponds to a large sum of influences. Relation (0.7) is due to Margulis and Russo. In [T], (0.4) is deduced from an inequality of the form
$$\mu_p(A)[1-\mu_p(A)] \leq C(p) \sum_{i=1}^n\ \frac{\mu_p(A_i)}{\log[1/\mu_p(A_i)]}. \tag{0.8}$$
This last inequality and its proof relies on the paper by Kahn, Kalai and Linial [KKL], where it is shown that always
$$\sup_{1\leq i\leq n}\ \mu_{1/2}(A_i) \geq c\, \frac{\log n}{n}. \tag{0.9}$$
Friedgut and Kalai [FK] used an extension of (0.9) given in [BKKKL] to show that for properties which are invariant under the action of a transitive permutation group the threshold interval is $O(1/\log n)$ and proposed some conjectures on the dependence of the threshold interval on the group.

Our aim here is to obtain a refinement and strengthening of the preceding in the context of "G-invariant" properties. Let f be a $0,1$-valued function on $\{0,1\}^n$ and G a subgroup of the permutation group on n elements $\underline{n} = \{1, 2, \dots, n\}$. Say that f is G-invariant provided

$$f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)}) \text{ for all } x \in \{0,1\}^n, \pi \in G. \tag{0.10}$$

Given G, define for $1 \leq t \leq n$

$$\phi(t) = \phi_G(t) = \min_{S \subset \underline{n}, |S|=t} \log(\#\{\pi(S) | \pi \in G\}) \tag{0.11}$$

and for all $\tau > 0$

$$a_\tau(G) = \sup\{\phi(t) | \phi(t) > t^{1+\tau}\}. \tag{0.12}$$

Observe that since $\phi(t) \leq \log \binom{n}{t}$, necessarily $a_\tau(G) \lesssim (\log n)^{1/\tau}$.

Theorem 1. *Assume G transitive and A a monotone G-invariant property. Then for all $\tau > 0$*

$$\frac{d\mu_p(A)}{dp} > c_\tau a_\tau(G)\, \mu_p(A)[1 - \mu_p(A)] \tag{0.13}$$

provided $p(1-p)$ stays away from zero in a weak sense say

$$\log[p(1-p)]^{-1} \lesssim \log \log n. \tag{0.14}$$

It follows that in particular the threshold interval is at most

$$C_\tau a_\tau(G)^{-1} \quad \textit{for all} \quad \tau > 0. \tag{0.15}$$

Previous results as mentioned above only yield estimates of the form $(\log n)^{-1}$ and the main point of this work is to provide a method going beyond this.

Theorem 1 is deduced from (0.7) and the following fact, independent of monotonicity assumptions.

Theorem 2. *Assume A G-invariant and (0.14) holds. Then for all $\tau > 0$*

$$\sum \mu_p(A_i) > c_\tau a_\tau(G)\mu_p(A)[1-\mu_p(A)]. \tag{0.18}$$

Comments.

(1) For every group G one can always exhibit a G-invariant property with threshold interval $\geq \frac{1}{a(G)}$. For primitive permutation groups except when $G = S_n$ is the full permutation group (or equivalently the alternating group), the result is nearly optimal in the sense that one may always exhibit a G-invariant property with threshold interval $\sim \frac{1}{a(G)}$. For $G = S_n, A_n$ a threshold interval $\lesssim \frac{1}{\sqrt{n}}$ is obtained,(*) while Theorem 1 only $(\log n)^{-M}$ (M arbitrary) predicts. In fact, the analysis as presented below permits to take $\tau \sim \frac{1}{\sqrt{\log t}}$ but may conceivably be further refined. It is possible that the theorem holds for $\tau \sim \frac{\log t}{t}$ and even an improvement to $\tau \sim \frac{1}{\log t}$ will imply a precise (up to a multiplicative constant) answer for all primitive groups.

(2) In the particular case of monotone graph properties on N vertices, we get $n = \binom{N}{2}$ and G is induced by permuting the vertices. One gets essentially

$$\phi(t) \sim \log\binom{N}{\sqrt{t}} \tag{0.16}$$

in this situation and the conclusion of Theorem 1 is that any threshold interval is at most $C_\tau(\log N)^{-2+\tau}$, $\tau > 0$. This is essentially the sharp result, since, fixing $M \sim \log N$, the property for a graph on N vertices to contain a clique of size M yields a threshold interval $\sim (\log N)^{-2}$.

For primitive permutation groups Theorem 1 implies a close to complete description of the possible threshold interval of a G-invariant property, depending on the structure of G. (Recall that a permutation group $G \subset S_n$ is primitive if it is impossible to partition $\underline{n}$ to blocks $B_1, \dots B_t$, $t > 1$ so that every element in G permute the blocks among themselves.)

(*) Consider for instance $A = \{x \in \{0,1\}^n \mid \sum x_i > \frac{n}{2}\}$, with threshold interval $\sim \frac{1}{\sqrt{n}}$

It turns out that there are some gaps in the possible behaviors of the largest threshold intervals. This interval is proportional to $n^{-1/2}$ for S_n and A_n but at least $\log^{-2} n$ for any other group. The worst threshold interval can be proportional to $\log^{-c} n$ for c belonging to arbitrary small intervals around the following values: $2, 3/2, 4/3, 5/4 \ldots$ or for c which tends to zero as a function of n in an arbitrary way. This (and more) is summarized in the next theorem. First we need a few definitions. For a permutation group $G \subset S_n$ let

$$T_G(\epsilon) = sup\{q - p : \mu_p(A) = \epsilon, \mu_q(A) = 1 - \epsilon\},$$

where the supremum is taken over all monotone subsets of $\{0,1\}^n$ which are invariant under G. A composition factor of group G is a quotient group H/H' where H is a normal subgroup of G and H' is a normal subgroup of H. A section of G is a quotient H/H' where H is an arbitrary subgroup of G and H' is a normal subgroup of H.

Theorem 3.

Let $G \subset S_n$ be a primitive permutation group.

1. If $G = S_n$ or $G = A_n$ then $T_G(\epsilon) = \log(1/\epsilon)/n^{1/2}$.

2. If $G \neq S_n, A_n$, $T_G(\epsilon) \geq c_1 \log(1/\epsilon)/\log^2 n$.

3. For every integer $r > 0$ and reals $\delta > 0, \epsilon > 0$ if $T_G(\epsilon) \leq c_2 \log(1/\epsilon)/(\log n)^{(1+1/(r+1))}$ then already $T_G(\epsilon) \leq c_3(\delta) \log(1/\epsilon)/(\log n)^{(1+1/r-\delta)}$.

4. If G does not involve as composition factors alternating groups of high order then $T_G(\epsilon) \geq \log(1/\epsilon)/\log n \log\log n$.

5. Let $n = \binom{m}{r}$ and G is S_m acting on r-subsets of $[m]$. Then for every $\delta > 0$

$$(\log(1/\epsilon)/\log^{(1+1/(r-1))} n) \leq T_G(\epsilon) \leq c(\delta)(\log(1/\epsilon)/\log^{(1+1/(r-1)-\delta)} n)$$

6. For $G = PSL(m,q)$ acting on the projective space over F_q, for fixed q,

$$T_G(\epsilon) = O(\log(1/\epsilon)/\log n \log\log n)$$

7. For every function $w(n)$ such that $\log w(n)/\log\log n \to 0$ there are primitive group $G_n \subset S_n$ such that $T_{G_n}(\epsilon)$ behaves like $\log(1/\epsilon)/\log n \cdot w(n)$.

8. For every $w(n) > 1$ such that $w(n) = O(\log\log n)$ there are primitive group $G_n \subset S_n$ which do not involve alternating groups of high order as composition factors such that $T_{G_n}(\epsilon)$ behaves like $\log(1/\epsilon)/(\log n \cdot w(n))$.

9. If G does not involve as sections alternating groups of high order then $T_G(\epsilon) \geq O(\log(1/\epsilon)/\log n)$.

Sections 1-3 are devoted to the proof of Theorem 2 with $p = \frac{1}{2}$. In Section 4 we prove Theorem 3. We give the proof of (0.18) for $p = \frac{1}{2}$. The general case, assuming (0.14), is done completely similarly, replacing the $\{\varepsilon_i\}_{i=1,\dots n}$ variables and the usual Walsh system $(w_S)_{S\subset \underline{n}}$

$$w_S(x) = \prod_{i\in S} \varepsilon_i(x) \tag{0.19}$$

by the coordinate variables

$$\begin{cases} r_i(x) & = \sqrt{\frac{1-p}{p}} \quad \text{if} \quad x_i = 1 \\ r_i(x) & = \sqrt{\frac{p}{1-p}} \quad \text{if} \quad x_i = 0 \end{cases} \tag{0.20}$$

satisfying $\int r_i d\mu_p = 0$, $\int r_i^2 d\mu_p = 1$, and the corresponding orthonormal basis $(r_S)_{S\subset \underline{n}}$ of $L^2(\{0,1\}^n, \mu_p)$

$$r_S(x) = \prod_{i\in S} r_i(x). \tag{0.21}$$

This is the same procedure as in [T], used to adjust the [KKL] argument.

As in most of these arguments, the key property of the system needed is some moment inequality comparing L^2 and L^q-norms, $q > 2$, on the linear subspaces $[r_S \,|\, |S| = k]$. One has in the present setting (see [T], Lemma 2.1)

Lemma 0.22. *Denote*

$$\theta = [p(1-p)]^{-1/2}. \tag{0.23}$$

Then for all $q \geq 2$, $k \geq 1$ and scalars $(a_S)_{|S| \leq k}$

$$\Big\| \sum_{|S| \leq k} a_S r_S \Big\|_q \leq (q-1)^{k/2}\ \theta^k \Big(\sum_{|S| \leq k} a_S^2 \Big)^{1/2}. \tag{0.24}$$

For $p = \frac{1}{2}$, (0.24) results from the standard hypercontractivity result. See [T] for the general case. We use (0.24) with a fixed $q > 2$. If (0.14) holds, the factors C^k need to be replaced by $C^{k.(\log \log n)}$ which is harmless in the subsequent analysis.

We may clearly assume

$$|A|(1 - |A|) > (\log n)^{-1/\tau}. \tag{0.25}$$

Denote $f = \chi_A$ and

$$f(\varepsilon) = \sum_{S \subset \underline{n}} f_S . w_S(\varepsilon) \tag{0.26}$$

its expansion in the Walsh system. Let

$$f(i) = \frac{1}{2}[f(\varepsilon_1, \dots, \varepsilon_{i-1}, 1, \varepsilon_{i+1}, \dots, \varepsilon_n) - f(\varepsilon_1, \dots, \varepsilon_{i-1}, -1, \varepsilon_{i+1}, \dots, \varepsilon_n)] = \sum_{i \in S} f_S w_{S \setminus \{i\}} \tag{0.27}$$

and

$$I(f) = \sum \|f_{(i)}\|_2^2 = \sum |S| f_S^2 \tag{0.28}$$

corresponding to the left member of (0.18), (multiplied by a factor $p(1-p)$ in the p-case).

1. First reduction of the problem

Since $f = \chi_A$, we have

$$\sum_{|S| > 0} f_S^2 = |A|(1 - |A|) = \rho \tag{1.1}$$

assuming (0.25).

Fix K. Assume

$$\sum_{0 < |S| \leq K} f_S^2 > \frac{\rho}{10}. \tag{1.2}$$

Define

$$g(\varepsilon) = \sum_{0<|S|\leq K} f_S\, w_S(\varepsilon). \tag{1.3}$$

¿From (1.2), (0.27)

$$\frac{\rho}{10} < \sum_{0<|S|\leq K} f_S^2 \leq \sum_{|S|\leq K} |S| f_S^2 = \sum |S|\, f_S g_S = \sum\nolimits_i \int f_{(i)} g_{(i)} \leq \sum\nolimits_i \|f_{(i)}\|_{4/3}\, \|g_{(i)}\|_4. \tag{1.4}$$

One has

$$\|f_{(i)}\|_{4/3} = \left(\int |f_{(i)}|^{4/3}\right)^{3/4} \sim \left(\int |f_{(i)}|^2\right)^{3/4} = \|f_{(i)}\|_2^{3/2} \tag{1.5}$$

since $f_{(i)}$ ranges in $\{0, 1, -1, \frac{1}{2},\ -\frac{1}{2}\}$

and

$$\|g_{(i)}\|_4 \leq C^K\, \|g_{(i)}\|_2 \tag{1.6}$$

by (0.22).(*)

¿From (1.4), (1.5), (1.6)

$$\begin{aligned} \frac{\rho}{10} < C^K\, \Sigma_i \|f_{(i)}\|_2^{3/2}\, \|g_{(i)}\|_2 &< C^K \max_i \|g_{(i)}\|_2^{1/2}.\ \sum\nolimits_i \|f_{(i)}\|_2^{3/2}\, \|g_{(i)}\|_2^{1/2} \\ &< C^K \left[\max_i\, \|g_{(i)}\|_2^{1/2}\right].\ I(f)^{3/4}\, I(g)^{1/4} \\ &< C^K \left[\max_i\ \|g_{(i)}\|_2^{1/2}\right].\ I(f). \end{aligned} \tag{1.7}$$

Estimate $\|g_{(i)}\|_2$ using group action. From the invariance assumption

$$f_S = f_{\pi(S)} \quad \text{for} \quad \pi \in G.$$

Fix i. By transitivity of G, one may take $\pi_1, \ldots, \pi_n \in G$ with $\pi_j(i) = j$. Then

$$\|g_{(i)}\|_2^2 = \sum_{\substack{i\in S\\ |S|\leq K}} f_S^2 = \frac{1}{n}\sum_{j=1}^n \sum_{\substack{i\in S\\ |S|\leq K}} f_{\pi_j(S)}^2 = \frac{1}{n}\sum_{j=1}^n \sum_{\substack{j\in S'\\ |S'|\leq K}} f_{S'}^2 = \frac{1}{n}\sum_{|S'|\leq K} |S'|\, f_{S'}^2 < \frac{K}{n}\rho. \tag{1.8}$$

(*)We will use C to indicate possibly different constants.

¿From (1.7), (1.8)

$$\frac{\rho}{10} < n^{-1/4}\ I(f)C^K \tag{1.9}$$

$$I(f) > C^{-K}\, n^{1/4}.\rho \tag{1.10}$$

This means that either $I(f) > n^{1/4}\rho$ or $K \gtrsim \log\, n$. We assume the second alternative, thus

$$\sum_{|S|>\log\, n} f_S^2 > \frac{\rho}{10}. \tag{1.11}$$

(2) Improving the logarithmic estimate

Choose $K \gtrsim \log\, n$ such that

$$\sum_{|S|>K} f_S^2 > \frac{\rho}{10} \quad \text{and} \quad \sum_{|S|\sim K} f_S^2 > \frac{\rho}{\log\, n}. \tag{2.0}$$

Our aim is to improve the lower bound on K. Before describing a more efficient scheme we give first a simpler version of it which already yields an improvement of the $\log\, n$-lower bound.

Let $v = v(k) < K$ be an integer to be specified.

Let $I \subset \{1, \ldots, n\}$ be a random set of size $\sim \frac{v}{K} \cdot n$. Thus $I = I_\omega$ is generated as

$$I_\omega = \{j = 1, \ldots, n \mid \xi_j(\omega) = 1\} \tag{2.1}$$

where $\{\xi_j\}_{j=1,\ldots,n}$ are independent 0,1-valued random variables (= selectors) of expectation

$$\int \xi_j = \frac{v}{K}. \tag{2.2}$$

For given $S \subset \{1, \ldots, n\}$, one has

$$|S \cap I_\omega| = \sum_{j\in S} \xi_j(\omega)$$

hence, by (2.2)

$$\frac{1}{v}\ \Big|\ |S\cap I_\omega| - \frac{v}{K}\ |S|\ \Big| = \frac{1}{v}\ \Big|\sum_{j\in S}\Big(\xi_j(\omega) - \int \xi_j\Big)\Big| \tag{2.3}$$

and, by (2.3)

$$\mathbb{E}_\omega\ \left[\frac{1}{v}\ \Big||S\cap I_\omega| - \frac{v}{K}\ |S|\Big|\ \right] \sim \frac{1}{v}\,\mathbb{E}_\omega\ \left[|S\cap I_\omega|^{1/2}\right] \le v^{-1/2}. \tag{2.4}$$

Define

$$\mathcal{S} = \mathcal{S}_I = \Big\{S\subset\{1,\dots,n\}\ \Big|\ \frac{1}{2}\ \frac{v}{K}\,|S| < |S\cap I| < 2\,\frac{v}{K}\,|S|\Big\}. \tag{2.5}$$

Thus

$$\sum_{|S|\sim K,\, S\notin\mathcal{S}_{I_\omega}} f_S^2 \lesssim \sum_{|S|\sim K} f_S^2 \left[\frac{1}{v}\ \Big|\ |S\cap I_\omega| - \frac{v}{K}\,|S|\Big|\ \right] \tag{2.6}$$

and averaging in ω yields by (2.4)

$$\mathbb{E}_\omega\ \Big[\sum_{|S|\sim K,\, S\notin\mathcal{S}_{I_\omega}} f_S^2\Big] \lesssim v^{-1/2} \sum_{|S|\sim K} f_S^2. \tag{2.7}$$

Hence, there is ω such that $I = I_\omega$ fulfills

$$\sum_{|S|\sim K,\, |S\cap I|\sim v} f_S^2 \sim \sum_{|S|\sim K} f_S^2 \equiv \Gamma > \frac{\rho}{\log\, n}. \tag{2.8}$$

This is a preliminary construction. Write $\varepsilon = (\varepsilon^1, \varepsilon^2) = (\varepsilon_j|_{j\in I},\, \varepsilon_j|_{j\notin I})$ according to the decomposition $\{1, 2, \dots, n\} = I\cup I^c$.

Define for $S\subset I$

$$F_S(\varepsilon^2) = \sum_{S'\cap I = S} f_{S'}w_{S'\setminus S} \quad \text{and} \quad G_S(\varepsilon^2) = \sum_{\substack{S'\cap I=S\\ |S'|\sim K}} f_{S'}w_{S'\setminus S}. \tag{2.9}$$

Hence, from (2.8)

$$\sum_{S\subset I,\, |S|\sim v} \int F_S G_S = \Gamma > \frac{\rho}{\log\, n}. \tag{2.10}$$

Observe also that

$$\sum_{S\subset I} F_S^2 = \|f\|^2_{L^2(\varepsilon^1)} \leq 1. \tag{2.11}$$

Fix $\delta > 0$, M to be specified and define

$$\chi_i(\varepsilon^2) = \chi_{\{(\sum_{\substack{i\in S\\ |S|\sim v}} G_S^2)^{1/2} > \delta\}} \quad \text{for} \quad i \in I$$

$$\chi = \chi_{\{(\sum G_S^2)^{1/2} < M\}}. \tag{2.12}$$

Hence

$$\sum_{i\in I} \chi_i . \chi < \delta^{-2} \left[\sum_{i\in I} \left(\sum_{\substack{i\in S\\ |S|\sim v}} G_S^2\right)\right] \chi < \delta^{-2} v. \left(\sum G_S^2\right) \chi < \delta^{-2}.v.M^2 \tag{2.13}$$

$$\int (1-\chi) d\varepsilon^2 < M^{-2} \int \left[\sum G_S^2(\varepsilon^2)\right] \leq M^{-2}. \tag{2.14}$$

One has by (2.10)

$$\frac{\rho}{\log\, n} < \int \sum_{|S|\sim v} |F_S|.|G_S| <$$

$$\int \sum_{|S|\sim v} |F_S|\ |G_S|\, \chi. \prod_{i\in S} \chi_i \tag{2.15}$$

$+$

$$\int \sum\ |F_S|\ |G_S|\ (1-\chi) \tag{2.16}$$

$+$

$$\int \sum_i \sum_{\substack{|S|\sim v\\ i\in S}} |F_S|\ .\ |G_S|\, .\ (1-\chi_i). \tag{2.17}$$

Estimation of (2.15).

By (2.13)

$$\sum_{|S|\sim v} \chi. \prod_{i\in S} \chi_i < \left(\sum_{i\in I} \chi_i\ .\ \chi\right)^{2v} < (\delta^{-2} v.M^2)^{2v} \tag{2.18}$$

hence

$$(2.15) \leq (\delta^{-2}v.M)^{2v} \cdot \int \max_{|S|\sim v} |F_S|.|G_S| < (\delta^{-2}v.M^2)^{2v}. \int \max_{|S|\sim v} |G_S| d\varepsilon^2. \tag{2.19}$$

By (2.9) and (0.22)

$$\int \max_{|S|\sim v} |G_S|\, d\varepsilon^2 < \Bigg(\sum_{\substack{S\subset I\\ |S|\sim v}} \|G_S\|^4_{L^4(\varepsilon_2)} \Bigg)^{1/4}$$

$$< C^K \Bigg(\sum_{\substack{S\subset I\\ |S|\sim v}} \|G_S\|^4_{L^2(\varepsilon_2)} \Bigg)^{1/4}$$

$$< C^K \max_{\substack{S\subset I\\ |S|\sim v}} \Bigg[\sum_{\substack{S'\cap I = S\\ |S'|\sim K}} f^2_{S'} \Bigg]^{1/4}. \tag{2.20}$$

$$< C^K \max_{|S|\sim v} \Bigg(\sum_{\substack{S'\supset S\\ |S'|\sim K}} f^2_{S'} \Bigg)^{1/4}. \tag{2.21}$$

Fix $S \subset \{1, \dots , n\},\ |S| = v$. Estimate again $\sum_{\substack{S'\supset S\\ |S'|\sim K}} f^2_{S'}$ using the group action.

Recall that

$$e^{\phi(v)} = \min_{|S|=v} \left(\sharp\{\pi(S)\,|\,\pi \in G\}\right). \tag{2.22}$$

Then, choosing again a system $(\pi_\alpha)_{\alpha\leq A}$ in G with $\pi_\alpha(S)$ mutually different, $A = e^{\phi(v)}$, we get from the invariance

$$\sum_{\substack{S'\supset S\\ |S'|\sim K}} f^2_{S'} = \frac{1}{A} \sum_{\alpha=1}^{A} \sum_{\substack{S'\supset S\\ |S'|\sim K}} f^2_{\pi_\alpha(S')} = \frac{1}{A} \sum_{\alpha=1}^{A} \sum_{\substack{S'\supset \pi_\alpha(S)\\ |S'|\sim K}} f^2_{S'}$$

$$< \frac{1}{A} \sum_{|S'|\sim K} \binom{|S'|}{|S|} f^2_{S'} < e^{-\phi(v)} K^{2v} \ (2.23)$$

Substituting (2.23) in (2.21) and (2.19) yields thus

$$(2.15) < (\delta^{-2} v\, M^2)^{2v}\ C^K\ e^{-\frac{1}{4}\,\phi(v)} K^{\frac{v}{2}}. \tag{2.24}$$

Estimation of (2.16).

Estimate by Hölder's inequality and (2.11), (2.14)

$$\begin{aligned}\int \sum |F_S|\, |G_S|\, (1-\chi) &\le \int \Big(\sum F_S^2\Big)^{1/2} \Big(\sum G_S^2\Big)^{1/2} (1-\chi)\\ &\le \int \Big(\sum G_S^2\Big)^{1/2} (1-\chi)\\ &\le \Big(1-\int \chi\Big)^{1/2}\\ &< M^{-1}. \end{aligned} \tag{2.25}$$

Estimation of (2.17).

By Cauchy-Schwartz

$$\begin{aligned}(2.17) &< \int d\varepsilon^2 \sum_i \Big(\sum_{\substack{|S|\sim v\\ i\in S}} F_S^2\Big)^{1/2} \Big(\sum_{\substack{|S|\sim v\\ i\in S}} G_S^2\Big)^{1/2} (1-\chi_i)\\ &< \delta^{1/2} \int \sum_{i\in I} \Big(\sum_{\substack{|S|\sim v\\ i\in S}} F_S^2\Big)^{1/2} \Big(\sum_{\substack{|S|\sim v\\ i\in S}} G_S^2\Big)^{1/4} \text{ (by (2.12), definition of } \chi_i)\\ &< \delta^{1/2} \sum_{i\in I} \Big\| \Big(\sum_{\substack{|S|\sim v\\ i\in S}} F_S^2\Big)^{1/2} \Big\|_{L^{4/3}(\varepsilon^2)} \Big\| \Big(\sum_{\substack{|S|\sim v\\ i\in S}} G_S^2\Big)^{1/4} \Big\|_{L^4(\varepsilon_2)}\\ &< \delta^{1/2} C^{2v} \sum_i \Big\| \|f_{(i)}\|_{L^{4/3}(\varepsilon^1)} \Big\|_{L^{4/3}(\varepsilon^2)} \Big(\sum_{i\in S', |S'|\sim K} f_{s'}^2\Big)^{1/4} \text{ (dualizing (0.22))}\\ &< \delta^{1/2} C^{2v} \sum_i \|f_{(i)}\|_2^{3/2} |f_{(i)}\|_2^{1/2}\\ &< \delta^{1/2} C^{2v} I(f). \end{aligned} \tag{2.26}$$

Collecting (2.24), (2.25), (2.26) yields from (2.15)-(2.17)

$$\frac{\rho}{\log\, n} < (\delta^{-2}\, v\, M^2)^{2v}\, C^K\, e^{-\frac{1}{4}\phi(v)}\, K^{v/2} + \frac{1}{M} + \delta^{1/2}\, C^v.\; I(f). \tag{2.27}$$

Recall that $\log 1/\rho \sim \log\log n$.

Taking $\log M \sim \log\log\, n$, $\log\,\frac{1}{\delta} \sim v \gg \log\log n$ gives thus

$$1 < C^{v^2+K}\ e^{-\frac{1}{4}\phi(v)} + 2^{-v}\ I(f). \tag{2.28}$$

Choose $v = t$ such that

$$\phi(t) > C'\, t^2. \tag{2.29}$$

(2.28) implies that either

$$K \gtrsim t^2 \quad \text{or} \quad I(f) > 2^t$$

and hence certainly

$$I(f) \gtrsim t^2. \tag{2.30}$$

In the application to graphs, one has

$$\phi(t) > \log\ \binom{\sqrt{n}}{\sqrt{t}} \sim \sqrt{t}.\ \log\, n. \tag{2.31}$$

Hence, in (2.29), we may let $t \sim (\log\, n)^{2/3}$ and we get

$$I(f) > (\log\, n)^{4/3} \tag{2.32}$$

from (2.30), improving on the $\log n$ lower bound.

Our next purpose is to improve on estimate (2.28). Our aim is to replace the exponent $Cv^2 - \phi(v)$ by a better one. The main idea is to carry out a finite iteration process, (2) represents one step off.

(3) Proof of Theorem 2.

Let r be an arbitrary large but fixed constant. Let v be an integer such that

$$v > (r \log\log n)^{10}, \qquad v^{r+2} < K \tag{3.1}$$

where K satisfies (2.1).

We introduce a tree of subsets $(I_c)_{c\in\{1,\dots,v\}^{r'}, r'<r}$ of $\{1,\dots,n\}$ (of length $r-1$) of refining partitions of

$$I = I_\phi = I_1 \cup I_2 \cup \cdots \cup I_v \tag{3.2}$$

$$I_i = \bigcup_{i'=1}^{v} I_{i,i'} \quad (i = 1,\dots,v) \tag{3.3}$$

and in general

$$I_c = \bigcup_{i=1}^{v} I_{c,i} \quad (|c| \le r-2) \tag{3.4}$$

such that

$$\sum_{S\in\mathcal{S}} f_S^2 \sim \sum_{|S|\sim K} f_S^2 \ge \frac{\rho}{\log\, n} \tag{3.5}$$

where

$$\mathcal{S} = \{S \subset \{1,2,\dots,n\} \bigm| |S| \sim K,\ \ |S\cap I_c| \sim v^{r-|c|} \quad \text{for all} \quad c \in \{1,\dots,v\}^{r'},\, r' < r\}. \tag{3.6}$$

Clearly it suffices to satisfy

$$|S \cap I_c| \sim v \quad \text{for} \quad c \in \{1,\dots,v\}^{r-1}. \tag{3.7}$$

To achieve (3.7), consider for $(I_c)_{|c|=r-1}$ a family of disjoint random subsets of $\{1,\dots,n\}$ of size $\frac{v}{K}\cdot n$ and observe that for fixed S, $|S| \sim K$, the expectation of

$$\max_{|c|=r-1} \frac{1}{v} \left|\ |S\cap I_c| - \frac{v}{K}\,|S|\ \right| \tag{3.8}$$

is bounded by (from (3.1))

$$(\log\, v^{r-1})^{1/2}.\ v^{-1/2} < v^{-1/3} \tag{3.9}$$

instead of (2.4). One may then easily deduce (3.5) as in section 2 for (2.8).

After this preliminary construction, we now perform an inductive process (with r steps) along the lines of section 2.

Step 1.

Write $\varepsilon = (\varepsilon^1, \varepsilon^2) = (\varepsilon_j|_{j\in I_\phi},\ \varepsilon_j|_{j\notin I_\phi})$ and $\varepsilon^1 = (\varepsilon^{1,1}, \dots, \varepsilon^{1,v})$ where $\varepsilon^{1,i} = \varepsilon_j|_{j\in I_i}$.

Define

$$\mathcal{S}_\phi = \{S\cap I_\phi | S\in\mathcal{S}\} \tag{3.10}$$

$$\subset \left\{ S\subset I_\phi \;\middle|\; |S\cap I_c| \sim v^{r-|c|} \quad \text{for all} \quad c\in\{1,\dots,v\}^{r'},\ r'<r \right\} \tag{3.11}$$

Define

$$F_S = \sum_{S'\cap I_\phi = S} f_{S'} w_{S'\setminus S} \quad \text{and} \quad G_S = \sum_{\substack{S'\cap I_\phi = S\\ S'\in\mathcal{S}}} f_{S'}\ w_{S'\setminus S}. \tag{3.12}$$

One has

$$\sum\ F_S^2 = \|f\|^2_{L^2(\varepsilon^1)} \leq 1. \tag{3.13}$$

By (3.5), one gets

$$\sum_{S\in\mathcal{S}_\phi} \int F_S G_S\, d\varepsilon^2 = \sum_{S'\in\mathcal{S}} f^2_{S'} > \frac{\rho}{\log\, n}. \tag{3.14}$$

Decomposing $I_\phi = I_1\cup I_2\cup\cdots\cup I_v$, write for $S\in\mathcal{S}_\phi$

$$S = S_1\cup S_2\cup\cdots\cup S_v. \tag{3.15}$$

Define

$$\chi = \chi(\varepsilon^2) = \chi_{[(\sum\ G_S^2)^{1/2} < M]} \tag{3.16}$$

and for $i = 1,\dots,v$

$$\chi^i_{S_i} = \chi^i_{S_i}(\varepsilon^2) = \chi_{\left[\left(\sum_{S\cap I_i = S_i} G_S^2\right)^{1/2} > \delta_1\right]}. \tag{3.17}$$

Hence, from (3.16), (3.17), for $i = 1,\dots,v$

$$\sum_{S_i} \chi^i_{S_i}\cdot\chi < \delta_1^{-2}\, \left(\sum G_S^2\right)\chi < \delta_1^{-2}\, M^2 \tag{3.18}$$

and

$$1 - \int \chi < M^{-2} \int \sum G_S^2 < M^{-2}. \tag{3.19}$$

With

$$\sum_{S\in\mathcal{S}_\phi} \int F_S G_S = \sum \int F_S G_S \, (1 - \chi^1_{S_1})$$

$$+$$

$$\sum \int F_S G_S \, \chi^1_{S_1} \, (1 - \chi^2_{S_2})$$

$$+$$

$$\vdots$$

$$+$$

$$\sum \int F_S G_S \prod_{i'<i} \chi^{i'}_{S_{i'}} \, (1 - \chi^i_{S_i}) \tag{3.20}$$

$$+$$

$$\vdots$$

$$+$$

$$\sum \int F_S G_S \prod_{i=1}^{v} \chi^i_{S_i}. \tag{3.21}$$

Estimate (3.21) as

$$\sum \int |F_S| \, |G_S| \, (1-\chi) + \Sigma \int |F_S| \, |G_S| \prod_{i=1}^{v} \chi^i_{S_i} \cdot \chi$$

by (3.13), (3.18)

$$\leq \int \Big(\sum G_S^2\Big)^{1/2} (1-\chi) + (\delta_1^{-2} M^2)^v \int \max |G_S|$$

by (3.19), (3.6), (3.12), 0.22)

$$\leq M^{-1} + (\delta_1^{-2} M^2)^v \, C^K \max_{S\in\mathcal{S}_\phi} \|G_S\|^{1/2}_{L^2(\varepsilon^2)}. \tag{3.22}$$

where

$$\|G_S\|_2 \leq \Big(\sum_{\substack{S \subset S' \\ |S'| \sim K}} f_{S'}^2 \Big)^{1/2}. \tag{3.23}$$

Recall that $S \in \mathcal{S}_\phi$, hence $|S| \sim v^r$. Using the group action as in section 2, we get then that

$$\sum_{\substack{S \subset S' \\ |S'| \sim K}} f_{S'}^2 < e^{-\phi(v^r)} \binom{2K}{v^r} < e^{-\phi(v^r)} \, K^{2v^r}. \tag{3.24}$$

Hence, we get

$$(3.21) < M^{-1} + (\delta_1^{-2} M^2)^v \, C^K \; e^{-\frac{1}{4}\phi(v^r)} \; K^{\frac{1}{2}v^r} \tag{3.25}$$

and letting $M = \rho^{-1} (\log\, n)^2$

$$(3.21) < \frac{\rho}{(\log\, n)^2} + \delta_1^{-2v} \, C^K \, K^{v^r} \; e^{-\frac{1}{4}\phi(v^r)}. \tag{3.26}$$

Assume

$$\delta_1^{-2v} \, C^K \, K^{v^r} \; e^{-\frac{\rho}{4}\phi(v^r)} < \frac{\rho}{(\log\, n)^2}. \tag{3.27}$$

Then one of the terms (3.20) is at least $\frac{\rho}{v.\log\, n}$, say

$$\sum \int F_S G_S \prod_{i' < i} \chi_{S_{i'}}^{i'} \, (1 - \chi_{S_i}^{i}) > \frac{\rho}{v.\log\, n} \tag{3.28}$$

for some $i = 1, \ldots, v$. We now replace I_ϕ by I_i and let

$$\mathcal{S}_i = \{ S \cup I_i | S \in \mathcal{S}_\phi \} \tag{3.29}$$

$$\subset \{ S \subset I_i \;\big|\; |S \cap I_c| \sim v^{r - |c|} \quad \text{for all} \quad I_c \subset I_i \} \tag{3.30}$$

Define

$$F_{S_i} = \sum_{S' \cap I_i = S_i} f_{S'} w_{S' \backslash S_i} = \sum_{S \cap I_i = S_i} F_S \, (\varepsilon_j|_{j \notin I_\phi}) \; w_{S \backslash S_i} \, (\varepsilon_j|_{j \in I_\phi \backslash I_i}). \tag{3.31}$$

and redefine G_{S_i} as

$$G_{S_i} = \sum_{S\cap I_i=S_i} G_S \prod_{i'<i} \chi^{i\prime}_{S_{i'}} (1-\chi^i_{S_i})\ w_{S\setminus S_i} \tag{3.32}$$

where $G_S \prod\limits_{i'<i} \chi^{i'}_{S_{i'}}(1-\chi^i_{S_i})$ only depends on $\varepsilon_j|_{j\notin I_\phi}$.

Hence, by (3.28)

$$\sum_{S_i\in\mathcal{S}_i} \int F_{S_i}G_{S_i} = \sum_{S\in\mathcal{S}_\phi} \int F_S G_S \prod_{i'<i} \chi^{i'}_{S_{i'}}(1-\chi^i_{S_i}) > \frac{\rho}{v\ \log\ n}. \tag{3.33}$$

Also

$$\sum\nolimits_{S_i} \|G_{S_i}\|_2^2 \le \sum \|G_S\|_2^2 \le 1. \tag{3.34}$$

Estimate next

$$\|G_{S_i}\|_4^4 = \int \Big(\prod_{j\notin I_\phi} d\varepsilon_j\Big)\ (1-\chi^i_{S_i}) \int \Big(\prod_{j\in I_\phi\setminus I_i} d\varepsilon_j\Big) \Big|\sum_{S\in\mathcal{S}_\phi, S\cap I_i=S_i} G_S \prod_{i'<i} \chi^{i'}_{S_{i'}}\ w_{S\setminus S_i}\Big|^4. \tag{3.35}$$

Since $|S|\le v^r$ for $S\in\mathcal{S}_\phi$, (0.22) yields

$$(3.35) < C^{v^r} \int \Big(\prod_{j\notin I_\phi} d\varepsilon_j\Big)\ (1-\chi^i_{S_i}) \Big(\sum_{S\cap I_i=S_i} G_S^2 \prod_{i'<i} \chi^{i'}_{S_{i'}}\Big)^2$$

by (3.17)

$$< C^{v^r}\ \delta_1^2\ \|G_{S_i}\|_2^2$$

hence

$$\|G_{S_i}\|_4 < C^{v^r}\ \delta_1^{1/2}\ \|G_{S_i}\|_2^{1/2}. \tag{3.36}$$

Put

$$C^{v^r}\delta_1^{1/2} = \gamma_1 \tag{3.37}$$

hence

$$\|G_{S_i}\|_4 < \gamma_1\ \ \|G_{S_i}\|_2^{1/2} \tag{3.38}$$

and condition (3.27) becomes

$$\gamma_1^{-4v}.C^{v^{r+1}+K} \ . \ e^{-\frac{1}{4}\phi(v^r)} < \frac{\rho}{(\log\, n)^2}. \tag{3.39}$$

Step $\ell < \mathbf{r}$.

We estimate, cf. (3.33)

$$\frac{\rho}{v^{\ell-1}\log\, n} < \sum_{S\in\mathcal{S}_c} \int \ F_S G_S \tag{3.40}$$

where $|c| = \ell - 1$ and

$$\mathcal{S}_c \subset \{S \subset I_c \ \big| \ |S\cap I_{c'}| \sim v^{r-|c'|} \quad \text{for all} \quad c' \text{ with } I_{c'} \subset I_c\} \tag{3.41}$$

$$\sum \ \|G_S\|_2^2 \leq 1 \tag{3.42}$$

$$\|G_S\|_4 < \gamma_{\ell-1} \ \|G_S\|_2^{1/2} \tag{3.43}$$

(cf. (3.34), (3.38)).

Decompose $I_c = I_{c,1}, \cup \cdots \cup I_{c,v}$ and $S = S_1 \cup S_2 \cup \ldots \cup S_v$ for $S \in \mathcal{S}_c$.

Define again

$$\chi = \chi(\varepsilon_j|_{j\notin I_c}) = \chi_{[(\sum\ G_S^2)^{1/2}<M]} \tag{3.44}$$

$$\chi_{S_i}^i = \chi_{S_i}^i\ (\varepsilon_j|_{j\notin I_c} = \chi_{\left[\left(\sum\limits_{S\cap I_{c,i}=S_i} G_S^2\right)^{1/2} > \delta_\ell\right]} \tag{3.45}$$

and proceed as before, letting

$$M = \rho^{-1}v^{\ell-1}(\log\, n)^2. \tag{3.46}$$

Repeating (3.22), estimating $\int \max|G_S| \leq (\sum\ \|G_S\|_4^4)^{1/4}$, (3.42), (3.43), (3.46) yields the following estimate on the (3.21) term

$$\frac{\rho}{v^{\ell-1}(\log\, n)^2} + (\delta_\ell^{-2}v^{2(\ell-1)}(\log\, n)^4\rho^{-2})^v\ \gamma_{\ell-1}. \tag{3.47}$$

We require

$$(\delta_\ell^{-2}\, v^{2(\ell-1)}\, (\log\, n)^4 \rho^{-2})^v\, \gamma_{\ell-1} < \frac{\rho}{v^{\ell-1}(\log\, n)^2} \tag{3.48}$$

which by (3.1) is satisfied for

$$\gamma_{\ell-1} < e^{-v^2}\, \delta_\ell^{2v}. \tag{3.49}$$

Then again one of the terms (3.20) is at least $\frac{\rho}{v^\ell \log\, n}$, say

$$\sum_{S\in\mathcal{S}_c} \int\, F_S G_S\, \prod_{i'<i} \chi^{i'}_{S_{i'}} (1-\chi^i_{S_i}) > \frac{\rho}{v^\ell \log\, n} \tag{3.50}$$

for some $i = 1, \ldots, v$. We define for $S_i \in \mathcal{S}_{c,i} = \{S \cap I_{c,i} | S \in \mathcal{S}_c\}$

$$F_{S_i} = \sum_{S'\cap I_{c,i}=S_i} f_{S'}\, w_{S'\backslash S} = \sum_{S\cap I_{c,i}=S_i} F_S\, (\varepsilon_j|_{j\notin I_c})\, w_{S\backslash S_i}\, (\varepsilon_j|_{j\in I_c\backslash I_{c,i}}) \tag{3.51}$$

and redefine G_{S_i} as

$$G_{S_i} = \sum_{S\cap I_{c,i}=S_i} G_S\, \prod_{i'<i} \chi^{i'}_{S_{i'}}\, (1-\chi^i_{S_i})\, w_{S\backslash S_i} \tag{3.52}$$

with $G_S \prod\limits_{i'<i} \chi^{i'}_{S_{i'}}\, (1-\chi^i_{S_i})$ only dependent on $\varepsilon_j|_{j\notin I_c}$.
Hence, by (3.50)

$$\sum_{S_i\in\mathcal{S}_{c,i}} \int\, F_{S_i} G_{S_i} = \sum_{S\in\mathcal{S}_c} \int\, F_S G_S\, \prod_{i'<i} \chi^{i'}_{S_{i'}} (1-\chi^i_{S_i}) > \frac{\rho}{v^\ell \log\, n}. \tag{3.53}$$

One has, repeating the calculation of (3.35) with I_ϕ(resp I_i) replaced by I_c (resp. $I_{c,i}$) and taking (3.41), (3.45) into account

$$\|G_{S_i}\|_4^4 < C^{v^{r-\ell+1}}\, \delta_\ell^2\, \|G_{S_i}\|_2^2. \tag{3.54}$$

Hence

$$\|G_{S_i}\|_4 < \gamma_\ell\, \|G_{S_i}\|_2^{1/2} \tag{3.55}$$

with

$$\gamma_\ell = C^{v^{r-\ell+1}}\, \delta_\ell^{1/2}. \tag{3.56}$$

Condition (3.49) becomes thus

$$\gamma_{\ell-1} < \gamma_\ell^{4v}\ C^{-v^{r-\ell+2}}. \tag{3.57}$$

Last Step.

Assume

$$\frac{\rho}{v^{r-1}\log n} < \sum_{S\in\mathcal{S}_c} \int F_S G_S \tag{3.58}$$

where $|c| = r-1$ and F_S, G_S depend only on $(\varepsilon_j)_{j\notin I_c}$,

$$\sum \|G_S\|_2^2 \le 1 \tag{3.59}$$

$$\|G_S\|_4 < \gamma_{r-1}\ \|G_S\|_2^{1/2}. \tag{3.60}$$

Repeat the estimate from section 2, taking $M = \rho^{-1}v^{r-1}(\log n)^2$, $\delta = \delta_r$ in (2.12). Estimate in (2.19)

$$\int \max_{S\in\mathcal{S}_c} |G_S| < \Big(\sum\ \|G_S\|_4^4\Big)^{1/4} < \gamma_{r-1}\ \Big(\sum\ \|G_S\|_2^2\Big)^{1/4} < \gamma_{r-1} \tag{3.61}$$

from (3.59), (3.60). Hence

$$(2.15) < (\delta_r^{-2}\, v\, M^2)^{2v}\ \gamma_{r-1}. \tag{3.62}$$

Estimate

$$(2.17) < \delta_r^{1/2}\, C^v\ I(f). \tag{3.63}$$

In order to get a contradiction, we require thus that

$$(3.62) + (3.63) < \frac{\rho}{v^{r-1}(\log n)^2}. \tag{3.64}$$

Hence, let

$$\delta_r < C^{-v}\ I(f)^{-2} \tag{3.65}$$

and

$$\gamma_{r-1} < C^{-v^2}\, I(f)^{-8v}. \tag{3.66}$$

Recall (3.57)

$$\gamma_{\ell-1} < \gamma_\ell^{4v} \cdot C^{-v^{r-\ell+2}}. \tag{3.57}$$

Assuming

$$\log I(f) < v \tag{3.67}$$

(3.66), (3.57) yield thus the condition

$$\gamma_\ell < C^{-(v)^{r-\ell+1}}\ \gamma_{r-1}^{(4v)^{r-\ell-1}};\ \gamma_\ell < C^{-(4v)^{r-\ell+1}} \text{ for } \ell < r-1. \tag{3.68}$$

Hence, (3.39)

$$\gamma_1^{-4v}\, C^{v^{r+1}+K}\ e^{-\frac{1}{4}\phi(v^r)} < \frac{\rho}{(\log\, n)^2} \tag{3.69}$$

yields a contradiction for

$$\gamma_1 = C^{-(4v)^r}. \tag{3.70}$$

Consequently,

$$(4v)^{r+1} + K \gtrsim \phi(v^r). \tag{3.71}$$

Recall also assumption (3.1)

$$K > v^{r+2}. \tag{3.72}$$

Letting $v = \frac{1}{5}K^{1/r+2} > (\log\, n)^{1/r+2}$, it follows from (3.67), (3.71) that either

$$\log I(f) > (\log\, n)^{1/r+2} \tag{3.73}$$

or

$$K \gtrsim \phi\left(5^{-r} K^{\frac{r}{r+2}}\right). \tag{3.74}$$

Recall (0.12) and thus

$$a_\tau(G) = \phi(t_0) \tag{3.75}$$

where t_0 is defined by

$$\phi(t) > t^{1+\tau} \text{ for } t < t_0. \tag{3.76}$$

Thus, from (3.74), (3.76)

$$K \gtrsim (5^{-r} K^{\frac{r}{r+2}})^{1+\tau} \tag{3.77}$$

provided $5^{-r} K^{\frac{r}{r+2}} < t_0$. Given $\tau > 0$, a choice of sufficiently large r contradicts (3.77). Hence $5^{-r} K^{\frac{r}{r+2}} \geq t_0$ and (3.74), (3.75) imply

$$K \gtrsim a_\tau(G) \tag{3.78}$$

and by (0.28), (2.0)

$$I(f) \gtrsim a_\tau(G).\rho. \tag{3.79}$$

This is obviously also true if (3.73), proving (0.18) (for $p = \frac{1}{2}$).

4. Orbits of primitive groups on large sets

Lemma. *[Friedgut] Let A be a monotone family so that all minimal sets in A have cardinality at most K. Then $I(A) \leq k\mu_p(A)(1 - \mu_p(A))$.*

Proof: (Compare also [M]) For $S \in A$ let $h(S)$ denotes the number of neighbors of S which are not in A. $I(A) = \int h(A) d\mu_p$. We will show that for every $S \in A$, $h(S) \leq K$. Indeed, if $S \in A$ and $B \subset A$ is a minimal set then for every $i \in S \backslash B$ we have that $S \backslash \{i\}$ contains B and hence belongs to A. Therefore, $h(S) \leq K$.

We will prove now that for every permutation group G there is a G-invariant monotone family A such that $T_G(\epsilon) \gtrsim \log(1/\epsilon)1/a(G)$. Consider a set S of minimal size so that $log|G(S)| \geq |S| + 1$, and the family of subsets of $[n]$ which contain a set of the form $g(S)$ for some $g \in G$. Now for $p = 1/2$ the expected number of sets in the orbit of S which are contained in a random set is at most $1/2$. Therefore the critical probability q for which $\mu_q(A) = 1/2$ satisfies $q \geq 1/2$. But by the previous Lemma $I(A) \leq \mu_p(A)(1 - \mu_p(A))|S|$ and therefore the length of the threshold interval of A is at least $\sim \log(1/\epsilon)1/|S|$.

In the rest of this section we give upper bounds on the sum of influences for certain G-invariant families. We need to study the of sizes of orbits of permutation groups on sets of unbounded cardinality, which seems to complement the vast knowledge on the orbit-size of sets of bounded cardinality, and thus being of independent interest. We refer the reader to [C,P] for related material on permutation groups.

For a permutation group $G \subset S_n$ and $0 \leq t \leq n$ recall that $\phi(t) = \phi_G(t)$ is the minimal size of an orbit of a t-subset of $[n]$ under G. Let S_t be a set of cardinality t whose orbit size is $\phi(t)$. Consider the family A_t of those subsets U of $\{0,1\}^n$ which contain a set in the orbit of S_t. It is reasonable to guess that A_t will have in some asymptotic sense smallest influence among G-invariants families.

We will first describe the value of $a_\tau(G)$ for the case of graph properties, the more general case of properties of k-uniform hypergraphs and the case where $G = GL(q,m)$ acting on F_q^m. (F_q is the field with q elements.)

Lemma.

1. Let $G = S_m$ acting on $\binom{m}{k}$. If t is of the form $\binom{r-1}{k} < t \leq \binom{r}{k}$, and $tl \leq n - r$ then $\phi_G(t) \geq \binom{m}{r}$.

2. Let $G = GL(q,m)$ acting on F_q^m. If $[\binom{m}{r-1}](q) < t \leq [\binom{m}{r}](q)$, and $r < m/2$ then $\phi_t(G) \leq [\binom{m}{r}](q)$.

Proof: (1) Let T be a set with $|T| = t$ which supported by u points. Then $u < n - r$ so the orbit of T is at least $\binom{n}{r}$. (2) Let T be a t-subset of F_q^m, and let U be the subspace of F_q^n spanned by T. Clearly $dimU \geq r$. If $dimU \leq m - r$ we are done. Otherwise the orbit of T is at least $|GL(q, m-r)|/t!$, and this number is larger than $[\binom{m}{r}](q)$ in the range of the Lemma.

Corollary 4.1.

(1) Let $k \leq \log\log m$ and let $G = S_m$ acting on $\binom{m}{k}$, (thus $n = \binom{m}{k}$.) Then $a(G) =$

$\log^{1+1/(k-1)} m$ *and* $a_\tau(G) = O(\log^{1+(1-\tau)/(k-1)} m)$*. (2) Let* $G = GL(q,m)$ *acting on* F_q^m *(thus* $n = q^m$*) then* $a(G) = a_\tau(G) = O(\log n(1 + \log_q \log n))$*.*

Proof: (1)If $\log \binom{m}{r} = \binom{r}{k}^{1+\tau}$ then $r \log m = \binom{r}{k}^{1+\tau}$ and $\log m = r^{k(1+\tau)-1}$ so that $\binom{r}{k} = \log m^{k/(k+\tau k-1}$ and $a_\tau(G) = \binom{r}{k}^{1+\tau} = (\log m)^{(k+\tau k)/(k+\tau k-1)}$. (2) If $\log[\binom{m}{r}](q) = (q^r)^{1+\tau}$, then $r = \log_q m + \log_q \log_q m + \log_q \log_2 q$ and $a_\tau(G) = q^{r(1+\tau)} = O(\log_2 n \cdot (1 + \log_q \log_2 n))$, for every $\tau \geq 0$.

We will continue now to discuss general primitive permutation groups. We need the following Theorem from Cameron [C] This theorem relies on the classification of finite simple groups and specifically on the O'nan-Scott classification theorem for primitive groups. It is quite possible that by a more delicate group-theoretic argument via the O'nan-Scott theorem it will be possible to identify the values of $a_\tau(G)$ for every primitive permutation group.

Theorem. *[Cameron]*

There is a constant c *such that if* G *is a primitive permutation group of order* n *then one of the following holds:*

(i) G *has an elementary abelian regular normal subgroup, in other words* G *is a subgroup of* $AGL(n,q)$ *acting on* F_q^n*.*

(ii) G *is a subgroup of* $Aut(T)WrS_l$*, where* T *is an alternating group acting on* k*-element subsets, and the wearth product has the product action.*

(ii') G is a subgroup of $Aut(T)WrS_l$*, where* T *is a classical simple group acting on an orbit of subspaces or (in case* $T = PSL(d,q)$*) pairs of subspaces of complementary dimensions, and the wearth product has the product action.*

(iii) $|G| \leq n^{c \log\log n}$*.*

Proof of Theorem 3: We will first prove that for all groups of type (i),(ii)' and (iii) $a(G) \leq O(\log n \log\log n)$. Next we will describe completely the value of $a_\tau(G)$ for groups of type (ii).

Note that clearly $a(G) \leq \log |G|$, therefore for groups of type (iii) $a(G) \leq O(\log n \log\log n)$. If $G = AGL(m, q)$ acting on F_q^m then by the same argument as the proof of (4.1) we get that $a(G) \leq O(\log n(1 + \log_q \log n))$.

In case (ii') we first consider the case $l = 1$. It follows by a case by case checking that the action of $H = Aut(T)$ has $a(H) \leq \log n \log\log n$. First note that $Out(T) = Aut(T)/T$, is always very small. More precisely, if G is of Lie type $G = X(m, q)$ where m is the dimension and the field is of size $q = p^k$, then $Out(T)$ has order $O(mk)$ and consists of so called field automorphisms, diagonal automorphisms, and "diagram automorphisms", see [KL]. Therefore, if you multiply $T(S)$ by $O(dk)$ to get a bound for $Aut(T)(S)$ the change in the orbit size is negligible.

We first consider the case where $G = PSL(m, q)$. (It make no difference to consider $GL(m, q)$ and the action on F_q^m was studied above.) We will consider now the action on k-dimensional subspaces of F_q^m. If $k < \sqrt{\log m}$ consider the orbit of all k-dimensional spaces of some r-dimensional space, where $r \sim \log_q m + \log_q \log m + \log_q \log q$. If r is larger consider two disjoint spaces V_1 and V_2 of dimensions a and b respectively and consider the orbit of the set of all k-subspaces which contain V_1 and have a $(k-a)$-dimensional intersection with V_2. A simple adjusting of the parameters shows that in both case $a(G) \leq \log n \log\log n$. (When k get larger than $\log m$, $b = k + 1 - a$ and in this case $a(G) \sim \log n$.) We have also to check the case of action on pairs of complementary subspaces and this works exactly like action on single subspaces.

Next, we have to check tha cases where $X(m, q) = PSL(m, q)$, $SP(m, q)$, $P\Omega^+(m, q)$, $P\Omega^-(m, q)$, $\Omega(m, q)$ and $PSU(m, q)$, linear, simplectic, orthogonal and unitary groups. In each such case a set of small orbits is obtained from an appropriate subspace. It is quite likely that $a(G)$ can be computed precisely for all these groups and all their primitive actions but we will describe a short verification of the fact that $a(G) \leq O(\log n \log\log n)$. First consider the case where X is acting on F_q^m or on 1-dimensional subspaces of F_q^m. In this case the result follows from the result for $GL(m, q)$ since the size of orbits of subspaces is maximal in this case. More generally X can act on either nonsingular or totally singular subspaces. The result still follows from those for $PSL(m, q)$ because the number of such subspaces of given dimension d (provided it is not 0) depends polynomially on the corresponding numbers for $GL(m, q)$. And in the ranges of interest to us these numbers of subspaces will be zero only if certain parity conditions holds. In short, the examples for $PSL(m, q)$ with perhaps changing the dimensions in question by 1, continue to apply for $X(m, q)$.

To see this last statement look at tables 3.5 in [KL] pp. 70 - 74 giving the isomorphic type of the point-stabilizer. (You should look at the line corresponding to C1). By looking at another table with the orders of classical groups - on p. 170, one can compute the orders of the groups G, the stabilizer H, hence the index $(G : H)$ which is the number of relevant subspaces. Doing this one finds a polynomial relation, as we wanted.

Now, let $H \subset S_m$ be a permutation group and $G = HWrS_l$ acting on $\underline{m}^l$ with the product action. If $l > \log m$ then $a(G) \leq l \log m = \log n$ (even if $H = S_n$).

If $l \leq \log m$ then from $a(H) \leq c \log m \log\log n$ it follows easily that

$$a(G) \leq 2cl \log m \log\log(m^l).$$

So the only case where $a(G)$ is bigger than $O(\log n \log\log n)$ is when $G \subset S_m WrS_l$, and S_m is acting on k-subsets of m and G contains A_m^l. (Thus $n = \binom{m}{k}^l$.). These cases are dealt with as (4.1) and it turns out that $a(G)$ is of the form $\Omega(\log n^{1+1/r})$, for $r = kl - 1$, and $a_\tau(G) \geq O(\log n^{1+(1-\tau)/(r)})$.

We will continue now in the proof of Theorem 3. Part 1 is well known. Part 2, 3 and 4 follow from Theorem 1 and the computations above. Parts 5-8 follows from the Theorem 1 and Corollary (4.1) For part 7 consider groups of the form S_m acting on $\binom{m}{k}$, where $k = k(m) \leq \log\log m$ depends on m in an arbitrary way and for part 8 consider the group $GL(m, q)$ acting on F_q^m, where $q = q(m) \leq \log m$ depends on m in an arbitrary way. Part 9, follows from the following Theorem of Babai, Cameron and Palfy [BCP].

Theorem. *[Babai, Cameron and Palfy] For a n integer D, let $G \subset S_n$ be a primitive* *permutation group which does not involve A_d as a section for $d > D$. Then $|G|$ is bounded by a polynomial in n (depending on D).*

This complete the proof of Theorem 3.

Remark: The hypothesis of the Babai, Cameron and Palfy theorem is equivalent to the following: in all the nonabelian decomposition factors of G the Lie rank and degree (of A_k) are bounded.

A theorem from [FK] asserts that for a monotone property A if the critical probability is q (namely, $\mu_q(A) = 1/2$) then the length of the threshold interval is at most $O(qlog(1/q)/\log n)$. (q can depend on n.) One can ask what are all the (abstract) groups for which this theorem is sharp for every primitive representation and every q. It is plausible that these groups are precisely the groups with no large alternating groups as factors.

Aknowledgement: We would like to thank Aner Shalev for his help on primitive permutation groups and Ehud Friedgut for fruitful discussions.

References

[BCP]. L. Babai, P.J. Cameron and P.P. Pálfy, *On the orders of primitive groups with restricted nonabelian composition factors*, J. Algebra 79 (1982), 161–168.

[B]. B. Bollobas, Academic Press, London, 1985.

[BKKKL]. J. Bourgain, J. Kahn, G. Kalai, Y. Katznelson and N. Linial, *The influence of variables in product spaces*, Israel J. Math. 77(1992), 55-64.

[C]. P. Cameron, *Finite permutation groups and finite simple groups,*, Bull. London Math. Soc. 13(1981), 1-22.

[FK]. E. Friedgut, G. Kalai, *Every monotone graph property has a sharp threshold*, Proc. Amer. Math. Soc, in press.

[KKL]. J. Kahn, G. Kalai, N. Linial, *The influence of variables on Boolean functions*, Proc. 29th IEEE FOCS 58-80, IEEE, NY (1988).

[KL]. P. Kleidman, M. Liebeck, Cambridge University Press, Cambridge, 1990.

[M]. G. Margulis, *Probabilistic characteristic of graphs with large connectivity*, Problems Info. Transmission, Plenum Press, NY (1977).

[P]. L. Pyber, *Asymptotic results for permutation groups,*, in *Groups and Computation*, eds: L. Finkelstein and W.M. Kantor, DIMACS Series on Discrete Math. and Theor. Computer Science **11** (1993), 197–219..

[R]. L. Russo, *An approximative zero-one law*, Z. Wahrsch. Verw. Gebiete 61, 129-139, (1982).

[T]. M. Talagrand, *On Russo's approximative zero-one law*, Annals of Prob., Vol.22, N3, 1576-1587, (1994).