How Quantum Computers Can Fail

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Abstract

We propose and discuss two postulates on the nature of errors in highly correlated noisy physical stochastic systems. The first postulate asserts that errors for a pair of substantially correlated elements are themselves substantially correlated. The second postulate asserts that in a noisy system with many highly correlated elements there will be a strong effect of error synchronization. These postulates appear to be damaging for quantum computers.

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1 Quantum computers and the threshold theorem

Quantum computers are hypothetical devices based on quantum physics. A formal definition of quantum computers was pioneered by Deutsch [1], who also realized that they can outperform classical computation. The idea of a quantum computer can be traced back to works by Feynman, Manin, and others, and this development is also related to reversible computation and connections between computation and physics that were studied by Bennett in the 1970s. Perhaps the most important result in this field and certainly a major turning point was Shor's discovery [2] of a polynomial quantum algorithm for factorization. The notion of a quantum computer along with the associated complexity class BQP is an exciting gift from physics to mathematics and theoretical computer science, and has generated a large body of research. Quantum computation is also a source of new, deep, and unifying questions in various areas of experimental and theoretical physics. For background on quantum computing, see Nielsen and Chuang's book [3].

Of course, a major question is whether quantum computers are feasible. An early critique of quantum computation (put forward in the mid-90s by Unruh, Landauer, and others) concerned the matter of noise:

[P0] The postulate of noise: Quantum systems are noisy.

The foundations of noisy quantum computational complexity were laid by Bernstein and Vazirani in [4]. A major step in showing that noise can be handled was the discovery by Shor [5] and Steane [6] of quantum error-correcting codes. The hypothesis of fault-tolerant quantum computation (FTQC) was supported in the mid-90s by the "threshold theorem" [7, 8, 9, 10], which asserts that under certain natural assumptions of statistical independence on the noise, if the rate of noise (the amount of noise per step of the computer) is not too large, then FTQC is possible. It was also proved that high-rate noise is an obstruction to FTQC. Several other crucial requirements for fault tolerance were also described in [11, 12].

The study of quantum error-correction and its limitations, as well as of various approaches to fault-tolerant quantum computation, is extensive and beautiful; see, e.g., [13, 14, 15, 16, 17, 18, 19]. Concerns about noise models with statistical dependence are mentioned in several places, e.g., [20, 21]. Specific models of noise that may be problematic for quantum error-correction are studied in [22]. Current FTQC methods apply even to more general models of noise than those first considered, which allow various forms of time- and space-statistical dependence; see [23, 24, 25].

The basic conjecture of this paper is that noisy highly correlated data cannot be stored or manipulated. On a heuristic level this conjecture is interesting for both the quantum and the classical cases.[1] The formal conjectures are restricted to the quantum case and refer to decoherence, namely the information loss of quantum systems.

Section 2 gives more background on noise and fault-tolerance. An informal description of our conjectures in Section 3 is followed by a mathematical formulation in Section 4. Section 5 is devoted to a discussion of the consistency of our conjectures with quantum mechanics and with the reality of classical error-correction and fault tolerance. We also discuss the challenge of finding concrete noise models satisfying our conjectures, as well as connections with computational complexity theory and with physics.

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[1] Note that in the classical case correlations do not increase the computational power. When we run a randomized computer program, the random bits can be sampled once they are created, and it is of no computational advantage in the classical case to "physically maintain" highly correlated data.

2 Noise and fault tolerance

The postulate of noise is essentially a hypothesis about approximations. The state of a quantum computer can be prescribed only up to a certain error. For FTQC there is an important additional assumption on the noise, namely on the nature of this approximation. The assumption is that the noise is "local." This condition asserts that the way in which the state of the computer changes between computer steps is statistically independent, for different qubits. We will refer to such changes as "qubit errors." In addition, the gates that carry the computation itself are imperfect. We can suppose that every such gate involves at most two qubits and that the gate's imperfection can take an arbitrary form, so that the errors (referred to as "gate errors") created on the two qubits involved in a gate can be statistically dependent. (Of course, qubit errors and gate errors propagate along the computation and handling this is a main difficulty in fault tolerance.)

The basic picture we have of a noisy computer is that at any time during the computation we can approximate the state of each qubit only up to some small error term ϵ. Nevertheless, under the assumptions concerning the errors mentioned above, computation is possible. The noisy physical qubits allow the introduction of logical "protected" qubits which are essentially noiseless.

The close analogy between the classical case and the quantum case for error correction and fault tolerance is very useful. For our purposes, a good way to understand the notions of quantum error-correction and fault tolerance is to draw the line not between classical and quantum information but between deterministic information (or even stochastic information where the elements are statistically independent) and stochastic highly correlated information (both classical and quantum). Thus, while the state of a digital computer having n bits is a string of length n of zeros and ones, in the (classical) stochastic version, the state is going to be a (classical) probability distribution on all such strings.

Quantum computers are similar to these (hypothetical) stochastic classical computers and they work on qubits (say n of them). The state of a single qubit q is described by a unit vector $u = a|0> + b|1>$ in a two-dimensional complex space U_q. (The symbols $|0>$ and $|1>$ can be thought of as representing two basis elements in U_q.) We can think of the qubit q as representing $'0'$ with probability $|a|^2$ and $'1'$ with probability $|b|^2$. The state of the entire computer is a unit vector in the 2^n-dimensional tensor product of these vector spaces U_q's for the individual qubits. The state of the computer thus represents a probability distribution on the 2^n strings of length n of zeros and ones. The evolution of the quantum computer is via "gates." Each gate g operates on k qubits, and we can assume $k \leq 2$. Every such gate represents a unitary operator on U_g, the (2^k-dimensional) tensor product of the spaces that correspond to these k qubits.

A simple (rather special) example of noise to keep in mind is that all qubit errors are independent random unitary operators for the individual qubits, and all gate errors are random unitary operators on the spaces U_g. If these errors are small (namely, if all these operators are sufficiently close to the identity), the threshold theorem will apply.

A main insight of quantum error-correction is that errors affecting a substantial but small fraction of — even highly correlated – bits/qubits can be handled. (For this, basic linearity properties of probability theory as well as of quantum physics are crucial.) Errors that exceed, with substantial probabilities, the capacity of the error-corrector are problematic. Under the independence assumptions of the threshold theorems, if the rate of errors is small the probability for exceeding the capacity of the error-corrector is extremely small. The crux of the matter is whether independent (or almost independent) errors on highly correlated elements is a possible or even a physically meaningful notion.

3 Noisy stochastic correlated physical systems

3.1 The postulate of noisy correlated pairs

The purpose of this section is to propose and discuss the following postulate:

[P1] In any noisy physical system, the errors for a pair of elements that are substantially statistically dependent are themselves substantially statistically dependent.

In particular, for quantum computers[2] this postulate reads:

[P1] In a quantum computer, the errors for a pair of substantially correlated qubits are substantially correlated.

Another way to put Postulate [P1] is: noisy correlated elements cannot be approximated up to almost independent error terms: if we cannot have an approximation better than a certain error rate for each of two correlated elements, then an uncorrelated or almost uncorrelated approximation is likewise impossible.

Remarks:

1. **Real-life examples: The weather and the stock market.** We can discuss Postulate [P1] for cases of (classical) stochastic systems with highly correlated elements. I am not aware of a case of a natural system with stochastic highly correlated elements that admits an approximation up to an "almost independent" error term. This is the kind of approximation required for fault-tolerant quantum computation.

Can we expect to estimate the distribution of prices of two very correlated stocks in the stock market up to an error distribution that is almost independent?

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[2] Our conjectures themselves come in (highly correlated) pairs. Each conjecture is formulated first for general noisy physical systems and then specified to quantum computers, which are physical devices able to maintain and manipulate highly entangled qubits.

Or take, for example, the weather. Suppose you wish to forecast the probabilities for rain in twenty nearby locations. We suppose these probabilities will be strongly dependent. Can we expect to have a forecast that is off by a substantial error that is almost statistically independent for the different locations?

To make this question a little more formal, consider not how accurately a weather forecast predicts the weather, but rather how it predicts (or differs from) a later weather forecast. Let $\mathcal{D}$ be the distribution that represents the best forecast we can give for the rain probabilities at time T from the data we have at time $T-1$. Let $\mathcal{D}'$ be the best forecast from data we have at time $T-1-t$. Suppose that $\mathcal{D}$ is highly correlated. Postulate [P1] asserts that we cannot expect that the difference $\mathcal{D}-\mathcal{D}'$ will be almost statistically independent for two locations where $\mathcal{D}$ itself is substantially correlated.

2. **The threshold theorem and pair purification.** The threshold theorem that allows FTQC has various remarkable applications, but our postulate can be regarded as challenging its simplest nontrivial consequence. The assumptions of the threshold theorem allow the errors on a pair of qubits involved in a gate to be statistically dependent. In other words, the outcome of a gate acting on a pair of qubits prescribes the position of the two qubits only up to an error that is allowed to exhibit an arbitrary form of correlation. The process of fault tolerance allows us to reach pairs of entangled qubits that, while still being noisy, have errors that are almost independent. This "purifying" nature of fault tolerance for quantum computation is arguably an element we do not find in fault tolerance for deterministic computation.

3. **Positive correlations for errors.** Consider a noisy classical computer on n bits and suppose that the overall error is given by taking the XOR of the n bits in the computer with a randomly chosen string e of n bits according to a probability distribution $\mathcal{E}$. Suppose that for every bit, the error probability is $1/1000$. If the errors are independent then the probability that e will have, say, $n/500$ error is very tiny as n grows. Positive correlations

between the errors for all (or most) pairs of bits will change this picture. For example, if for every two bits the probability that for e both these bits are 1 is around 1/50,000 (rather than 10^{-6}), then there will be a substantial probability that more than $n/100$ bits will be "hit" by the error. (The same conclusion will apply if for every triple of bits the probability that they will all be hit is, say, 10^{-7} rather than 10^{-9}.) This effect of positive correlation for errors is the basis for Postulate [P2] below.

4. **Leaks of information.** Rather than talk about errors and noise we can talk about information "leaked" from our physical systems to the outside world. For quantum computers leaking of information automatically amounts to noise and thus a strong form of Postulate [P1] for quantum computers is:

[P1'] For a noisy quantum computer, information leaks for two substantially correlated qubits have a substantial positive correlation.

For general stochastic systems [P1'] reads:

[P1'] In any noisy physical system, the information leaks concerning the states of two elements that are substantially statistically dependent have a substantial positive correlation.

Postulate [P1'] seems natural for systems where correlations are gradually created and information is gradually leaked. The central question is whether such an effect can be diminished via error correction.

3.2 The postulate of error synchronization

Suppose we have an error rate of ϵ. The assumptions of the various threshold theorems (and other proposed methods for quantum fault-tolerance) imply that the probability of a proportion of δ qubits being "hit" is exponentially small (in the number of bits/qubits) when δ exceeds ϵ. Error synchronization refers to an opposite scenario: there will be a substantial probability of a large fraction of qubits being hit.

[P2] In any noisy physical system with many substantially correlated elements there will be a strong effect of spontaneous error-synchronization.

[P2] In any quantum computer at a highly entangled state there will be a strong effect of spontaneous error-synchronization.

As remarked above, error synchronization is expected for a large system when the errors (or information leaks) are positively correlated. An even stronger form of error synchronization is considered in [26], where formal definitions for the quantum case can be found.

Remarks:

1. **Empiric.** Postulates [P1] and [P2] can be tested, in principle, for quantum computers with a small number of qubits (10-20). Even if such devices where the qubits themselves are sufficiently stable are still well down the road, they are to be expected long before the superior complexity power of quantum computers kicks in.

The rigorous form of Postulate [P1] (Section 4) can be suggested as a benchmark for quantum-computer engineers: to construct pairs of noisy entangled qubits with almost independent error-terms.

2. **Spontaneous synchronization for highly correlated systems.** The idea that for the evolution of highly correlated systems changes tend to be synchronized, so that we may witness rapid changes affecting large portions of the system (between long periods of relative calm), is appealing and may be related to other matters like sharp threshold phenomena and phase transition, the theory of evolution, the evolution of scientific thought, and so on.[3] We can examine the possibility of error synchronization for the examples considered above. Can we expect synchronized errors for weather forecasts? Can we expect stock prices, even in short time scales, to exhibit substantial probabilities for changes affecting a large proportion of stocks?

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[3] This idea is conveyed in the Hebrew proverb "When troubles come they come together."

Spontaneous synchronization is also related to the issue of pattern formation for correlated systems.

3. **Error synchronization and the concentration of measure phenomenon.** A mathematical reason to find spontaneous synchronization of errors an appealing possibility is that it is what a "random" random noise looks like. Talking about a random form of noise is easier in the quantum context. If you prescribe the noise rate and consider the noise as a random (say unitary) operator (conditioning on the given noise rate), you will see a perfect form of synchronization for the errors, and this property will be violated with extremely low probability.

Random unitary operators with a given noise rate are *not* a realistic form of noise. The qubits in a quantum computer are expected to be quite isolated, so that the errors are described by a "locally defined" process (namely, a process (stochastically) generated by operations on a small number of qubits at a time) — not unlike the (noiseless) evolution described by quantum computation itself.

While random unitary operators with a prescribed error rate appear to be unapproachable by any process of a "local" nature, the issue is whether some of their statistical properties may hold for such stochastic processes describing the errors. The fact that perfect error-synchronization is the "generic" form of noise may suggest that stochastic processes describing the noise will approach this "generic" behavior unless they have good reason not to. (One obstruction to error synchronization, pointed out by Greg Kuperberg, is time independence.)

4. **Correcting highly synchronized errors.** An observation that complements the discussion so far is that synchronized errors that are unbiased can be corrected to produce noiseless deterministic bits. Suppose we have a situation in which an error hits every bit with probability $(1-\epsilon)$ and when a bit is hit it becomes a random unbiased bit. (That is, a bit is flipped with probability $(1-\epsilon)/2$.) This type of noise can be corrected by representing

a 0 bit by a long string of 0's and a 1 bit by a long string of 1's. (If the noise hits a smaller fraction of bits, the condition of it being unbiased can be compromised.) If we start with qubits and again, with probability $(1-\epsilon)$, replace each qubit with a random (uniformly distributed) qubit, we can still extract noiseless *bits*. However, there is no quantum error-correction code for such noise.

This means that deterministic noiseless bits can prevail (for classical and quantum systems) even for some forms of highly correlated errors. (Our postulates do not imply a high correlation for the errors when the elements of the system are statistically independent, but mechanisms leading to our conjectural effects may still be relevant for the nature of noise for classical forms of storing information and computation.)

The method of "clone and sample" appears to be essentially the only error-correction method we find in nature. This method allows us to introduce gates where errors on the involved bits will be almost independent to start with, and thus will reduce "noise on gates" to "noise on bits." But this method is unavailable for quantum information of a general type.

5. **The censorship conjecture**. Notions of "highly correlated" or "highly entangled" systems are not easy to define. We will refer informally to systems that up to a small error are induced by their marginal distributions on small sets of elements as "approximately local."

[C] Censorship conjecture: Noisy stochastic physical systems are approximately local.

[C] The states of quantum computers are approximately local.

The rationale for this conjecture is that high forms of entanglement will necessarily be accompanied with a strong effect of error synchronization, which in turn will push the system towards approximate locality.

A suggested definition of "approximately local" (for the quantum case only), and a precise formulation of the conjecture are given in the next section (see also [26] for a different approach).[4]

4 A mathematical formulation

4.1 Measuring information leaks

In this section we give a mathematical formulation for Postulate [P1'] and present even stronger conjectures. Our setting is as follows. We have a quantum computer running on n qubits. The noise can be described by a unitary operator on the computer qubits and the neighborhood qubits or as a quantum operation E on the space of density matrices for these n qubits. The ideal state of the quantum computer is pure.

Consider the conjectures in this section in the following way. A noisy quantum computer is subject to noise described by a quantum operation E, such that the error rates for individual qubits are small but substantial and E satisfies the requirements described in this section. The operation E need not be the overall noise that describes the gap between the ideal state and the noisy state of the computer, but we assume that any damaging properties of E will not be remedied by additional noise of a different nature.

We denote by $L(a)$ the "amount of information the neighborhood has on the qubit a." More generally, for a set A of qubits we denote by $L(A)$ the "amount of information the neighborhood has on A." Next we propose mathematical definitions for these notions.

Let ρ be a state of the computer. For a set A of qubits let $\rho|_A$ be the induced state on A. When the state ρ is a tensor product pure state then for every set A of qubits, $S(\rho|_A) = 0$ and the information leak of the noise operator E from the set of qubits A can be measured by the entropy $S((E \circ \rho)|_A)$.

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[4] There are many measures for the "amount of entanglement" (and correlation) that can be used. It is also unclear if we should measure the entanglement of a single state or use a measure that depends on the variety of feasible states for a system. Leggett's early paper [27] and his "disconnectivity measure" (D-measure) seem relevant.

Here, $S(*)$ is the (von Neumann) entropy function see, e.g., [3]; Ch. 11. (We deem this entropy-based notion appropriate for our purposes, evan though the entropy does not capture every form of "information leak" attributable to a noise operator.)

I am unaware of any canonical way to make the "information leak" a measure of the noise operation E that does not depend on a specific choice for this tensor product state. In what follows we let $\rho_0 = (+)^{\otimes n} = (1/\sqrt{2}((|0> +|1>))^{\otimes n}$ and define $L(A) = L_E(A) = S(E(\rho_0|_A))$.

Remark: Let $\hat{\rho}$ be the state of the computer's qubit and the environment that is represented by a set N of qubits. A standard measure of the information that the environment has on the qubits in A is $L'(A) = S(\hat{\rho}|_A) + S(\hat{\rho}|_N) - S(\hat{\rho}|_{\{A \cup N\}})$. I would expect that $L'(A)$ can replace $L(A)$ for the formulation of the conjectures in this section.

4.2 Two qubits

We will state mathematically a version of Postulate [P1']. Our setting is as follows. Let ρ be the "ideal" state of the computer and consider two qubits a and b. We use as the (rather standard) measure of entanglement

$$ENT(\rho; a, b) = S(\rho|_a) + S(\rho|_b) - S(\rho|_{\{a,b\}}).$$

As a measure of correlation of information leaks we use

$$EL(a, b) = L(a) + L(b) - L(\{a, b\}).$$

Postulate [P1'] can be formulated as:

$$EL(a, b) \geq K(L(a), L(b)) \cdot ENT(\rho; a, b), \tag{1}$$

where K is a function of $L(a)$ and $L(b)$, which is substantially larger than their average $(L(a) + L(b))/2$. ($K(0, 0) = 0$, so that relation (1) tells us nothing about noiseless entangled systems.)

Remark: We are mainly interested in the case where the error rate is fixed, but the dependence of $K(L(a), L(b))$ on the error rates is also of interest.

4.3 Two qubits: A stronger version

We go on to describe and motivate an even stronger form of [P1'] and an extension to more than two qubits. These extensions go beyond Postulates [P1] and [P2] as discussed so far.

The expression $S(\rho|_a) + S(\rho|_b) - S(\rho|_{\{a,b\}})$ was used as a measure of entanglement between two qubits. We would like to replace it by a measure that can be called "emergent entanglement," which we are now going to define. This measure, denoted by $EE(\rho; a, b)$, captures (roughly) the expected amount of entanglement among the two qubits when we measure some other qubits, "look at the outcome," and condition on all possible outcomes for the measurement. It appears to be related to Briegel and Raussendorf's notion of "persistent entanglement" [28].

For every representation ω of $\rho|_{\{a,b\}}$ as a mixture (convex combination) of joint states

$$\rho|_{\{a,b\}} = \sum_{i=1}^{t} p_k \rho_k,$$

let

$$ENT_\omega(\rho; a, b) = \sum p_k ENT(\rho_k; a, b).$$

Define

$$EE(\rho; a, b) = \max ENT_\omega(\rho; a, b),$$

where the maximum is taken over all representations ω. (We can assume that ω is a mixture of pure joint states.)

A strong form of relation (1) is

$$EL(a, b) \geq K(L(a), L(b)) \cdot EE(\rho; a, b), \tag{2}$$

where, as before, K is a function of $L(a)$ and $L(b)$ which is substantially larger than their average $(L(a) + L(b))/2$.

The motivation for this strong version of Postulate [P1'] comes from considering the state of a quantum computer that applies a fault-tolerant computation. The state of the computer is t-wise independent for a large value of t; hence every two qubits are statistically independent and Postulate [P1'] does not directly apply. Consider an error-correcting code and let s be the minimal number of qubits whose state "determines" that of the others, so that once they are measured and their value are "looked at" the state of the other qubits is determined. When we measure and look at the values of $s-1$ qubits, we see a very strong dependence between every pair of the remaining qubits. Now, if we assume Postulate [P1'] and (just tentatively) also assume that "measuring and looking at" the contents of some qubits does not induce errors on other qubits (this is a standard assumption in current noise models), we see that the conclusion of Postulate [P1'] should apply for pairs of qubits in a quantum computer running FTQC even though pairs of qubits are independent.

4.4 More qubits

Here is a suggestion for an extension of the above conjecture from pairs of qubits to larger sets of qubits. This suggestion goes beyond Postulates [P1] and [P2] and is related to a strong form of error synchronization conjectured in [26].

For a set $A = \{a_1, a_2, \ldots, a_m\}$ of m qubits let

$$ENT(\rho; A) = -S(\rho) + \max S(\rho^*),$$

where ρ^* is a mixed state with the same marginals on proper sets of qubits as ρ, i.e., $\rho^*|_B = \rho|_B$ for every proper subset B of A.

Define in a similar way

$$EL(A) = -L_E(A) + L_{E^*}(A),$$

where E^* is a quantum operation which satisfies $E^*|_B = E|_B$ for every proper set B of A.

Using these definitions we will extend our conjectures, given by relations (1) and (2), from pairs of qubits to larger sets of qubits. Let ρ be an ideal state of the computer and let A be a set of m qubits. Extending (1) we conjecture that

$$EL(A) \geq K_m ENT(\rho|_A). \tag{3}$$

Here, $K_m = K_m(\{L(a) : a \in A\})$ is substantially larger than $\min\{L(a))) : a \in A\}$ and it vanishes when all the individual information leaks vanish.

Here again we further conjecture that for every representation ω of the state $\rho|_A$ as a convex combination $\rho|_A = \sum p_k \rho_k$ of pure joint states,

$$EL(A) \geq K_m \sum p_k ENT(\rho_k; A). \tag{4}$$

Remarks: 1. We expect that a quantum error-correcting code that corrects t-errors and has a fixed error rate will have a strong form of error-synchronization as t tends to infinity. Namely, the noise operation will have a similar effect to that of the following operation: with probability ϵ a $(1-o(1)$-fraction of qubits are being measured. (This is referred to as "devastating" noise in [26].) I expect that Postulate [P1'] as expressed by relation (2) will imply the weaker form of error-synchronization discussed in Section 3.2, while an extension for larger sets of qubits given by (4) will imply the stronger form.

2. The value of $ENT(\rho; A)$ is intended to serve as a measure of the additional information when we pass from "marginal distributions" on proper subsets of qubits to the entire distribution on all qubits.

4.5 Censorship

Here is a suggestion for an entropy-based mathematical formulation for Conjecture [C]. We remind the readers that in this section we always assume that the "ideal" state of the quantum computer (before the noise is applied) is a pure state. Some adjustments to our conjectures will be required when the ideal state itself is a mixed state.

Let ρ be a pure state on a set $A = \{a_1, a_2, \dots, a_n\}$ of n qubits. Define

$$\widetilde{ENT}(\rho) = \sum \{ENT(\rho; B) : B \subset A\}.$$

In this language a way to formulate the censorship conjecture is:

There is a polynomial P (perhaps even a quadratic polynomial) such that for any quantum computer on n qubits, which describes a pure state ρ,

$$\widetilde{ENT}(\rho) \leq P(n). \tag{5}$$

We will mention now some mathematical challenges. It will be interesting to prove relation (5) based on relation (3), and to formulate and prove weak and strong forms of error synchronization based, respectively, on relations (1) and (3).

A further goal would be to derive, based on the assumptions on noise for the physical qubits (relations (3) and (4)), the same relations as well as relation (5), for "protected" qubits, namely logical qubits represented by quantum error-correction.

Remarks: 1. It is interesting to study how the quantities $ENT(A; \rho)$ evolve in time for dynamical systems describing (quantum and classical) physical processes.

2. The additional conjectures of this section are meant to draw the following picture: we have an ideal notion of a quantum computer that has extraordinary physical and computational properties. Next come noisy quantum computers with an ideal notion of noise. If the noise rate is small then

FTQC is possible. Next come noisy quantum computers that satisfy relation (1). For those, fault tolerance will require controlling the error rate as well as K_2, which we expect to be much harder. This model is also an idealization as long as $K_3 = 0$ and so on. For such highly entangled states as those required in quantum algorithms, K_i will be more and more damaging for larger values of i.

5 Discussion

Our conjectures on the nature of noise for correlated systems appear to be damaging to the possibility of storing and manipulating correlated quantum or classical stochastic data and therefore for the possibility of computationally-superior quantum computers.

In Section 5.1 we ask if our conjectures are consistent with quantum mechanics. We also examine if they are consistent with the reality of classical error-correction and fault-tolerant classical computation. In Section 5.2 we ask if our conjectures can be supported by concrete models of noise. In Section 5.3 we discuss the computational complexity consequences of the conjectures, and in Section 5.4 we ask if there are any existing or expected counterexamples from physics.

5.1 Consistency

Causality

We do not propose that the entanglement of the pair of noisy qubits *causes* the dependence between their errors. The correlation between errors can be caused by the process leading to the correlation between the qubits, or simply by the ability of the device to achieve strong forms of correlation.

Linearity

Do our postulates violate the linearity of quantum physics? The plain simple answer is no. Again the analogy with classical stochastic processes is telling. The conjecture that in noisy systems like the weather substantially correlated events are subject to substantially correlated noise (or, in other words, can only be approximated up to error terms that are also substantially correlated) is perhaps bold and possibly false, but it is not remotely bold enough to violate the laws of probability theory. This is also so in the quantum case.

Probability, secrets, and computing

We will now describe a difficulty for our conjectures at least in the classical case. Consider a situation where Alice wants to describe to Bob a complicated correlated distribution $\mathcal{D}$ on n bits that can be described by a polynomial-size randomized circuit. Having a noiseless (classical) computation with perfect independent coins, Alice can create a situation where for Bob the distribution of the n bits is described precisely by $\mathcal{D}$. In this case the values of the n bits will be deterministic and $\mathcal{D}$ reflects Bob's uncertainty. Alice can also make sure that for Bob the distribution of the n bits will be $\mathcal{D} + \mathcal{E}$, where $\mathcal{E}$ describes independent errors of a prescribed rate.

Is this a counterexample to our Postulates [P1] and [P2]? One can argue that the actual state of the n bits is deterministic and the distribution represents Bob's uncertainty rather than "genuine" stochastic behavior of a physical device.[5] But the meaning of "genuine stochastic behavior of a physical device" is vague and perhaps ill-posed. Indeed, what is the difference between Alice's secrets and nature's secrets? In any case, the difficulty described in this paragraph cannot be easily dismissed.[6]

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[5] Compare the interesting debate between Goldreich and Aaronson [29] on whether nature can "really" manipulate exponentially long vectors.

[6] The distinction between the two basic interpretations of probability as either expressing human uncertainty or as expressing some genuine physical phenomenon is an important

However, note that as in the case of faraway qubits, the noisy distribution $\mathcal{D}+\mathcal{E}$ was based on the ability to achieve the noiseless distribution $\mathcal{D}$. Achieving the distribution $\mathcal{D}$ was based on noiseless classical computation to start with. For the case of quantum computers, we can still defend our Postulates [P1] and [P2] against this argument as follows. Even if nature can simulate Alice, and Bob's "mental" uncertainty can be replaced by a "real" physical situation where a highly correlated distribution is prescribed up to an independent error term, this approximation has been achieved via a noiseless computation to start with. Therefore, such an approximation cannot serve, in the quantum case, as a basis for fault tolerance.

The difficulties considered here motivated the formulation of our conjectures (Section 3.1, remark 4, and Section 4) in terms of information leaks.[7] The mathematical formulation of our postulates in Section 4 is thus restricted to the quantum case.[8]

Faraway qubits

Suppose we have two qubits that are far away from each other at a given entangled state at time T. Consider their state at time $T+t$. Is there any reason to believe that the changes will not be independent? And if t is small compared to the distance between the qubits isn't it the case that to implement a noise that is not independent we will need to violate the speed

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issue in the foundation of (classical) probability. See, e.g., Anscombe and Aumann [30]. Opinions range from not seeing any distinction at all between these concepts to regarding human uncertainty as the only genuine interpretation.

[7] A related idea, relevant also to the next item of faraway qubits, is to regard a stricter definition of a noisy quantum computer as such that at *any time* along the computation for every qubit, and *for every observer* (who extracts information from the computer) the noise rate for every qubit (namely, the difference between its ideal state and its actual state) is at least ϵ.

[8] For the classical case (or for a commutative fragment of noncommutative probability) our postulates [P1] and [P2], as well as the postulate of noise [P0] and even the notion of noise itself, are meaningful only on a heuristic or subjective level.

of light? And finally isn't this observation a counterexample to Postulate [P1]?

The answer to the last question is negative. There is no difficulty in conceding that changes over time in the states of two faraway entangled qubits will be independent. The problem with this critique is the initial assumption: we are *given* two qubits at time T at a given state. Starting with noiseless correlated elements, we may well reach correlated elements that can be described up to substantial but independent error terms. But for fault tolerance we may not assume noiseless pairs of entangled qubits to start with.

In this paper, as in other papers dealing with FTQC, we assume that at any time during the computation every qubit is noisy. Sometimes, a quantum computer that is only partially noisy is studied (e.g., [19]). In such a case we should reformulate Postulates [P1] and [P2] relative to the noiseless part.

Running a quantum algorithm with a "random" state at all times

A critique of the possibility of any systematic damaging relation between the state of the quantum computer and the noise was suggested by Ben-Or (see [26]) and is related to some works of Preskill and Shor. (A related concern was pointed out by Preskill. A detailed proof of such a result along with an interesting interpretation was recently offered by Aharonov [31].) Having a classical computer control a quantum computer makes it possible to run a variant of any quantum computer program where at the initial state we apply random Pauli operators on every qubit and modify the action of the gates accordingly. This interesting critique does not apply to the mathematical formulation given in Section 4 for Postulate [P1'], since the measures of entanglement we use are invariant under such an operation.

To sum up this part of the discussion, our conjectures, properly formulated in terms of decoherence of quantum systems, are consistent with the reality of classical fault-tolerance. They also appear to be fully consistent with quantum mechanics. On the other hand, as proposed properties of decoherence our conjectures are not in agreement with the common point of view regarding decoherence and the traditional decoherence models.

5.2 Models

A basic remaining challenge is to present concrete models of noise that support Postulates [P1] and [P2]. (Of course, there is a difference between showing that the type of behavior we are looking for is possible and showing that it is unavoidable.) A model for the noise that supports our postulates should already exhibit [P1] and [P2] for the "new errors" — either qubit-errors or gate-errors or both — and would thus be quite different from current models.

It is worth noting that error synchronization is a very familiar phenomenon for error propagation of (unprotected) quantum programs. It is instructive to see in this context how error synchronization is often created when we start with small independent errors and let them propagate along the steps of a computer program.

One way to view the noise is as represented by a rather primitive (but quick) stochastic program (or circuit) "running" along the actual program. We run the program $\mathcal{P}$ and we actually get $\mathcal{P}+\mathcal{N}$. The simplest explanation for why errors of correlated qubits are themselves correlated is that the noise $\mathcal{N}$ depends on $\mathcal{P}$, or can be described as a weak perturbation of the original program itself. But this is not the only possibility. It may be the case that $\mathcal{N}$ does not depend on $\mathcal{P}$ but rather represents a certain form of "generic" quantum program. In both these cases we think of $\mathcal{N}$ as a quantum program with many steps for each computer cycle. This hypothetical "noise program" partially achieves one familiar "computational task" for a distributed system: synchronization.

Recently, Klesse and Frank [34] described a physical system in which qubits (spins) are coupled to a bath of massless bosons. They reached (after certain simplifications) a noise model with error synchronization. (I am thankful to Robert Raussendorf for this information.) The earlier models suggested by Alicki, Horodecki, Horodecki, and Horodecki [22] appear to be relevant to our conjectures. Also relevant is Alicki's idea [32] (see also [33]) that "slow gates" (combined with the free evolution of the system) will be an obstacle to error correction.

There is a substantial interest in local stochastic behavior leading to spontaneous (collective) synchronization (e.g., [35, 36, 37, 38, 39]). The Glauber dynamics (a very simple locally defined "program") for the Potts model (e.g., [40]) can also be regarded in this way. There is also a substantial amount of work on emergence of patterns in stochastic (correlated) locally described systems.

Finally, let me mention the relevance of *cluster states* defined by Briegel and Raussendorf (see, [41]). (We will further discuss cluster states below.) The description of cluster states involves an array of qubits located on the vertices of a rectangular lattice in the plane (or in space). Cluster states are "generated" by local entanglements between pairs of nearby qubits on the lattice grid. They can be regarded as the quantum analogs of the Ising and Potts classical models. (Note that "cluster state" is a collective name for a large number of possible models.) There is some evidence in the literature (see [42]) that cluster states emerge in realistic situations.

Controlled creation and manipulation of cluster states can be very important for building quantum computers. On the other hand, cluster states (and decohered cluster states) can serve as a basis for concrete models of noise. Local processes leading (in reality) to cluster states may represent realistic models of decoherence. This possibility deserves further study. (I am thankful to Scott Aaronson for fruitful discussions concerning cluster states.)

5.3 Computation complexity

While it looks intuitively correct that our postulates are damaging for quantum computation, proving it, and especially proving a reduction all the way to the classical model of computation, is not going to be an easy task. This is an interesting question in computational complexity. In an earlier paper [26] some problems on the computational power of quantum computers with various hypothetical types of noise were considered. Reduction of low-rate noisy quantum computation to BPP is not known even for cases where the noise is provided by an adversary. A more realistic task would be to show that our postulates exclude fault tolerance based on linear quantum error-correction.

Let me first mention a few relevant earlier works. The problem of describing complexity classes of quantum computers subject to various models of noise was proposed by Peter Shor [43] in the 90s, but apparently was not picked up. See also [44]. It is even theoretically possible that deviating from the standard assumptions on noise (and, in particular, allowing dependence of the noise $\mathcal{N}$ on the program $\mathcal{P}$), will allow stronger computation power than BQP.

Aaronson [45] proposed a theory to study the computational-complexity effect of an arbitrary form of restriction on states of quantum computers, namely, some states are forbidden while others can be freely prepared and manipulated. Aaronson was motivated by several skeptical opinions on quantum computers, especially Levin's paper [27], and he proposed his theory as a way "to make the debate concerning quantum computers more scientific and less ideologic." The approach taken here is close, in broad strokes, to Aaronson's. (A notable difference in the rhetoric is that Aaronson equates the failure of quantum computers with the breakdown of quantum mechanics.)

Going back to the issue of how damaging our postulates on noisy quantum computers can be, we note that going below the computation power of logarithmic depth polynomial-size quantum circuits appears to be difficult, yet such circuits combined with classical computers are strong enough to allow a polynomial-time algorithm for factoring as follows from a recent result of Cleve and Watrous [46].

Aharonov, Ben-Or, Impallazio, and Nisan [12] proved that the computational power of noisy *reversible* quantum computers reduces to log-depth quantum computation. But it is not even known whether or not noisy reversible quantum computers (combined with a noiseless classical one), under the standard noise models, allow polynomial-time factoring, and it may well be the case that they do. Razborov [18] showed that, for a certain (standard) model of noise, if the qubit-error rate is 50% then the computation power reduces to log depth quantum computation. In this case too, a reduction to classical computation is unknown. These results suggest that also in our case it will be difficult to prove a reduction which excludes log-depth quantum circuits. Moreover, it is quite possible that log-depth quantum circuits prevail, with quasi-polynomial or even polynomial overhead, under rather general forms of noise. This is the case when the noise is described by a random unitary operator; the perfect error-synchronization allows log-depth circuits to work perfectly with a probability which is only polynomially small. Combining this observation with the application of low-overhead circuits for fault-tolerance may apply to *every* low-rate noise model for which the noise is invariant under permutations of the qubits. (Since constants in the depth translate to exponents in the overhead, such a result will not be practically useful.)

Complexity-theoretic reductions appear difficult and so is Scott Aaronson's nice challenge of a "Sure/Shor separator" [45]. A more realistic goal would be to prove that models of noise satisfying our conjectures do not allow for quantum linear error-correction, e.g., by deriving relations (1) and (5) for any form of "protected qubits" obtained by linear quantum error-correction.

5.4 Potential counterexamples from physics

Topological quantum computers and anyons

In the area of "topological quantum computers" [14, 47, 48] there is a beautiful and powerful "bilingual dictionary" between certain forms of combinatorial methods for fault tolerance and remarkable objects from physics. It is suggested that fault tolerance can be realized by *non-Abelian anyons*, which can be thought of as analogous to the physical realization of logical bits in a digital computer that are very robust to noise. The two "languages" of this dictionary are a combinatorial descriptions of a quantum error-correction with n qubits, and their physical realization as certain quasi-particles called "anyons." Only a small number of types of non-Abelian anyons are required to realize the full power of quantum computers.

Analyzing the stability of non-Abelian (and Abelian) anyons based on the assumption that the noise is "local" (statistically independent, as discussed above) reveals a remarkable phenomenon referred to as "mass gap." Below a certain temperature the anyon is going to be extremely stable. (Low temperature translates to low error rate.) The mathematical model predicts that as n grows the region of stability (in terms of temperature) will not become smaller and the "gap" will be maintained. Moreover, in this stable area the robustness to noise will be exponential in n and thus, on the physics side, we will obtain very robust qubits.[9]

The existence of very robust ("protected") qubits based on quantum error-correction via a highly entangled state, whether implemented by "software," say ion traps, or by "hardware," say anyons, runs counter to our conjectures. We can expect that when we study the effect of noise for the combinatorial model of anyons with a highly entangled state, using a perturbation method that reflects Postulates [P1] and [P2], the exponential robustness with n, or even the "mass gap," will disappear.

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[9] In the translation between a discrete combinatorial model with n qubits (or n elements) and a concrete physical object, it is unclear what the interpretation is on the physics side for the value of n, and the relevance of the behavior as n tends to infinity should not be taken for granted. Inner dependencies of the physical object appear to be relevant to the best value of n for the combinatorial model describing it.

Other possible counterexamples from physics

Superconductivity. Several people have suggested that our postulates, and especially Conjecture [C], are already in conflict with phenomena from physics, like superconductivity and Einstein–Bose condensation. Superconductivity and related phenomena are indeed physical systems with strong forms of (pairwise) entanglement that appear to be related to what is required for quantum fault-tolerance.

I tend to think that the form of entanglement for superconductivity is insufficient to refute Conjecture [C] since the entanglement in this case is "generated" (to a large extent) by dependencies of pairs of elements. Translating and testing Postulate [P1'] for the setting of superconductivity would be of interest.

Cluster states. Cluster states, defined by Briegel and Raussendorf (see [41]) are roughly quantum analogs of (low-temperature) probability distributions described by Ising and Potts models. There is some simulation-based evidence [42] that certain materials from solid-state physics exhibit a similar form of entanglement. Those materials can thus be potential candidates for checking empirically our conjectures on decoherence. On the other hand, the possibility of having universal quantum computation [41], and even fault-tolerant quantum computation [17], based on cluster states and single-qubits measurements may challenge the relevance of our conjectures on the FTQC hypothesis.[10] For cluster states $\widetilde{ENT}$ appears to be linear in the number of qubits.

2n bosons. Noisy quantum computers that respect our postulates are incapable of simulating hypothetical objects like non-Abelian anyons. But are they capable of simulating familiar, much simpler, objects from physics? A simple example to test the conjectures is to consider them for a state X of

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[10] What needs to be examined in this respect is the translation of non-Markovian noise models (like those satisfying our conjectures) on the cluster-state computer, back to the quantum circuit it simulates; see [17].

$2n$ bosons (n large) each having a ground state $|0>$ and an excited state $|1>$, so that $|0>$ has occupation number (precisely) n and $|1>$ has occupation number n. (A similar state Y where the occupation number has a binomial distribution can be simulated by a tensor product state.)

Other possible relations to physics

An obvious connection to physics is that a failure of computationally superior quantum computing would suggest that computations of quantum physics that are relevant to physical reality can efficiently be simulated on classical computers, and thus would question the relevance to reality of computations from quantum physics that appear to be computationally hard. We mention two other potential connections.

Perturbation methods. Another connection to physics may come from the perturbation methods used to analyze non-Abelian anyons. These methods are related to standard perturbative methods used in various other areas of physics and mainly in quantum field theory. Modifications of the perturbation method itself, which may amount to amending unjustified hidden probabilistic assumptions and may lead to a drastically different behavior for the extreme situation of (hypothetical) highly entangled systems like quantum error-correcting code and quantum computers, may be of interest also in more mundane situations from physics, where these perturbation methods are (rather successfully) used.

Thermodynamics. Connections between fault tolerance and thermodynamics were considered, e.g., in [33, 49], and were intensely debated. (The results and methods of [12] also have a clear thermodynamic flavor.)

For example, in a very recent paper, Alicki and Horodecki [49] propose the following line of thought: 1. Thermodynamics is relevant because very robust storage of quantum information requires large systems. 2. Meta-stable states for finite systems are necessarily manifested by equilibrium states of infinite systems. 3. Equilibrium states of infinite systems must have the form "probability measures over a set of states" and, in particular, cannot support even a single ("viable") qubit. Of the above points, the second is perhaps the most controversial, and in view of some potential counterexamples may be related to noise models and perturbative methods that are different from the standard ones.

The information-theoretic form of the mathematical formulation of our postulates (which, to a large extent, were required in order to respond to various points discussed in this section) suggests possible connections with thermodynamics. Of particular interest are connections with entropy-type measures of "high order" statistical dependence.

5.5 Conclusion

My belief is that the interesting question of the physically realistic "Church–Turing thesis" (put forward mainly by Deutsch) and, in particular, the feasibility of computationally superior quantum computers will have a convincing solution, and that, whatever this solution is, the asymptotic approach — namely, the relevance of the asymptotic behavior of complexity to real-life computation — will prevail.

The question "How can (computationally superior) quantum computers fail"[11] is as important a part of the quantum information and quantum computers endeavor, as the question "How can (computationally superior) quantum computers succeed." As a matter of fact, the two questions are the same.

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[11] While the possibility of computationally superior quantum computers certainly captures the imagination, it is worth noting that implementing even simple computations on quantum systems can be important for applications, such as enhancing the performance of medical NMR [50, 51].

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