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COMBINATORICS WITH A GEOMETRIC FLAVOR: SOME EXAMPLES

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Dedicated to the Memory of Rodica Simion

Abstract

In this paper I try to present my field, combinatorics, via five examples of combinatorial studies which have some geometric flavor. The first topic is Tverberg's theorem, a gem in combinatorial geometry, and various of its combinatorial and topological extensions. McMullen's upper bound theorem for the face numbers of convex polytopes and its many extensions is the second topic. Next are general properties of subsets of the vertices of the discrete n-dimensional cube and some relations with questions of extremal and probabilistic combinatorics. Our fourth topic is tree enumeration and random spanning trees, and finally, some combinatorial and geometrical aspects of the simplex method for linear programming are considered.

Introduction

There is a delicate balance in mathematics between examples and general principles, and in this paper I try to present my field, combinatorics, via five examples of combinatorial studies which have some geometric flavor.

In order to make the presentation self-contained, detailed and interesting, the choice of material (even within the individual sections) is subjective and nonuniform. For an unbiased and comprehensive point of view the reader is referred to the many links and references. I have tried to include many open problems and to point out various possible connections, some of which are quite speculative.

Section 1 deals with configurations of points in Euclidean spaces and specifically with Tverberg's theorem which asserts that every set of $(r-1)(d+1)+1$ points in $\mathbb{R}^d$ can be divided into r parts whose convex hulls have nonempty intersection. A principal question is to find conditions which will guarantee the conclusion of Tverberg's theorem for a smaller set of points.

Section 2 is devoted to McMullen's upper bound theorem for convex polytopes which asserts that among all d-polytopes with n vertices the cyclic polytope has the maximal number of faces of any dimension. Sharp and general forms of this theorem and what will take to prove them are discussed.

The topic of section 3 is the cube: the combinatorics of subsets of the vertices of the discrete cube, discrete isoperimetric relations and especially the notion of influence. General facts about Boolean and real valued functions defined on the discrete cube are useful for various problems in extremal combinatorics, probability and mathematical physics.

In section 4, I discuss some recent results concerning random spanning trees and tree enumeration and mention the recent emerging picture of random spanning trees of grids in the plane.

The principal problem in the final section, §5 is to find a polynomial-time version of the simplex algorithm for linear programming. Combinatorial and geometric aspects of the problem are considered.

Although there are relations between the five sections they can be read in any order. The reader can safely skip any place where she or he feels that the mathematics becomes too heavy-going. Probably these places reflect the fact that either the mathematics should be improved or my understanding of it should.

In the (unusual) style of the "Vision in Mathematics" meeting, each section concludes with brief comments of a philosophical nature.

1 Combinatorial Geometry: An Invitation to Tverberg's Theorem

1.1 Radon's theorem and order types (oriented matroids).

Theorem 1.1 (Radon's Theorem). *Every $d + 2$ points in $\mathbb{R}^d$ can be partitioned into two parts such that the convex hulls of these parts have nonempty intersection.*

A pair of disjoint subsets of X whose convex hulls intersect are called a *Radon partition*. The points in the intersection of the convex hulls are called *Radon points*.

Radon's theorem follows at once from the fact that $d + 2$ points in $\mathbb{R}^d$ are always affinely dependent. It implies at once another basic theorem on convex sets – Helly's theorem: *For every finite family of convex sets, if every $d + 1$ of its members have a point in common then all sets in the* *family have a point in common.* The reader is referred to [14], [20] for much information on Helly type theorems.

Given n points on the line, the (minimal) Radon partitions determine (up to orientation of the line) the ordering of these points. In a similar way we can classify configurations of points in the plane or in $\mathbb{R}^d$ according to their Radon partitions. This leads to the theory of oriented matroids or order types (see [10]).

1.2 Tverberg's theorem.

Theorem 1.2 (Tverberg's Theorem). *Every $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ can be partitioned into r parts such that the convex hulls of these parts have nonempty intersection.*

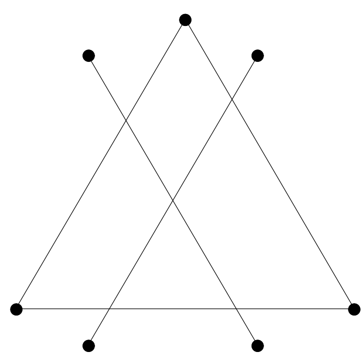


Figure 1: Seven points in the plane and their Tverberg partion.

Proofs of Tverberg's theorem were given by Tverberg ('66) [27], Doignon and Valette ('77), Tverberg ('81), Tverberg and Vrecica ('92), Sarkaria ('92) [25], and Roudneff ('99) [23]. While the original proof was quite difficult, the proofs of Sarkaria and Roudneff are remarkably simple.

Roudneff's recent proof is by minimizing the *sum of squares of the r distances* between a point x and the convex hulls of r pairwise disjoint subsets of the points. It turns out (under mild assumptions of genericity) that if this minimum is positive, it is attained without using one of the points and this extra point can be used to push this minimum down.

While the recent proofs of Tverberg's theorem give an algorithm to find the partition, the computational complexity of finding such a partition is not known.

Problem 1.1. Find a polynomial-time algorithm to obtain a Tverberg partition when Tverberg's theorem applies.

Note that (as will be clear below) deciding for a configuration of points of less than $2d+3$ in $\mathbb{R}^d$ if a Tverberg partition to 3 parts exists is an **NP**-complete problem. However, it is possible that when the number of points is large enough to guarantee a partition then finding such a partition is computationally feasible.

1.3 Topological versions.

CONJECTURE 1.2 (The Topological Tverberg Conjecture). *Let f be a continuous function from the m-dimensional simplex σ^m to $\mathbb{R}^d$. If $m \geq (d+1)(r-1)$ then there are r pairwise disjoint faces of σ^m whose images have a point in common.*

The case $r=2$ was proved by Bajmoczy and Bárány using the Borsuk-Ulam theorem. The case where r is a prime number was proved in a seminal paper of Bárány, Shlosman and Szücs [8]. The prime power case was proved by Ozaydin (unpublished), Volovikov [30] and Sarkaria. For this case the proofs are quite difficult and are based on computations of certain characteristic classes.

If f is a linear function this conjecture reduces to Tverberg's theorem. For a discussion of the topological extensions of Tverberg's theorem in a larger context, see [32]. It turns out that topological methods are crucial for proving various Tverberg type theorems even for linear maps.

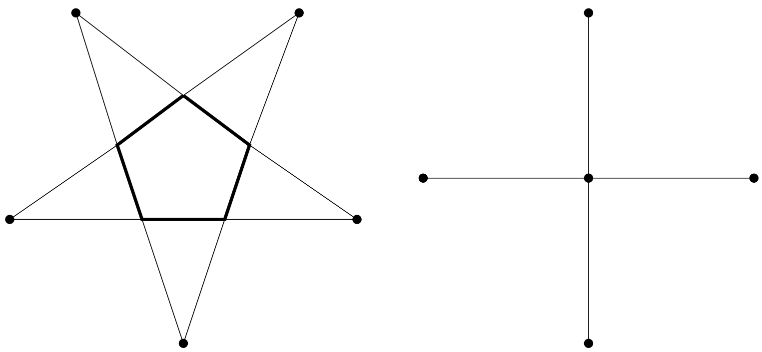
1.4 The dimension of Tverberg's points. For a set A, denote by $T_r(A)$ those points in $\mathbb{R}^d$ which belong to the convex hull of r pairwise disjoint subsets of X. We call these points *Tverberg points of order r*.

If we have $(d+1)(r-1)+1+k$ points in $\mathbb{R}^d$, then we expect that the dimension of Tverberg points of order r will be at least k. This is so in the "generic" case. Reay conjectured that it is enough to assume the points are in general position. Various special cases were recently proved by Roudneff [23], [24].

In another direction, I conjectured that failing to have the "right" dimension for the Tverberg points of order r implies the existence of a Tverberg point of order $r+1$.

CONJECTURE 1.3 (Kalai, 1974). *For every $A \subset \mathbb{R}^d$,*

$$\sum_{r=1}^{|A|} \dim T_r(A) \geq 0\,.$$



(a) $T_1=2$, $T_2=1$, $T_i=-1$, $i\geq 3$ (b) $T_1=2$, $T_2=T_3=0$, $T_4=T_5=-1$.

Figure 2: Two planar configurations of five points.

Note that $\dim\emptyset = -1$. This conjecture includes Tverberg's theorem as a special case: if $|A| = (r-1)(d+1)+1$ $\dim A = d$ and $T_r(A) = \emptyset$, then the sum in question is at most $(r-1)d + (|A|-r+1)(-1) = -1$.

It may even be true that Conjecture 1.3 holds if we replace $T_r(A)$ by the minimum of $T_r(A')$ over all configurations A' of the same order type as A.

Kadari proved (around 1980) Conjecture 1.3 for planar configurations. Crucial to his proof is the fact that in the plane (but not in higher dimensions), the convex hull of Tverberg points of order r is precisely the $(r-1)$-core of A: The intersection of all subsets of A of cardinality $|A|-(r-1)$. (Of course, every Tverberg point of order r belongs to the $(r-1)$-core.)

1.5 Conditions for Tverberg partitions and graph colorings.

1.5.1 Conditions for a Tverberg partition into 3 parts. The following problem seems important.

Problem 1.4. Find conditions on the order type for a configuration A of m points ($m < 2d+3$) in $\mathbb{R}^d$ that guarantee the existence of a Tverberg partition into three parts.

Note that deciding the existence of Tverberg partitions into three parts when $m < 2d+3$ is **NP**-complete, as will become evident below, and does not depend only on the order type of the configuration. However, I do expect that there are useful topological sufficient conditions. Conjecture 1.3 gives one such condition: $\dim T_2(A) < |A| - d - 2$.

1.5.2 Point configurations from graphs. For a graph $G = \langle V, E\rangle$ consider the configuration of points in R^V which are the incidence vectors of edges of the graph. Thus, the vector associated to an edge $\{u, v\}$ has the value '1' in the coordinates that correspond to u and v and the value '0' in all other coordinates.

PROBLEM 1.5. What can be said about affine dependencies and Radon points (and Tverberg points) of such point configurations?

The Radon partitions of such configurations arising from graphs, especially regular graphs, seem to be related to matching theory for graphs [17].

Note that a proper 3-coloring for the edges of a connected cubic graph G is *equivalent* to the existence of a Tverberg partition into 3 parts for the point configuration corresponding to G. Indeed, given a Tverberg partition into 3 parts, color every edge according to the part it belongs to. Every vertex which is incident to one colored edge must be incident to three edges colored with the 3 different colors and therefore the colored edges describe a proper 3-coloring of some cubic subgraph. Since G is connected this must be the entire graph.

1.5.3 The four color theorem. The four color theorem (Appel-Haken, 1977, see [28]) asserts that every planar map is four colorable. An equivalent formulation of the four color theorem is: Every 2-connected cubic planar graph is 3-edge colorable. (A cubic graph or a 3-regular graph is a graph all of whose vertices have degree 3. A graph is 2-connected if it remains connected after deleting every vertex.)

Now, consider a configuration of points P corresponding to a cubic planar graph with n vertices. Note, we have $3n/2$ points in a $(n-1)$-dimensional space. (If G is bipartite, these points are in a $(n-2)$-dimensional subspace.) Finding sufficient conditions for the existence of Tverberg partitions when the number of points is smaller than $2d+3$ may thus be relevant to finding new avenues towards the 4-color theorem (and its many open generalizations).

REMARKS. 1. The idea of trying to relate Tverberg's theorem and the four color theorem (in a different way) goes back to Tverberg himself.

2. There are, of course, 2-connected cubic graphs which are not 3-edge colorable. The most famous example is the Petersen graph (identify pairs of antipodal vertices in the graph of the dodecahedron). It is worth noting that the 2-core of point configurations associated to 2-connected cubic graphs is always nonempty. (The 2-core is the intersection of all convex hulls of all

but two of the points.)

3. The Radon partitions of a set A of $d+2+k$ points in $\mathbb{R}^d$ correspond to the faces of a k-dimensional zonotope. Every point in the boundary of this zonotope corresponds to a (normalized) affine dependence of the points and is mapped to a Radon point of A. This map maps two antipodal points on the zonotope to the same Radon point and thus induces a map from RP^k to $\mathbb{R}^d$ whose image is the Radon points of A.

For generic $3n/2$ points in $\mathbb{R}^{n-1}$ the Radon points form an embedding of the $(n/2-1)$-dimensional real projective space into $\mathbb{R}^{n-1}$. The case of configurations of points arising from graphs is, of course, highly non-generic.

1.6 Other problems and connections.

1.6.1 Halving hyperplanes and colored Tverberg's theorems. An important problem in combinatorial geometry is to determine the maximal number of ways a configuration of $2m$ points in $\mathbb{R}^d$ can be divided into two equal parts by a hyperplane. More generally, to determine the maximum number of ways a configuration of n points in $\mathbb{R}^d$ can be divided by a hyperplane to parts of sizes k and $n-k$ (see [6]). Equivalently, this is the minimal possible number of Radon partitions into two equal parts (or parts of prescribed sizes).

Even in the plane there is a substantial gap between the best known lower bound $C_1 n \cdot \exp(\sqrt{\log n})$ (Toth, [29]) and the upper bound $C_2 n^{4/3}$ (Dey, [13]).

The planar case of the problem is closely related to the following algebraic question: Given a reduced (=minimal) representation of a permutation in S_n as the product of adjacent transpositions, what is the maximum number of appearances of a specific transposition? To see the connection, project the points on a line and slowly rotate the line (see [15]).

For dimension d, it is easy to bound the maximal number of halving hyperplanes between n^{d-1} and n^d. Toth's lower bound extends to a lower bound of $n^{d-1} \cdot \exp(\sqrt{\log n})$ in any dimension. In space, the best known upper bound is $n^{5/2}$ [26]. In higher dimensions, the only known way for finding upper bounds for the halving hyperplane problem is via generalizations of Tverberg's theorem for colored configurations of points. Remarkably, the only proofs of these generalizations are by the topological method [33], [34]. This gives, in every dimension d, an upper bound for the number of halving hyperplanes of the form n^{d-c_d}, for some $c_d > 0$.

1.6.2 Eckhoff's partition conjecture. Ekchoff raised the possibility of finding a purely combinatorial proof of Tverberg's theorem based on Radon's theorem. He considered replacing the operation "taking the convex hull of a set A" by an arbitrary closure operation.

Let X be a set endowed with an abstract closure operation $X \to cl(X)$. The only requirements from the closure operation are: (1) $cl(cl(X)) = cl(X)$ and (2) $A \subset B$ implies $cl(A) \subset cl(B)$.

Define $t_r(X)$ to be the largest size of a (multi)set in X which cannot be partitioned to r parts whose closures have a point in common. Eckhoff conjectured that always

$$t_r \leq t_2 \cdot (r-1)\,.$$

Thus, if X is the set of subsets of $\mathbb{R}^d$ and $cl(A)$ is the convex hull operation then Radon's theorem asserts that $t_2(X) = d+1$ and Eckhoff's partition conjecture reduces to Tverberg's theorem.

1.7 Some links and references. The reader will find additional references to earlier works and survey papers in the more recent ones. Personal web sites (listed before the references at the end of the paper) will be cited by the name appearing in square brackets. Many of the papers in the references as well as related ones can be found there. The handbooks [3], [2], [1] contain many chapters which are relevant to this paper and we cite only a few.

Helly and Radon type theorems [14], [20]; topological proofs of Radon type theorems [8], [18], [31], [33], [34]; combinatorial geometry [22]; topological methods in combinatorics [9], [32]; oriented matroids [10]; halving lines and hyperplanes [6], [7], [26]; colorings of graphs [16], [5]; developments concerning the four color theorem [28]; matching theory [17]; graph theory [11]; a Radon type theorem of Larman which deserves simple proofs and better understanding [19].

Proofs, more proofs, "proofs from the book" and computer proofs

Science has a dual role: exploring and explaining. In mathematics, unlike other sciences, mathematical proofs are used as the basic tool for both tasks: to explore mathematical facts and to explain them.

The meaning of a mathematical proof is quite stable. It seems unharmed by the "foundation crisis" and the incompleteness results at the beginning of the 20th century, and unaffected by the recent notions of randomized and interactive proofs in theoretical computer science. Still, long and complicated proofs, as well as computerized proofs, raise questions about the nature of mathematical explanations.

Proofs are gradually becoming intolerably difficult. This may suggest that our days of successfully tackling a large percentage of the problems we pose will soon be over. This may also reflect the small incentives to simplify.

Be that as it may, we cannot be satisfied without repeatedly finding new connections and new proofs, and we should not give up hope of finding simple and illuminating proofs that can be presented in the classroom. For some "proofs from the book", see the lovely book by Aigner and Ziegler [4].

Some believe that computer proofs will take over [Zeilberger]. Appel and Haken's proof of the four color theorem was a landmark in this respect. The role of computers in exploring mathematical facts is already significant. As for explaining mathematical facts, it raises, for instance, the question "Explaining to whom? To humans, or to other computers?"

2 Polytopes and Algebraic Combinatorics: How General is the Upper Bound Theorem?

2.1 Cyclic polytopes and the upper bound theorem.

2.1.1 Cyclic polytopes. Consider the *moment curve* $x(t) = (t, t^2, \ldots, t^d) \subset \mathbb{R}^d$. The cyclic polytope $C(d, n)$ is the convex hull of n (distinct) points $x(t_1)$, $x(t_2), \ldots, x(t_n)$ on the moment curve. The face structure does not depend on the choice of these points.

Cyclic polytopes are *$d/2$-neighborly*, namely the convex hull of every set of k vertices forms a face of the polytope when $k \leq d/2$. Thus $f_k(C(d,n))$, the number of k-dimensional faces (in brief, k-faces) of $C(d,n)$ is $\binom{n}{k+1}$, whenever $k < d/2$. Cyclic polytopes were discovered by Carathéodory and were rediscovered by Gale, who described their face-structure.

2.1.2 The upper bound theorem. The upper bound theorem (UBT), conjectured by Motzkin in 1957, asserts that the face numbers of a d-polytope with n vertices are bounded from above by the face numbers of the cyclic d-polytope with n vertices. This conjecture is of special interest in connection with optimization, because it gives the maximum number of vertices that can be possessed by a d-polytope P defined by means of n linear inequality constraints; hence it represents the maximum number of local strict maxima that can be attained by a convex function over P.

The assertion of the upper bound theorem was proved for polytopes (McMullen, 1970 [73]), for simplicial spheres (Stanley, 1975 [79], [82]) and for simplicial manifolds with either vanishing middle homology or the same Euler characteristic as a sphere (Novik, 1998 [74]). It was also been proved when n is large w.r.t. d ($n \geq d^2/4$, will do) for all Eulerian simplicial complexes (Klee, 1964 [65]. (An Eulerian simplicial complex is a pure simplicial complex in which the link of each simplex has the same Euler characteristic as the sphere of the appropriate dimension.)

2.1.3 A stronger form of the UBT. A stronger version of the UBT (referred to, below, as SUBC: strong upper bound conjecture) was proved for simplicial d-polytopes and full dimensional subcomplexes of their boundary complexes by Kalai [62]. It asserts (roughly) that for every k, $0 \leq k < d-1$, if one fixes the number of k-dimensional faces, then the number of $(k+1)$-dimensional faces is maximized by a cyclic d-polytope. (More precisely, it gives a bound on the number of $(k+1)$-faces in terms of the number of k-faces that is similar in form to the Kruskal-Katona theorem, which provides a similar bound for arbitrary simplicial complexes.) I conjecture that the SUBC applies to arbitrary polytopes (and more general complexes considered below).

The SUBC was also motivated by a problem from optimization, namely by an attempt to show expansion properties of graphs of d-polytopes. However, applications in this direction were quite limited.

2.2 Stanley-Reisner rings and their generic initial ideals (algebraic shifting). Stanley's proof of the upper bound theorem for triangulation of spheres relies on the notion of the Stanley-Reisner ring associated to a simplicial complex and on the fact that this ring is Cohen–Macaulay. We will describe below an algebraic statement concerning generic initial ideals of the Stanley-Reisner rings which implies the strong upper bound theorem.

Let me first explain the situation informally. The Stanley-Reisner ring is constructed by associating to each vertex i of a simplicial complex a variable x_i, and considering the ring of monomials which "live" on the complex. Consider next generic linear combinations of these variables y_1, $y_2, \dots, y_n$ and a Gröbner basis for this ring w.r.t. monomials in the new variables. This construction associates to every simplicial complex K a basis of monomials $GIN(K)$ (in the new variables) which record many topological, combinatorial and algebraic properties of K.

An algebraic statement for a $(d-1)$-dimensional simplicial complex K which immediately implies the UBT, and in fact also the SUBC, is that $GIN(K)$ is a subset of $GIN(C(d,n))$, where n is the number of vertices of K and $C(d,n)$ is the boundary complex of a cyclic d-polytope with n vertices. When K is isomorphic to the boundary complex of a simplicial polytope this relation follows from the Hard Lefschetz Theorem for toric varieties; see [62].

Here is a more accurate description of the Stanley-Reisner ring and $GIN(K)$. Associate to each vertex i of a simplicial complex K a variable x_i and consider the quotient

$$R(K) = R[x_1, x_2, \ldots, x_n]/I\,,$$

where I is the ideal spanned by monomials $x_{i_1} \cdot x_{i_2} \cdots x_{i_r}$ with $\{i_1, i_2, \ldots, i_r\} \notin K$.

Consider now $y_1, y_2 \ldots, y_n$, which are n generic linear combinations of $x_1, x_2 \ldots x_n$ and construct the Gröbner basis $GIN(K)$ w.r.t. the lexicographic order on the monomials in the y_i's. (Clearly, all monomials in the y_i's span the ring $R(K)$.) Thus, a monomial m belongs to $GIN(K)$ if and only if its image $\widetilde{m}$ in $R(K)$ is not a linear combination of (images of) monomials which are lexicographically smaller. (Recall that the lexicographic order is defined as follows: $m_1 <_L m_2$ if the variable with smallest index which divides precisely one of the two monomials divides m_1. Thus $y_1^2 <_L y_1y_2 <_L y_1y_3 <_L \cdots <_L y_1y_n <_L y_2^2 <_L \cdots$.)

2.3 How general is the upper bound theorem?

2.3.1 Witt spaces. Witt spaces [56], [77], [50] are orientable triangulated pseudomanifolds K such that for every K' which is an even-dimensional (proper) link of a face of K, the (middle perversity) intersection homology $IH_{dim K'/2}(K')$ vanishes. For these spaces middle perversity intersection homology is defined and satisfies Poincaré duality. These spaces include all (real) manifolds and (complex, possibly singular) algebraic varieties.

We come now to the main conjecture of this section.

CONJECTURE 2.1. *(i) For every triangulation K of a Witt space with vanishing middle intersection homology*

$$GIN(K) \subset GIN\big(C(d,n)\big)\,. \tag{2.1}$$

(ii) The strong upper bound conjecture holds for arbitrary polyhedral complexes (and even for all regular cell complexes whose face-poset form a lattice) whose underlying space is a Witt space with vanishing middle intersection homology.

What seems to be needed for a proof is an interpretation of intersection homology for simplicial pseudomanifolds in terms of the Stanley-Reisner ring and generic initial ideals. For the polyhedral case, what is needed is a suitable analog of the Stanley-Reisner ring. For this purpose too, intersection homology may play a crucial role. Intersection homology of toric varieties already plays an important role in the combinatorial study of (rational) polytopes [51], [81] (see below).

2.3.2 Embeddability. The upper bound theorem seems closely related to questions concerning embeddability. At the root of things is the assertion that K_5, the complete graph on 5 vertices, cannot be embedded in the plane.

Van Kampen proved that the r-skeleton of σ^{2r+2} (the $(2r+2)$-dimensional simplex) cannot be embedded in $\mathbb{R}^{2r}$. It seems that this property and corresponding local properties (for links of faces) would imply the assertions of the UBT, SUBC and relation (2.1). To understand such a connection it will be useful to know if the Van Kampen theorem holds when $\mathbb{R}^{2r}$ is replaced by any $2r$-dimensional manifold with vanishing middle homology, or even by any Witt space with vanishing middle intersection homology.

2.3.3 An upper bound conjecture for j-sets. Emo Welzl [88] has recently proposed another far-reaching extension for the upper bound theorem. Given a configuration A of n points in general position in $\mathbb{R}^d$ consider the set of all hyperplanes, $\mathcal{H}^j$, which are determined by points in A and have at most j vertices in one of their (open) sides (compare section 1.6.1). For $j=0$ these are supporting hyperplanes for $conv(A)$. Next, let $a_r^j(A)$ be the number of r-dimensional simplices which are determined by point in A and belong to a hyperplane in $\mathcal{H}^j$. (For $j=0$ these are just r-faces of $conv(A)$.) Welzl asked whether $a_r^j(A)$ is maximized for every r and j by n points on the moment curve in $\mathbb{R}^d$. For $j=1$ this is just the UBT and it is also known to be true for every j when $d=2$ (Alon and Gyori) and when $d=3$ (Welzl).

2.4 Duality and h-numbers. I have described very general cases for which I conjecture that the UBT and even the SUBC hold, and a strong property of $GIN(K)$ needed to prove these conjectures. However, these strong algebraic and combinatorial conjectures are known only in very limited cases. For the known cases of the UBT, weaker combinatorial and algebraic statements are sufficient if certain duality relations are also used.

2.4.1 The Dehn–Sommerville relations. For a $(d-1)$-dimensional simplicial manifold K define its h-numbers by the relation:

$$\sum_{k=0}^{d} h_k(K)x^{d-k} = \sum_{k=0}^{d} f_{k-1}(K)(x-1)^{d-k}. \tag{2.2}$$

The Dehn–Sommerville Relations asserts that if K is the boundary complex of a simplicial polytope then

$$h_k(K) = h_{d-k}(K). \tag{2.3}$$

In fact, these relations hold whenever K is *Eulerian* simplicial complex, namely K and all links of faces of K have the same Euler characteristics as a sphere of the same dimension.

2.4.2 The Cohen–Macaulay property. For Stanley's proof of the UBT when K is a simplicial sphere we need to know in addition to the Dehn–Sommerville relations that $R(K)$ is a Cohen–Macaulay ring. When $R(K)$ satisfies the Cohen–Macaulay property, then $h_k(K)$ is the number of monomials of degree k in $GIN(K)$ which use only the variables $y_{d+1}, y_{d+2}, \dots, y_n$. Novik [74] used $GIN(K)$ to prove the UBT for several classes of simplicial manifolds and she relied on the fact that $R(K)$ is still close enough to being a Cohen–Macaulay ring (the technical term is *Buchsbaum* ring). In addition, she needed the analogs of Dehn–Sommerville relations and Poincaré duality. We would like to have a better understanding of these duality relations in terms of $GIN(K)$ and for more general classes of simplicial complexes.

2.4.3 Partial unimodality and the Braden–MacPherson theorem. The face numbers of polytopes are not unimodal. Indeed, the face numbers of the cyclic polytope are highly concentrated near dimension $3d/4$ and therefore, by gluing a cyclic polytope and its dual, you will get two peaks at $d/4$ and at $3d/4$. To get a simplicial example glue a cyclic polytope to the cross polytope (with roughly the same total number of faces). You will get two peaks at $3d/4$ and at $2d/3$.

An appealing application (using an argument of Björner [47]) of the SUBC for general polytopes will be:

CONJECTURE 2.2. *The face numbers f_i of d-polytopes are nondecreasing for $i \leq [(d+3)]/4$ and nonincreasing for $i \geq [3(d-1)/4]$.*

It is possible that this conjecture as well as a suitable (weaker) version of the SUBC will follow in a purely combinatorial way from a recent result by

Braden and MacPherson [51] which relates the combinatorics of a polytope with that of faces and quotients.

McMullen's original proof of the upper bound theorem relied on the observation that for a simplicial polytope P and a vertex v,

$$h_k\big(lk(v,P)\big) \le h_k(P)\,. \tag{2.4}$$

Here, $lk(v,P)$ is the link of v in P and h_k is the h-number mentioned above.

The result of Braden and MacPherson is a sharpening as well as a far-reaching generalization of (2.4) for general polytopes. (It is proved, however, only for rational polytopes.) I will now state this result without explaining properly the background and I refer the reader to [51], [81], [63] for more. For a d-polytope P let

$$h_P(x) = \sum_{k=0}^{d} h_i(P)x^k\,, \quad g_P(x) = \sum_{i=0}^{[d/2]} g_k(P)x^k\,.$$

Here $h_k(P) = \dim IH_{2k}(T_P)$, and $g_k(P) = h_k(P) - h_{k-1}(P)$, where T_P is the toric variety associated to P and IH is intersection homology. (The quantities $\dim IH_k(T_P)$ can be described in a purely combinatorial way from the face structure of P and when P is simplicial this is just h_k.) Braden and MacPherson proved that for every rational polytope P and a face F of P,

$$g_P(x) \ge g_F(x)g_{P/F}(x)\,. \tag{2.5}$$

(Namely, every coefficient of the polynomial in the left hand side is at least as large as the corresponding coefficient on the right hand side.)

The Braden–MacPherson inequality has already been used by Bayer [41] to deduce a very sharp form of the UBT for general (rational) polytopes.

2.4.4 Other duality relations. The Dehn–Sommerville duality relations $h_k(P) = h_{d-k}(P)$ applies for arbitrary Eulerian simplicial complexes. For simplicial polytopes this numerical duality manifests Poincaré duality for the associated toric varieties. When we adopt the combinatorial formulas of intersection homology the relations $h_k = h_{d-k}$ extend even to arbitrary Eulerian partially ordered sets. For toric varieties associated to rational polytopes these duality relations manifest Poincaré duality for intersection homology. In commutative algebra this duality relations manifest the Gorenstein property for the Stanley-Reisner ring of homology spheres.

Another important notion of duality is duality between polytopes given by the polar polytope. (Thus, the cube is dual to the octahedron, the dodecahedron is dual to the icosahedron and the tetrahedron is self-dual.) In 1985 I observed some mysterious numerical formulas relating h-numbers of a polytope and those of its dual. The simplest non-trivial relation of this type asserts that for every 4-dimensional polytope $g_2(P) = g_2(P^*)$ [61]. Some extensions were proved by Bayer and Klapper and by Stanley [81] who realized the correct combinatorial context (incidence algebras) for understanding these formulas. Geometric or algebraic understanding of these relations is still missing but for very special cases of toric varieties (which give rise to Calabi–Yau manifolds) it turned out that these numerical relations manifest mirror symmetry [39].

Added in proof. Tom Braden has recently found an algebraic explanation for these duality relations via Koszul's duality.

Yet another important notion of duality is duality of oriented matroids which includes the notions of Gale transform and linear programming duality as special cases (see [10, Chapter 10]). The effect of this duality on the combinatorial notions discussed here (as well as on the algebraic and geometric ones) is yet to be explored.

2.5 Neighborliness.

2.5.1 Neighborly polytopes and spheres. For an extremal combinatorial problem, studying the cases of equality is often as important as proving the inequality. Equality for the upper bound theorem is attained by all neighborly d-polytopes, namely polytopes for which every $[d/2]$ vertices form a face.

Neighborly polytopes form an exciting but mysterious class of polytopes (see [76]). Their face numbers are determined by the number of vertices. It is conjectured that every simplicial polytope is the quotient (link) of an even dimensional neighborly polytope [67]. (The same conjecture can be made for simplicial spheres.) For a generalization of the notion of neighborly polytopes to the nonsimplicial case, see Bayer [40].

2.5.2 Triangulations of manifolds. Triangulations of $2k$-dimensional manifolds can be even $(k+1)$-neighborly. An example is the 6-vertex triangulation of the 2-dimensional projective plane obtained by identifying the opposite faces of the icosahedron. This is quite a fundamental combinatorial object and its dual graph is no other than the Petersen graph.

Heawood, who around 1890 studied colorings of graphs embedded on surfaces (in the context of extending the four color conjecture), conjectured

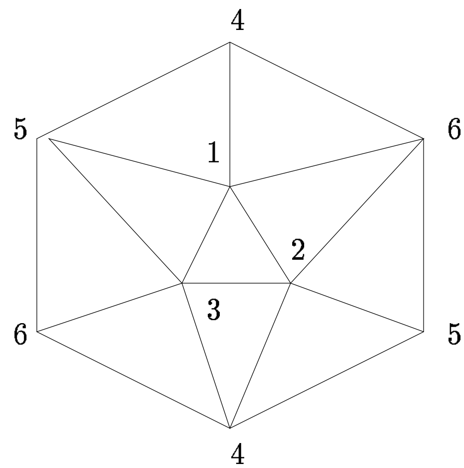


Figure 3: The 6 vertex triangulation of the real projective plane

that K_n (the complete graph on n vertices) can be embedded in a surface M (except for the Klein bottle) if and only if

$$n \leq \left(7 + \sqrt{49 - 24\chi(M)}\right).$$

(Here, $\chi(M)$ is the Euler characteristic of M.) Such embeddings giving 2-neighborly triangulations of M were indeed found in all cases by Ringel and Youngs (in some cases with other coauthors). See Ringel's book [75].

However, there are only a handful of examples of $(k+1)$-neighborly $2k$-manifolds, for $k > 1$ (see [68]). Perhaps the most famous example is the remarkable 9-vertex triangulation of the complex 2-dimensional projective space by Kühnel and Lassman (see [70], [69]).

In Kühnel's own words [68]: "To construct the triangulation we denote the nine vertices by 1,2,3, ..., 9 and take the union of the 4-dimensional simplices 12456 and 12459 under the action of a group of permutations H_{54} generated by: $\alpha = (147)(258)(369)$, $\beta = (123)(465)$ and $\gamma = (12)(45)(78)$. This group is a 2-fold extension of the Heisenberg group over Z_3. γ corresponds to the action of the complex conjugation, in fact its fixed point set is combinatorially isomorphic to the 6-vertex triangulation of the real projective plane."

Novik [74] proved an extension of the upper bound theorem for all triangulations of manifolds and it is plausible that this theorem, and a related conjecture by Kühnel concerning how large the Euler characteristic can be, apply to arbitrary triangulations of Witt spaces.

2.5.3 Neighborly embedded manifolds. The moment curve $x(t) = (t, t^2, \dots, t^d) \subset \mathbb{R}^d$ is an example of 1-dimensional $[d/2]$-neighborly manifolds in $\mathbb{R}^d$. Namely for every $[d/2]$ points on the curve there is a hyperplane which supports the curve precisely at these points.

While there are many different neighborly polytopes there is only one (in terms of order types) $[d/2]$-neighborly embedding of $\mathbb{R}$ into $\mathbb{R}^d$, for d even. Moreover, for d even, the moment curve is the only order type of an embedding of $\mathbb{R}$ into $\mathbb{R}^d$ where all points are in general position. This indicates that the as yet unexplored area of understanding the "order type" of nondiscrete subsets in $\mathbb{R}^d$ (such as embedded manifolds) may exhibit some simpler phenomena than the discrete (finite) case.

Perles asked: what is the smallest dimension $d(k,n)$ of the ambient space in which a k-neighborly n-dimensional manifold exists? A simple dimension count shows that we must have $d(k,n) \geq (k+1)n$. On the other hand, a straightforward extension of the moment curve gives a bound for $d(k,n)$ which is exponential. Kalai and Wigderson found a simple construction showing a polynomial upper bound on $d(k,n)$, and Vassiliev [86] showed by an intricate topological argument that $d(k,n) \geq 2kn - bin(n)$, where $bin(n)$ is the number of ones in the binary expansion of n.

2.6 Other problems and connections.

2.6.1 Clique complexes and spheres. Start from a graph G and consider its clique complex $K(G)$, a simplicial complex whose faces correspond to the complete subgraphs of G. Understanding the possible face numbers of such complexes is an important problem in extremal combinatorics related to Turan's theorem; see [49], [72]. Suppose that $K(G)$ is a triangulated sphere. What can be said then? Charney and Davis [52] formulated a conjecture concerning the face numbers of such complexes which is closely related to conjectures of Hopf on the Euler characteristic of manifolds M with nonpositive sectional curvature. For some recent developments, see [71].

2.6.2 Cubical upper bound theorems. Cubical complexes seem of equal importance yet quite different from simplicial complexes, and much less is known about them. (A structure of a cubical complex on a manifold seems to tell more on the geometry of the manifold.) Only recently some cubical analogs of the cyclic polytopes were constructed. Joswig and Ziegler [60] constructed d-polytopes with 2^n vertices with $[d/2]$-skeletons of the n-

dimensional cube. Previously, Babson, Billara and Chan [38] constructed cubical spheres with this property and found connections between questions on immersions of manifolds and the existence of certain cubical spheres. There are analogs for cyclic polytopes, but the analog of the upper bound theorem is false for spheres and probably also for polytopes. Adin [35] found the right notion of h-numbers, but a construction for a "cubical Stanley-Reisner ring" is yet unknown.

2.7 Some links and references. Polytope theory [89], [58], [59], [42], [66], [46], [78] [Ziegler]; face numbers and h-numbers of polytopes and complexes [43], [48], [45], [80], [63], open problems [85]; cubical spheres and polytopes [38], [60]; a continuous version of the UBT [87]; Kuhnel's CP^2 and other special triangulations [70], [69], [68]; Kruskal-Katona theorem and related results [54]; Turan type theorems [49], [55]; commutative algebra and combinatorics [82], [53] [Herzog], [Bayer]; generic initial ideals and algebraic shifting [48], [74], [37], [57], [53] [Herzog],[Bayer],[Kalai]; intersection homology [56], [50], and some combinatorial applications [81], [51]; h-numbers and polytope duality [81], [63], and mirror symmetry [39]; algebraic combinatorics à la Stanley [Stanley] [44], [82], [83], [84], [85]; Various proofs for the UBT: for Eulerian complexes with many vertices [65], for polytopes using shellability [73], a simple dual form using linear objective functions, [187], for spheres using the Cohen–Macaulay property [79], using shellability and the Cohen–Macaulay property [64], using shellability and a strong form of an extremal theorem of Bollobàs [36], for manifolds, using relations between face numbers and Betti numbers of Buchsbaum rings [74], a strong form for general polytopes using the Braden–MacPherson theorem [73].

Problems and conjectures

The posing of problems and conjectures is part of the process of exploring the factual matters as well as of proposing explanations for them. Is the development of mathematics shaped by problems? And what are good problems? Do they arise naturally like the sphere-packing conjecture, or are they perhaps sporadic and ingenious like Fermat's last theorem and the four color problem? To what an extent are good mathematical problems suggested by other sciences?

Modern combinatorics was greatly shaped by problems posed by Erdős, who was very cautious concerning our ability to predict the future of a problem.

3 Extremal and Probabilistic Combinatorics: the Discrete Cube and Influence of Variables

3.1 Influence of variables on Boolean functions.

3.1.1 The discrete cube. We consider the discrete cube $\Omega_n = \{-1,1\}^n$ and will try to understand real and Boolean functions defined on Ω_n. Boolean functions on Ω_n are of course in 1-1 correspondence with subsets of Ω_n. It turns out that many specific problems in extremal combinatorics, probability, mathematical physics and theoretical computer science can be formulated in terms of Boolean or real functions on Ω_n and that general properties of such functions are very useful.

For $x, y \in \Omega_n$ the Hamming metric $d(x,y)$ is defined by $d(x,y) = |\{i : x_i \neq y_i\}|$. Some related metrics will also be considered.

Denote by $\Omega_n(p)$ the discrete cube endowed with the product probability measure $\mathbf{P}_p$, where $\mathbf{P}_p\{x : x_j = 1\} = p$. Usually, we consider the uniform measure $p = 1/2$. (More general measures like FKG-measures should also be considered, but we will not attempt doing it here.)

Notation: In addition to the standard big O and little o notation we use the following notation: For positive real functions $f(x)$ and $g(x)$, we write $f(x) = \Theta(g(x))$ if, for some positive constants c_1 and c_2, $c_1 g(x) \leq f(x) \leq c_2 g(x)$, as x tends to infinity. We write $f(x) = \Omega(g(x))$ if for some positive constant c, $f(x) \geq c g(x)$.

3.1.2 Influence of variables. Consider an event $A \subset \Omega_n(p)$ and the associated Boolean function $f(x_1, x_2, \dots, x_n) = \mathcal{X}_A$, the characteristic function of A. The *influence* of the variable k on the Boolean function f, denoted by $I_k^p(f)$ (and also by $I_k^p(A)$), is the probability that flipping the value of x_k will change the value of f. The total influence $I^p(f)$ equals $\sum I_k^p(f)$. We define also $II^p(f) = \sum (I_k^p(f))^2$. (We will not use the superscript p for $p = 1/2$.)

Influence of variables (in a much greater generality) was introduced and studied by Ben-Or and Linial [99] in the context of "collective coin flipping", an important notion in theoretical computer science. The problem they considered is, in short: "Is there a protocol for a society of n processors to produce a random bit immune against a situation where a fraction of the processors is cheating?" Having each processor produce a single random bit, and using a Boolean function to produce the "collective bit" is a simple such protocol. But it turns out (from Theorem 3.3 below) that it can never be immune against $w(n)n/\log n$ cheaters, when $w(n)$ tends to infinity with

n. A multistage protocol immune against $\Omega(n)$ cheaters was found by Alon and Naor [92], see also Feige [110].

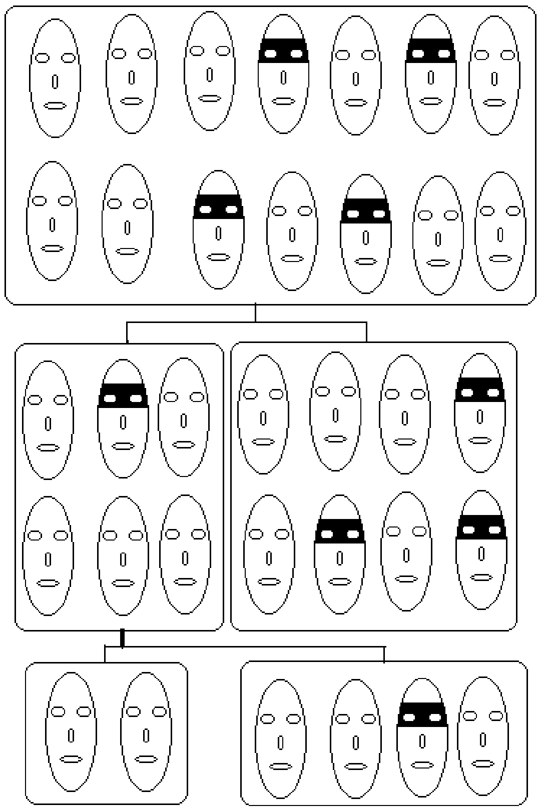


Figure 4: Two steps in Feige's protocol for a collective coin flipping. The agents enter a random room and the process continues with the room with the *least* number of agents.

$I_k(f)$ is essentially identical to the Banzhaff value in game theory. In [99], influence of larger sets of variable is also considered.

A function f is monotone if its value does not decrease when we flip the value of a variable from -1 to 1. Some basic facts on influences are given by:

Theorem 3.1 (Loomis-Whitney, Hart, Harper).

$$\sum I_k(f) \geq \mathbf{P}(A) \log\left(1/\mathbf{P}(A)\right).$$

Theorem 3.2 (Banzhaff). *For monotone Boolean functions f, $II(f) \leq 1$.*

The following result of Kahn, Kalai and Linial (KKL) has a central role in this section.

Theorem 3.3 (Kahn, Kalai and Linial, [119]).

$$\max_k I_k(f) \geq K\mathbf{P}(A)\log n/n\,.$$

Here, K is an absolute positive constant. In fact, $K = 1/2$ will do. Note that this theorem implies that when all individual influences are the same (e.g., when A is invariant under the induced action from a transitive permutation group on $[n]$), then the total influence is larger than $C\log n$. For the ultimate sharpening of this result,

$$\sum_{k=1}^{n} I_k^p(A)/\big(\log(I_k^p(A)\big) \geq K\mathbf{P}_p(A)\,;$$

see Talagrand [131].

3.1.3 Russo's lemma and threshold intervals. For a monotone event $A \subset \Omega_n$ (i.e., $\mathcal{X}_A$ is a monotone function), let $\mathbf{P}_p(A)$ be the measure of A with respect to the product measure $\mathbf{P}_p$. Note that $\mathbf{P}_p(A)$ is a monotone function of p. Russo's lemma (see [115]) asserts that

$$\tfrac{d\mathbf{P}(A)}{dp} = I^p(f)\,.$$

Given a small real number $\epsilon > 0$, consider the *threshold interval* $[p_1, p_2]$ where $\mathbf{P}_{p_1}(A) = \epsilon$ and $\mathbf{P}_{p_2}(A) = 1 - \epsilon$. Denote by p_C the value so that $\mathbf{P}_{p_C}(A) = 1/2$, and call it the critical probability for the event A. A basic result of Bollobás and Thomasson asserts that the threshold interval is always bounded by a constant times the critical probability. By Russo's lemma, large total influence around the critical probability implies a short threshold interval.

3.1.4 Fourier-Walsh expansion. Consider a Boolean function f on $\Omega_n(1/2)$. Consider the Fourier-Walsh expansion $f = \sum_{S\subset[n]} \widehat{f}(S)u_S$, where $u_S = \prod_{i\in S} x_i$. (For the case of general p, see [131].)

Now, $\|f\|_2^2 = \mathbf{P}(A) = \sum \widehat{f}^2(S)$ (Parseval) and one can show that $I(f) = \sum_{S\subset[n]} \widehat{f}^2(S)|S|$. The result of Kahn, Kalai and Linial (Theorem 3.3) follows using certain hypercontractive estimates of Bonamie and Beckner [95], [101]. In recent years, harmonic analysis on Z_2^n plays an important role in extremal and probabilistic combinatorics and in complexity theory. Bonamie's and related hypercontractive estimates are crucial for the proofs of several of the results discussed in this section.

3.1.5 Noise sensitivity. The effect of random changes in the variables is called the *noise sensitivity* of f. A class of functions is *uniformly noise stable* if for every $\epsilon > 0$ there is $\delta > 0$ such that if you flip the values of δn randomly chosen variables, the correlation of the new value of f with the original value is at least $1 - \epsilon$. It can be shown that this is equivalent to the property that most of the 2-norm of f is concentrated in small Fourier coefficients (i.e. f is well approximated (in ℓ_2) by a small degree polynomial.) A sequence of Boolean functions f_m is (asymptotically) noise sensitive if for every $\delta > 0$ the correlation just defined tends to zero as m tends to infinity. These concepts were introduced by Benjamini, Kalai and Schramm [97] (we will mention some of their results below) and (in a different language and motivations) by Tsirelson, who described remarkable relations and applications in [140], [128], [139] [Tsirelson].

Start a simple random walk (SRW) from a random point in A. How quickly will you converge to the uniform distribution on Ω_n? If $\mathbf{P}(A)$ is bounded away from 0 and 1, the answer is $O(n)$. For a sequence A_m of such events, the answer is $o(n)$ if and only if they are asymptotically noise sensitive.

3.2 Other general properties of subsets of the discrete cube.

3.2.1 Discrete isoperimetric inequalities. For $A \subset \Omega_n$ and $x \in A$ let $h(x)$ be the number of neighbors of x which are not in A. The vertex boundary of A denoted by $\partial_v(A)$ is the set of $x \in A$ with $h(x) > 0$.

Theorem 3.4. *Consider $A \subset \Omega_n$.*

1. *Given the size of A the vertex boundary of A is minimized for Hamming balls.*
2. *More generally, for every fixed $T > 0$, the number of the points whose distance from A is at least T is maximized when A is a Hamming ball. Therefore,*

$$\mathbf{P}\big\{x \in \Omega_n : d(x, A) > t\sqrt{n}\big\} \leq \exp(-t^2/2)/\mathbf{P}(A)\,. \tag{3.1}$$

Talagrand [133], [135], [134] found several deep extensions and surprising applications of the isoperimetric inequality. A very useful sharpening is to replace $d(x, A)$ by the Euclidean distance $d_T(x, A)$ from x to the *convex hull* of the points in A (considered as points in $\mathbb{R}^d$).

Theorem 3.5 (Talagrand isoperimetric inequality)**.**

$$\int_{\Omega_n(p)} \exp\left(\frac{1}{4} d_T^2(x, A)\right) \leq \frac{1}{\mathbf{P}_p(A)}\,. \tag{3.2}$$

This inequality (in a dual formulation) is extremely useful for proving tail-estimates [133], [135], [130]. Inequalities (3.1), (3.2) manifest the "concentration of measure phenomenon". We use the term "hyperconcentration" in cases where asymptotically stronger inequalities are valid.

For $A \subset \Omega_n(p)$ let $B(A) = \int_{x \in A} \sqrt{h(x)}$. Talagrand proposed $B(A)$ as the "correct" notion of boundary for A and proved it in [132] (sharpening a result by Margulis).

Theorem 3.6 (Margulis-Talagrand [132]). *Let A be an event in $\Omega_n(p)$, $t = \mathbf{P}_p(A)$ and $C_p = \min(p,q)/\sqrt{pq}$, where $q = 1-p$. Then*

$$B(A) \geq KC_p t(1-t)\sqrt{\log(t(1-t))}\,.$$

For related results see Bobkov and Götze [102].

3.2.2 FKG and Shearer's lemma. The simplest form of the FKG inequality states that two monotone events in Ω_n have a nonnegative correlation. There are many extensions, variations and applications; see [115], [106].

Let J_i be subsets of $[n]$ so that every element in $[n]$ is covered at least r times. Let $A \subset \Omega_n$ be an event, $H(A)$ its entropy and $H(A_{|J_i})$ the entropy function of A conditioned on the set of coordinates J_i. Shearer's lemma (for some applications, see [109], [117], [113], [127]) asserts:

$$\sum H(A) \leq \frac{1}{m}\sum_{i=1}^{m} H(X_{|J_i})\,.$$

(When $J_i = [n]\backslash\{i\}$, Shearer's inequality is essentially the Loomis-Whitney inequality (Theorem 3.1, Part 1).)

3.3 Advanced theorems on influences. The following theorem describes in loose terms the main advanced general theorems about influences. Recall that $II(f)$ is the sum of the squares of the influences.

Theorem 3.7 (Vague formulations). 1. [Talagrand], [136] *For two monotone events A and B, a large inner product of their influence vectors implies a stronger FKG-inequality.*

2. [Talagrand], [137] *A small value of $II(f)$ implies a large Margulis-Talagrand boundary.*

3. [Bourgain and Kalai], [105] *High symmetry implies large total influence.*

4. [Friedgut, [111]] *For a constant p, if the total influence is small, then the function is determined (approximately) by few coordinates.*

5. [Friedgut, [112] and Bourgain, [104]] *When p tends to zero, bounded total influence implies that f is "local".*

6. [Benjamini, Kalai and Schramm], [97] *For monotone events, sensitivity to noise is equivalent to having a small value of $II(f)$.*

7. [Talagrand], [138] *Small total influence implies hyperconcentration in the mean for the Hamming metric.*

8. [Benjamini, Kalai and Schramm], [98] *A small value of $II(f)$ implies hyperconcentration in terms of the second moment for Talagrand's metric.*

Van Vu and Jeong Kim [143], [121] proved hyperconcentration results for events which can be expressed by low-degree polynomials with small coefficients.

3.4 Two basic problems.

Problem 3.1. 1. Characterize Boolean functions for which $I(f)$ is small. Can such functions be always approximated by small-depth small-size Boolean circuits? (See section 3.5.10 below.)

2. Find general conditions for $I(f) \geq n^{\beta}$. Is this always the case when there is a notion of a scaling limit [90], [139], [164]?

3. Characterize Boolean functions whose Fourier coefficients have a small support. (For example, functions for which most of the 2-norm is concentrated on a polynomial number in n of Fourier coefficients.)

Problem 3.2. 1. Let f be a real function on the discrete cube. Under which (combinatorial) conditions can we guarantee that the distribution of f is close to a normal distribution?

2. Under which conditions can you expect a distribution which is more concentrated than normal? What kind of other distributions can you encounter from "natural" functions?

3.5 Examples. We consider now some examples of real and Boolean functions defined on the discrete cube. Given a real function f, consider the Boolean functions $S_T(f) = sign(f(x) - T)$. (Here, $sign(x) = +1$ if $x \geq 0$ and $sign(x) = -1$, otherwise.) Consider also the important special case $M(f) = S_T(f)$, where T is the median value of f.

3.5.1 Weighted majority. For weights $w_1, \dots, w_n$ consider the functions $f(x) = \sum w_i x_i$. The functions $S_T(f)$ are called *weighted majority functions.* The usual majority function (all w_i's are equal and $T = 0$) is a special case. For this example the influence of each variable is $C/\sqrt{n}$. Another special case is the function $f(x_1, x_2, \dots, x_n) = x_1$. Here $I_f(1) = 1$

and $I_f(k) = 0$ for $k \neq 1$. Weighted majority functions are uniformly noise-stable [97]. The Talagrand isoperimetric inequality is sharp for these functions and so is the Margulis-Talagrand inequality. More general examples are low degree polynomials and their signs. See also Bruck and Smolensky, [107].

3.5.2 Majority of majorities, tribes, runs. Of course, we can partition the set into parts and consider the majority of majorities with various parameters. An important example is the tribe example of Ben-Or and Linial [99]: Divide the variables into "tribes" of size $\log n - \log\log n$ and set $f = 1$ iff there is a tribe all of whose variables has the value 1.

A related function is: Let f be the size of the longest run of '1's' and consider $M(f)$. For these functions the influence of every variable is $\Theta(\log n/n)$ (which is optimal by Theorem 3.3).

3.5.3 Recursive majorities. Let n be a power of 3. Divide the set of variables into three parts, divide each part into 3 parts, and continue $\log_3 n$ steps. Let f be the majority of majorities ... of majorities of the 3-element sets and let $A \subset \Omega_n$ be the corresponding event [99]. If you start a SRW from a random vector of A you will come very close to a uniform distribution on Ω_n in $n^{\log 2/\log 3}$ steps. Mossel and Peres pointed out that replacing 3 by a larger (but constant) t the number of steps required is reduced to $n^{1/2+\delta}$ where δ tends to 0 as t tends to infinity.

3.5.4 Random subsets of Ω_n and error correcting codes. Consider A, which is not necessarily monotone. The parity function $f(x_1, x_2, \dots, x_n) = x_1 x_2 \cdots x_n$ is the most noise-sensitive with $\mathbf{P}(A) = 1/2$.

When $\log_2 |A| = sn$, $0 < s < 1$, we can ask: how quickly can a SRW from a random uniformly chosen point of A reach a distribution that is almost uniform on Ω_n? A reasonable guess is that the best choice would be to take A itself to be random. This is closely related to the conjecture that the Gilbert-Varshamov bounds for codes are optimal (see [142]).

3.5.5 Cliques in graphs. This time, let the variables correspond to the $n = \binom{m}{2}$ edges of the complete graph with m vertices. Every assignment of values to the variables corresponds to a graph: the graph whose edges correspond to the variables with value 1. Let f be the size of the largest clique in this graph.

Again consider $M(f)$. (The median value is $\Theta(\log n)$.) In this example the influence of each variable is $\Theta(\log^2 n/n)$. The threshold interval for having a clique of size $a \log n$ is $\Theta(1/\log^2 n)$.

3.5.6 General graph properties. More generally, every property of graphs on m vertices (connected, planar, have diameter ≥ 3, etc.) corresponds to a Boolean function on this set of $\binom{m}{2}$ variables. Bourgain and Kalai [105] used their study of influences under symmetry (Theorem 3.7, Part 3) to show that for every graph property the threshold interval is of size at most $c_\tau 1/\log^{2-\tau} n$, for every $\tau > 0$.

The most important cases for random graph properties are when the critical probability p_C itself depends on n. Already, Erdős and Renyi in their paper which introduced random graph theory showed that many graph properties (such as connectivity) have sharp threshold behavior, namely the threshold interval is $o(p_C)$.

Friedgut's theorem (Theorem 3.7, Part 5, [112]) for graph properties can be stated as follows:

"If a graph property does not have a sharp threshold then it can be approximated by the property of having a subgraph from a given finite list".

For example, the property "to have a complete subgraph with 4 vertices" has a coarse threshold. But the "connectivity" property has a sharp threshold since it cannot be approximated by having a subgraph from a finite list. Friedgut's theorem has many important applications for showing sharp threshold behavior.

3.5.7 Random formulas, the 3-SAT problem. Consider a Boolean formula with n variables of the form

$$(x_{11} \vee \bar{x}_{12} \vee x_{13}) \wedge (\bar{x}_{21} \vee \bar{x}_{22} \vee x_{23}) \wedge \cdots \wedge (x_{m1} \vee x_{m2} \vee \bar{x}_{m3}) .$$

It is an **NP**-complete problem to determine if the formula is satisfiable. The problem of satisfiability of random formulas has drawn a lot of attention recently. Friedgut [112] used his general criteria for bounded influence to show that there is a sharp threshold between values of m for which the formula is almost surely satisfiable and those for which it is almost surely unsatisfiable.

3.5.8 Crossing events in percolation. Consider the graph of the k by m grid in the plane and let the variables correspond to the edges (so n is roughly $2km$). Assume that the ratio k/m is bounded and bounded away from zero. Consider the event of having a left-to-right crossing. The influence of a variable is known in percolation theory as "the probability for an edge to be pivotal". $I(f)$ is an important "critical exponent" of percolation. It is conjectured that $I(f) = \Theta(n^{3/8})$.

A principal problem in percolation theory is that the probability for having a crossing tends to a limit if the ratio of k and m is fixed and k tends to infinity. This is a special case of the (conjectured) existence of a "scaling limit". In [97] it is shown that the crossing event is (asymptotically) sensitive to noise.

3.5.9 First passage percolation. A simple (but representative) variant of first passage percolation (FPP) can be described as follows. Consider the plane grid on which the length of each edge is assigned the value 0 with probability 1/4 (any $p < 1/2$ will do) and 1 with probability 3/4. We would like to know: What is be the distribution of the distance D from the origin to the point (m, m)? Here the distance is the minimum over all paths from the origin to (m, m) of the sum of lengths of edges in that path. D is a real random variable defined on n Boolean variables where n corresponds to all the edges of the grid in some large region containing $(0, 0)$ and (m, m). There are many exciting geometric and probabilistic problems concerning FPP.

Kesten showed that the variance of D is $O(m)$, and another simple proof with sub-Gaussian tail estimates was given by Talagrand [133] using his isoperimetric inequality. Influences and noise sensitivity (parts 5 and 7 of Theorem 3.7) are used in [98] to show that for FPP "on the torus" (and on a large class of symmetric graphs) the variance of D is actually $o(m)$. It is suggested by physicists that the variance behaves like $m^{2/3}$ and it is even speculated that the distribution of D and the large deviation properties are related to the distribution of the largest eigenvalues of certain random matrices; see [116].

3.5.10 Boolean functions expressed by bounded depth Boolean circuits. Consider a function described by a Boolean circuit (whose gates are: and, or, and negation) of depth c and size M. Linial, Mansour and Nisan showed that the Fourier coefficients of such functions decay exponentially [123]. Boppana [103] showed that for such a function f

$$I(f) \leq \log^{c-1} M .$$

Note that this bound applies to the examples of tribes, runs and cliques in graphs mentioned above (see also [107], [97]). An important example based on certain random depth-3 circuits was given by Ajtai and Linial [91].

3.5.11 Determinants, eigenvalues. (Suggested by I. Benjamini.) Consider an m by m matrix with entries ± 1 and consider its determinant D, the sign of D, its largest eigenvalue, etc. All these can be regarded as (nonmonotone) real functions on Ω_n with $n = m^2$. The corresponding Boolean functions are also of interest. What can be said about influences, noise-sensitivity and the Fourier-Walsh coefficients? Is there a notion of scaling limit?

3.5.12 Signed combination of vectors. Given n vectors $v_1, ..., v_n$ in some Euclidean space, with $\|v_i\|_2 \leq 1$, write $f(\epsilon_1, \ldots, \epsilon_n) = \| \sum \epsilon_i v_i \|_\infty$. The distribution of the values of f is of great interest and, in particular, a conjecture of Komlos asserts that f always attains a value below some absolute constant.

3.5.13 Linear objective functions. Consider a convex d-polytope which is combinatorially equivalent to the d-dimensional cube and let f be given by the values on the vertices of a linear functional on $\mathbb{R}^d$. Such functions extend weighted majority functions considered above and are of great importance in the theory of linear programming (see section 5).

3.6 Some links and references. The standard source for probabilistic combinatorics is [93], its second edition will treat various further topics discussed here. Various papers in [106] that we will not cite individually are good references for some probability topics discussed in this section and in the next one.

Influences and collective coin flipping [99], [119], [91], [92], [122]; Talagrand's method and applications [133], [135], [130]; extremal combinatorics [54]; Boolean circuit complexity [93], [107], [103], [123]; random graphs [100], [93]; combinatorial problems on the discrete cube [129]; noise sensitivity [97], [98], [140], [128], [139], [141] [Tsirelson][Schramm]; FKG variations and applications [115], [106]; entropy and Shearer's lemma [109], [113], [117]; percolation and first passage percolation [125], [126], [106]; random matrices and combinatorial connections [116], [118]; Komlos conjecture [94];

Examples

It is not unusual that a single example or a very few shape an entire mathematical discipline. Examples are the Petersen graph, cyclic polytopes, the Fano plane, the prisoner dilemma, the real n-dimensional projective space and the group of two by two nonsingular matrices. And it seems that overall, we are short of examples. The methods for coming up with useful examples in mathematics (or counterexamples for commonly believed conjectures) are even less clear than the methods for proving mathematical statements.

4 Enumerative Combinatorics and Probability: Counting Trees and Random Trees

4.1 Kirchhoff, Cayley, Kasteleyn and Tutte. Cayley proved that the number of trees on n labeled vertices is n^{n-2}. There are many beautiful proofs which demonstrate various principal techniques in enumeration theory and, amazingly, new proofs are still being found. See [162], [84] [Stanley]. The matrix tree theorem, asserting that the number of spanning trees for a graph G is (essentially) the determinant of the Laplacian of G, is even earlier and is attributed to Kirchhoff. Of the vast knowledge on tree enumeration, let me mention two additional results. Kasteleyn (see [17]) found fundamental relations between the number of perfect matchings of planar graphs and tree enumeration. Tutte [166] considered graphs drawn on the 2-dimensional sphere, which has the property that the antipodal map induces an order-reversing bijection between the faces (or dimensions 0, 1 and 2) of this graph. In particular, the graph is self-dual (but this is not sufficient). He proved that the number of spanning trees of such a graph is a perfect square and the square root is equal to the number of self-dual trees.

4.2 Random spanning trees and loop erased random walk. It is now understood that there is an intimate connection between exact or approximate enumeration of certain objects and between the problem of finding (exactly or approximately) a random element among them. What can be said about a random spanning tree of a graph G and how can you generate such an object?

Broder [148] and Aldous [145] proposed a very simple way to generate a random spanning tree in a finite graph: Start a random walk and add to the tree all edges in the walk which do not close a circle when first traversed. Wilson [168] found a remarkable algorithm with superior performances and important theoretical aspects. His algorithm is related to the discussion that follows.

Lawler [157], attempting to understand a self-avoiding random walk, considered the following (different) model. Given two vertices x and y, start from x a random walk until reaching y and erase all loops. Pemantle showed that the distribution on $x - y$ paths in Lawler's model of loop erased random walk is precisely the distribution of paths between x and y in a random spanning tree.

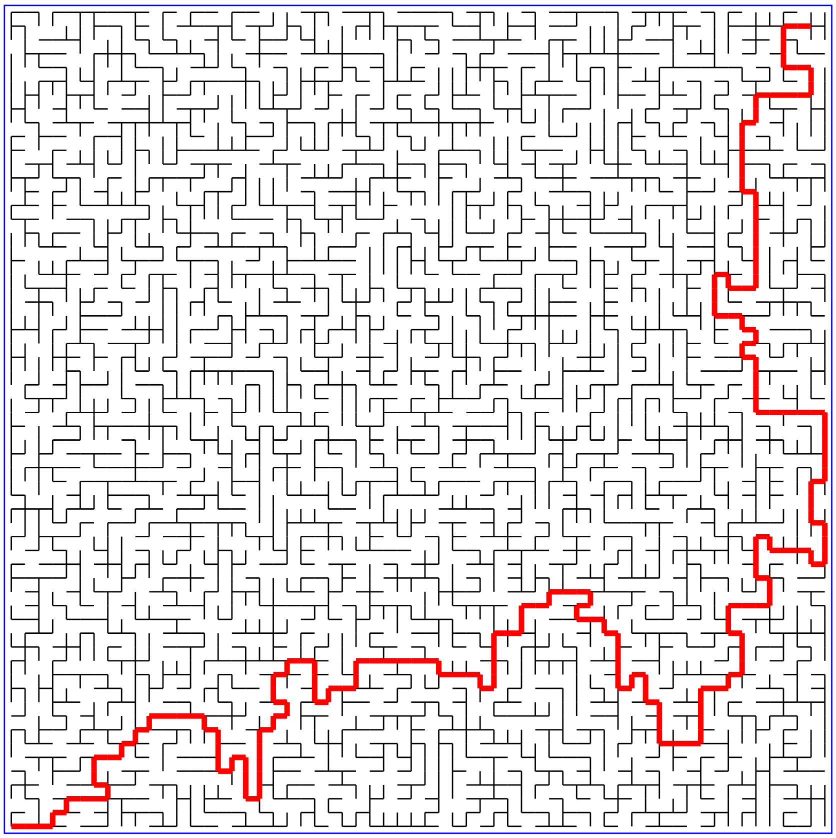


Figure 5: A random spanning tree and a loop erased random walk.

4.3 Random spanning trees II. Rick Kenyon [152], [153], [154], [155] was recently able to compute critical exponents, to prove conformal invariance and to establish the existence of scaling limits for models based on random spanning trees of planar grids. His ingenious and involved proofs use Kasteleyn's correspondence between matchings and spanning trees for planar grids in a crucial way. Here are two of Kenyon's results:

- The expected length of the path between the origin and the boundary of the n by n grid in a random spanning tree of the standard plane grid (equivalently, the expected length of the loop erased random walk) is $\Theta(n^{4/3})$.

- Consider a smooth planar figure F and three points x, y and z on its boundary, and the (unique) meeting point u of these three points in a random spanning tree of a fine planar grid. In the limit, u is a distribution on points inside F. Kenyon proved that this distribution is invariant under conformal maps of the plane.

4.4 Random spanning trees III. Oded Schramm [164] assumed a strong version of conformal invariance to show that various limiting objects for random spanning trees in planar grids are described by a certain stochastic process. Schramm constructed a remarkable class SLE_κ of stochastic processes depending on a parameter κ and showed that (assuming the existence of scaling limits and conformal invariance) these processes for $\kappa = 2$ describe the limiting paths of the loop erased random walk, and for $\kappa = 6$ the scaling limit of critical percolation cluster boundaries. For $\kappa = 8$ they are related to random Peano curves arising from random spanning trees, and for $\kappa = 4$ it is speculated that they describe the domino difference model (introduced and studied by Kenyon). This established surprising connections between objects which previously seemed different.

Here is a short description of Schramm's construction in his own words: "Consider a path γ in the closed unit disk $\mathbb{U}$ from the boundary to 0, which does not cross itself (and does not contain a nontrivial arc on the unit circle). Consider an initial arc β of γ. Let $q(\beta)$ be the endpoint of β which is not the initial point of γ. By Riemann's mapping theorem, there is a unique conformal map $g = g_\beta : \mathbb{U} - \beta \to \mathbb{U}$ normalized by $g(0) = 0$ and $g'(0) > 0$. Set $t(\beta) = -\log g'(0)$. The map g extends continuously to the closure of $\mathbb{U} - \beta$. In particular, $\delta = g(q(\beta))$ is a well defined point on the unit circle. We may think of β, and hence of δ as functions of t, $\delta : [0, \infty) \to \partial\mathbb{U}$. Loewner's slit mapping theorem shows that the collection of maps $g_\beta = g_t$ can be reconstructed from the function $\delta(t)$, by solving a differential equation with $\delta(t)$ as a parameter.

In the SLE_κ process, we take $\delta(t) = B(\kappa t)$, where κ is a constant and B is Brownian motion on the unit circle. By Loewner's theorem (and its extensions) this gives sufficient information to reconstruct g_t, hence γ."

Lawler, Schramm and Werner [159] related these objects to planar Brownian motion. Using conformal invariance which is known for Brownian motions and some earlier results of Lawler and Werner on the 'universality' of certain critical exponents for a large class of processes they managed to compute critical exponents anticipated by Duplantier-Kwon and Mandelbrot for Brownian motion in the plane.

These developments are a wonderful symphony of probabilistic, geometric, analytical and combinatorial reasoning.

4.5 Higher dimensions. Only a small fraction of the charm and importance of trees survives when we consider higher dimensional acyclic complexes. Yet in some contexts (e.g., buildings) such generalizations are useful. When it comes to tree enumeration it turns out that Cayley's formula does extend easily with a little twist: The weighted sum of $\mathbb{Q}$-acyclic k-dimensional acyclic complexes on n vertices with a complete $(k-1)$-dimensional skeleton is $n^{\binom{n-2}{k}}$, where the weight of a complex K is $|H_{k-1}(K)^2|$. Thus, for $n=6, k=2$ the formula gives 6^6 and there is a single type of complex, which is counted more than once (4 times): the 6-vertex triangulation of the real projective plane (Figure 3). The proof relies on extending the matrix-tree theorem and identifying the eigenvalues of certain Laplacians.

Very soon, as n grows, the weights in this formula become much larger than the number of summands. See Kalai [151] and Adin [144]. Analogs for Kasteleyn's and for Tutte's theorems mentioned above are expected but not known. Kenyon suggested that an appropriate extension of Kasteleyn's theorem to subcomplexes of the 3-dimensional grid may be useful in extending some of his explicit computations of critical exponents to 3 dimensions.

4.6 Haiman's diagonal harmonics. Consider the graded polynomial ring $H_n = \mathbb{C}[x_1, \dots, x_n, y_1, \dots y_n]/I$, where I is the ideal generated by all polynomials in the x_i's and y_i's which are invariant under the diagonal action of the symmetric group S_n on the variables.

Haiman [150], based on experimentation with Macaulay [158], conjectured that the dimension of H_n is $(n+1)^{n-1}$, the number of labeled trees on $n+1$ variables.

Further experimentation showed that finer statistics on the grading of H_n (the total degree or the degrees according to the variables x_i alone) turned out to be related to classical enumeration statistics of trees. Moreover, using the well known correspondence between trees and parking functions, it was possible to identify the representation of the symmetric group S_n on H_n. Haiman's conjecture turned out to be related to exciting issues in algebraic geometry and representation theory. Many tried to solve it, but very recently Haiman himself proved his conjecture [Haiman].

4.7 Some links and references. Tree enumeration [162], [84], [156]; trees and probability [161], [106]; random spanning trees and forests [146], [160], [163], [106]; enumerative combinatorics [83], [84], [147], [Zeilberger], [Stanley]; Schramm's processes, Brownian motion [164], [159], [167] Haiman's conjectures [150], [Haiman]; Macaulay [158], [Bayer]; approximate enumeration [183], [186]; eigenvalues of Laplacians of high dimensional complexes [Reiner].

Our community

Like musicians who can enjoy and understand complicated scores even in a world with no sound, for us mathematics is a source of delight, excitement and even controversy. This is hard to share with nonmathematicians.

In our small world we should seek new ways for communication and interaction and for the right balance between competition and solidarity, criticism and empathy, exclusion and inclusion.

5 Optimization: How Good is the Simplex Algorithm?

5.1 The simplex method. Linear programming is the problem of maximizing a linear objective function $\phi = b_1x_1+b_2x_2+\cdots+b_dx_d$ subject to n linear inequalities in the d variables $x_1, x_2, \ldots, x_d$. Linear programming and Danzig's simplex algorithm are among the most important applications of mathematics in the 20th century. The set Q of solutions for the inequalities is called the *feasible polyhedron*. The maximum of ϕ on Q (if ϕ is bounded on Q) is attained at a face of Q and, in particular, there is a vertex v for which the maximum is attained.

The *simplex algorithm* is a method to solve a linear programming problem by repeatedly moving from one vertex v to an adjacent vertex w of the feasible polyhedra so that in each step the value of the objective function is increased. The specific way to choose w, given v, is called the *pivot rule*. Klee and Minty [182], and later others, showed that various standard pivot rules may require exponentially many pivot steps in the worst case. On the other hand, Khachiyan [178], [190], [177] showed that linear programming is in **P** (namely, there is a polynomial-time algorithm for linear programming), and various authors (for several notions of a random linear programming problem) showed that the simplex algorithm requires a polynomial number of pivot steps on average [171].

It is an important open problem to decide if there is a variant of the simplex algorithm whose worst-case behavior takes polynomial time. Related problems are to show that there is a polynomial time algorithm for linear programming "over the reals" or to find a "strongly polynomial" algorithm (over the rationals) in the usual Turing model. These problems of fundamental importance in both complexity and optimization lie (as the problems of factorization of integers and graph isomorphism) in the grey area between **P** and **NP**, where surprising new results and insights are expected.

In the early 90s randomized subexponential simplex algorithms were found independently by Kalai [180] and by Matoušek, Sharir and Welzl [185]. These development as well as a related result on the diameter of graphs of polytopes [179] apply in a very general abstract combinatorial context. While it is possible that further improvements and even polynomial simplex algorithms can be found in this generality, the main point I would like to raise in this section is: can geometry help?

5.2 The combinatorics of linear programming. The following property is crucial:

- (local=global) ϕ takes its maximum on a vertex v of Q if and only if v is a *local maximum*, i.e., $\phi(v) \geq \phi(w)$ for every neighbor vertex w of v.

An ordering of the vertices of a polytope Q is an *abstract objective function* if the property (local=global) holds for Q and all its faces. (It is possible to consider also abstract linear programming problems in even greater generality; see [188], [179].)

For our purposes, there is no loss of generality in assuming that the feasible polyhedron Q is bounded, that the linear objective function is generic and that the problem is nondegenerate which, in other words, says that Q is a simple polytope: every vertex has d neighboring vertices, or equivalently every vertex belongs to exactly d facets.

The combinatorics of the problem involves the combinatorial structure of the polytope Q and the combinatorics of the total ordering on the vertices of Q induced by the linear objective function ϕ. The combinatorics of Q is relevant because the diameter of the graph of Q gives a lower bound on the number of pivot steps needed. However, I feel that the main difficulty lies in the combinatorics of objective functions and that understanding the case where Q is combinatorially isomorphic to the d-dimensional cube will go a long way towards solving the problem.

An interesting connection with section 2 is the following: Given a simple polytope Q (think about the cube!) and a linear objective function ϕ, we can consider for every vertex v its degree $deg(v)$, which is the number of neighbors u of v with $\phi(u) > \phi(v)$. It turns out that the distribution of the degrees of vertices does not depend on the objective function. The number of vertices of degree k is precisely the h-number h_k (considered in section 2). This applies to all abstract objective functions and, in fact, characterizes this class of orderings of the vertices of Q. This implies at once the Dehn–Sommerville relations $h_k = h_{d-k}$ (replace ϕ with $-\phi$), and relation (2.4).

The product measure $\mathbf{P}_p$ on the discrete cube (section 3) has a natural extension in this context for arbitrary simple polytopes Q by assigning to a vertex v the measure $(1-p)^{d-deg(v)}p^{deg(v)}$.

The combinatorics of the full arrangement of hyperplanes which correspond to points which satisfy one of the inequalities as equality, is also of great importance. In particular one can read from the hyperplane arrangement (and the ordering given by ϕ on its vertices) the combinatorics the *dual* linear programming problem.

5.3 Some classes of pivot rules. We will consider now various pivot strategies of a combinatorial nature for the simplex algorithm. Given a vertex v of the feasible polyhedron Q our aim is to reach the vertex of Q at the top.

1. (`"Random improving edge"`) Choose a random improving neighbor.
2. (`"Random facet"`) Choose a random facet F containing v and run the algorithm inside F until reaching the optimal vertex in that facet. Then repeat. The expected number of pivot steps is bounded above by $\exp(C\sqrt{n \log d})$ [185], [180].
3. (`"Universal instructions"`) For every vertex v of Q you are given an ordering of its neighbors. In each step of the algorithm you move to the first neighbor on the list which improves ϕ.

3(R). (`"Random universal instructions"`) The same as 3, except that you have a distribution on such an assignment of orderings and you choose a random one. (The best known randomized variant of the simplex algorithm in terms of worst case expected behavior is of this type [179].)

4. (Taking into account how well the neighbors improve) The same as 3 (or 3(R)) except the ordering in the vertex can depend on the ordering between the ϕ values of the neighbors of v. One of the earliest pivot rules using the most improving neighbor is, of course, a special case.
5. (Adaptive rule) The same as 3 (or 3(R) or 4 or 4(R)) except the

ordering in the vertex can depend on the history of the algorithm up to this stage.

6. ("`Random walk`") The basic operation is: Given two vertices v and u with $\phi(v) \geq \phi(u)$ start from a vertex v and perform a simple random walk on all vertices whose ϕ value is larger than $\phi(u)$. Then update v and u. (Note: here, we allow steps which decrease the value of the objective function.)

All these rules apply for abstract objective functions and for some of them (1-3(R)) all that is needed is the relation between the values of the objective function on neighboring vertices.

There are important pivot rules which depend in a stronger way on the geometry and cannot be described in terms of abstract objective functions. An important practical example is to always choose the steepest edge (towards the objective function) leaving the vertex. An important pivot rule from a theoretical point of view is the shadow-boundary rule, which is based on projecting the polyhedra on a two-dimensional space. For this (and only this) rule, it is known that the simplex algorithm is polynomial for an average problem [171].

5.4 Can geometry help? I: There are few objective functions. The first information about geometric objective functions is that there are not too many of them. It is not difficult to see that the number of abstract objective functions on the vertices of the d-cube is larger than 2^{2^d}. (For example, consider the orientation of edges of the discrete cube which corresponds to the linear objective function $x_1 + x_2 + \cdots + x_d$. Then consider a matching between the vertices of the two middle levels of the cubes. You may switch the orientation of any subset of edges of this matching and the property (local=global) will still hold (see, [89]).)

In sharp contrast,

Theorem 5.1. *The number of different possible geometric objective functions of the discrete d-dimensional cube is at most $\exp(d^3 \log d)$. Moreover, the number of combinatorial types of pairs (Q, ϕ), where Q is a d-polytope with n facets and ϕ a linear objective function on Q, is at most $\exp(K d^2 n \log n)$.*

The proof of this theorem relies on a theorem of Warren from real algebraic geometry on the number of sign patterns determined by a set of polynomials. The basic argument is due to Goodman and Pollack, [176], [169]. (The result applies to the number of combinatorial types of arrangements of n hyperplanes in $\mathbb{R}^d$ and the orderings given by a linear objective function on the vertices of the arrangement.)

A hereditary class of abstract objective functions on discrete cubes is *small* if the number of distinct orderings on the vertices of the d-cube is only exponential in a polynomial of d.

PROBLEM 5.1. 1. Given a small hereditary class of abstract objective functions, is it possible to prove the existence of an oblivious process such as `"Random universal instructions"` which works well for all abstract objective functions in this class?

2. Is it possible, to improve algorithms like `"Random facet"` by "learning" information from the low level recursion, in a way which will dramatically reduce the running time for *all* small hereditary classes of abstract objective functions?

The second option we raised, that of learning, needs further explanation. The situation seems to resemble what is called a bounded VC-dimension [192], which is an extremely useful condition for various combinatorial and algorithmic applications. Specifically, it is possible that applying an algorithm (such as `"Random facet"`) where the choices of the algorithm higher up in the recursion are positively correlated with successful choices in lower levels will perform well on *every* small class of abstract objective functions.

A case worthy of study (even experimentally) is a remarkable class of abstract objective functions on the discrete d-cube described by Matoušek [184]. For an average abstract objective function in this class the expected number of pivot steps needed for `"Random facet"` is indeed $\exp(K\sqrt{d})$. This class is hereditary and small: the number of abstract objective functions of this class on the d-cube is exponential in d^2.

Gärtner [174] showed that for Matoušek's class the geometry does help as `"Random facet"` itself requires only a quadratic number of pivot steps for geometric objective functions in Matoušek's class. Gärtner used only the conditions for geometric objective functions on the 3-dimensional faces.

Abstract objective functions on 3-dimensional polytopes were recently characterized by Mihalisin and Klee [181]. The orientations of graphs of 3-dimensional polytopes induced by a geometric objective function (each edge is oriented from the smaller vertex to the larger) are precisely the acyclic orientations with a unique source and a unique sink which admits three disjoint independent monotone paths from the source to the sink.

5.5 Can geometry help? II: How to distinguish geometric objective functions. We should also try to find concrete ways to distinguish between abstract and geometric objective functions. Consider the set A_v of all vertices u in the feasible polyhedron with $\phi(u) \geq \phi(v)$. Which properties does A_v satisfy?

A recent important result by Morris and Sinclair [186] shows that for the standard cube and weighted majority functions, A_v has (mild) expansion properties which implies that a SRW on A_v reaches an approximate-uniform distribution in n^8 steps.

For an arbitrary linear programming problem, can the graph induced on A_v always be divided into a polynomial (in d and n) number of parts, each of which is (mildly) expanding? (Mildly expanding means that the expansion constant is $1/p(d,n)$ for some polynomial $p(d,n)$.) Since every ordering of the vertices of the cube given by a monotone real function is an abstract objective function and since monotone subsets of Ω_n may have dismal expansion properties, the geometry must be used.

Monotone functions on Ω_n are not a real challenge for the simplex algorithm as any (improving) pivot rule will reach the maximum vertex in at most n steps. But understanding the class of orderings and events of the type A_v which come from an objective function on a polytope combinatorially isomorphic to the n-cube is nevertheless of much interest also from the point of view of complexity theory. Going back to the examples in section 3 it seems that repeated weighted majorities of various types can be realized.

Perhaps the strongest known result towards a strongly polynomial algorithm for linear programming is by Eva Tardos [191]. Fixing the feasible polyhedron (in fact, only the matrix of coefficients), she described a strongly polynomial algorithm independent of the objective function. Proving this result with a simplex type algorithm (even randomized) will already be a major achievement (see [173] for a special case).

Added in proof. Spielman and Teng have recently made substantial progress towards a polynomial (not yet strongly polynomial) version of the simplex algorithm. They showed that the shadow-boundary pivot rule needs a polynomial number of steps for a small random gaussian perturbation of a linear programming problem.

5.6 Some links and references. Linear programming [190], average case behavior of the simplex method [171], randomized pivot rules [172], [173], [189], [175], [180], [188], [185], computational geometry [187] algorithmic applications of random walks [183], [186], the diameter problem for polyhedra [179].

Applications and Expectations

Judging from the conference on 'Vision in Mathematics', mathematicians have a strong desire to interact and influence other sciences, as well as technology, industry, and even economic life. The trends towards isolationism have been reversed, and there is a greater understanding of the subtleties of applying mathematics to and interacting with other fields.

The general public knows very vaguely what mathematicians do. At the same time people have quite clear expectations from mathematics. More than other sciences, and certainly much more than law, religion, politics and the media, mathematics is expected to be rigorous and precise in telling its uninteresting, irrelevant, and uncomforting truths. The value of mathematics for society goes far beyond its applications through technology; it is indeed a pillar of human culture. After a century of amazing technological development along with rising influence of pseudosciences and the occult, this value is important.

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Rodica Simion, to whose memory this paper is dedicated, was Professor of Mathematics at George Washington University until her untimely death on January 7, 2000.

Personal web sites

Alon,	http://www.math.tau.ac.il/~noga/
Barvinok,	http://www.math.lsa.umich.edu/~barvinok/
Bayer,	http://www.math.columbia.edu/~bayer/vita.html
Benjamini,	http://www.wisdom.weizmann.ac.il/~itai/
Friedgut,	http://www.ma.huji.ac.il/~ehudf/
Haiman,	http://math.ucsd.edu/~mhaiman/
Herzog,	http://www.uni-essen.de/~mat300/
Kalai,	http://www.ma.huji.ac.il/~kalai/
Kenyon,	http://topo.math.u-psud.fr/~kenyon/
Lovasz,	http://www.cs.yale.edu/~lovasz/
Lyons,	http://php.indiana.edu/~rdlyons/
Matoušek,	http://www.ms.mff.cuni.cz/~matousek
Peres,	http://www.ma.huji.ac.il/~peres
Reiner,	http://www.math.umn.edu/~reiner/
Schramm,	http://www.wisdom.weizmann.ac.il/~schramm/
Stanley,	http://www-math.mit.edu/~rstan/

Talagrand, http://felix.proba.jussieu.fr/users/talagran/
Thomas, http://www.math.gatech.edu/~thomas/
Tsirelson, http://www.math.tau.ac.il/~tsirel/
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