First Passage Percolation Has Sublinear Distance Variance

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Abstract

Let $0 < a < b < \infty$, and for each edge e of $\mathbb{Z}^d$ let $\omega_e = a$ or $\omega_e = b$, each with probability $1/2$, independently. This induces a random metric dist_ω on the vertices of $\mathbb{Z}^d$, called first passage percolation. We prove that for $d > 1$ the distance $\mathrm{dist}_\omega(0, v)$ from the origin to a vertex v, $|v| > 2$, has variance bounded by $C\,|v|/\log|v|$, where $C = C(a, b, d)$ is a constant which may only depend on a, b and d. Some related variants are also discussed.

1 Introduction

Consider the following model of first passage percolation. Fix some $d = 2, 3, \dots$, and let $E = E(\mathbb{Z}^d)$ denote the set of edges in $\mathbb{Z}^d$. Also fix numbers $0 < a < b < \infty$. Let $\Omega := \{a, b\}^E$ carry the product measure, where $\mathbf{P}[\,\omega_e = a\,] = \mathbf{P}[\,\omega_e = b\,] = 1/2$ for each $e \in E$. Given $\omega = (\omega_e : e \in E) \in \Omega$ and vertices $v, u \in \mathbb{Z}^d$, let $\mathrm{dist}_\omega(v, u)$ denote the least distance from v to u in the metric induced by ω; that is, the infimum of $\sum_{e\in\alpha} \omega_e$, where α ranges over all finite paths in $\mathbb{Z}^d$ from u to v. Let $|v| := \|v\|_1$ for vertices $v \in \mathbb{Z}^d$.

Theorem 1. *There is a constant $C = C(d, a, b)$ such that for every $v \in \mathbb{Z}^d$, $|v| \ge 2$,*

$$\mathrm{var}\big(\mathrm{dist}_\omega(0, v)\big) \le C\,\frac{|v|}{\log|v|}\,.$$

In [14] Kesten used martingale inequalities to prove $\operatorname{var}\big(\operatorname{dist}_\omega(0,v)\big) \le C\,|v|$ and proved some tail estimates. Talagrand [18] used his "convexified" discrete isoperimetric inequality to prove that for all $t > 0$,

$$\mathbf{P}\Big[\,|\operatorname{dist}_\omega(0,v) - M| \ge t\sqrt{|v|}\,\Big] \le C\,\exp(-t^2/C)\,, \tag{1}$$

where M is the median value of $\operatorname{dist}_\omega(0,v)$. (Both Kesten's and Talagrand's results apply to more general distributions of the edge lengths ω_e.) "Novice readers might expect to hear next of a central limit theorem being proved," writes Durrett [8], describing Kesten's results, "however physicists tell us ... that in two dimensions the standard deviation ... is of order $|v|^{1/3}$." Recent remarkable work [1, 11, 7] not only supports this prediction, but suggests what the limiting distribution and large deviation behavior is. The case of a certain variant of **oriented** first passage percolation is settled by Johansson [11]. For lower bounds on the variance in two-dimensional first passage percolation see Newman-Piza [15] and Pemantle-Peres [16].

As in Kesten's and Talagrand's earlier results, the most essential feature about first passage percolation which we use is that the number of edges $e \in E$ such that modifying ω_e increases $\operatorname{dist}_\omega(0,v)$ is bounded by $C\,|v|$.

Another essential ingredient in the current paper is the following extension by Talagrand [17] of an inequality by Kahn, Kalai and Linial [12]. Let J be a finite index set. For $j \in J$, and $\omega \in \{a,b\}^J$ let $\sigma_j\,\omega$ be the element of $\{a,b\}^J$ which is different from ω only in the j-th coordinate. For $f : \{a,b\}^J \to \mathbb{R}$ set

$$\rho_j f(\omega) := \frac{f(\omega) - f(\sigma_j\,\omega)}{2}\,.$$

Talagrand's [17, Thm. 1.5] inequality is

$$\operatorname{var}(f) \le C \sum_{j\in J} \frac{\|\rho_j f\|_2^2}{1 + \log\big(\|\rho_j f\|_2/\|\rho_j f\|_1\big)}\,, \tag{2}$$

where C is a universal constant. (In Section 4 we supply a direct proof of (2) from the Bonami-Beckner inequality. A very reasonable upper bound for C can be obtained from this proof.)

The basic idea in the proof of Theorem 1 is to apply this inequality to $f(\omega) := \operatorname{dist}_\omega(0,v)$. For this, we wanted to show that, roughly, $\mathbf{P}\big[\,\rho_e f(\omega) \ne 0\,\big]$ is small, except for a small number of edges e. However, since we were not able to prove this, we had to resort to an averaging trick.

Theorem 1 should hold for other models of first passage percolation, where the edge lengths have more general distributions. However, we chose to prefer simplicity to generality. There are some nontrivial difficulties to obtain some of the expected generalizations. The relevant result of [12], as well as (2) rely on the Bonami-Beckner [6, 2] inequality, which holds for $\{a, b\}^n$, but fails on some more general product spaces. However, [5] does extend some of these results to general product spaces. See also [9]. Talagrand's [17, Thm. 1.5] applies to product measures on $\{a, b\}^n$, which are not necessarily uniform.

To first illustrate the basic technique in a simpler setting we will also prove the following theorem about the variance of the first passage percolation diameter in graphs with symmetries. If $G = \big(V(G), E(G)\big)$ is a finite connected graph, and $\omega : E(G) \to \{a, b\}$, define

$$\operatorname{diam}_\omega(G) := \max_{v,u \in V(G)} \operatorname{dist}_\omega(v, u)\,.$$

(Although we have defined $\operatorname{dist}_\omega$ only for $\mathbb{Z}^d$, the definition obviously extends to arbitrary graphs.) Also, let $\operatorname{diam}(G)$ denote the diameter of G in the graph metric; that is, $\operatorname{diam}(G) = \operatorname{diam}_1(G)$.

Theorem 2. *Let $G = \big(V(G), E(G)\big)$ be a finite connected graph, and let Γ be the group of automorphisms of G. Set $Y := \min\big\{|\Gamma e| : e \in E(G)\big\}$; that is, the cardinality of the smallest orbit of edges under Γ. Let $\omega \in \{a, b\}^{E(G)}$ be random-uniform. Then*

$$\operatorname{var}\big(\operatorname{diam}_\omega G\big) \le C\, \frac{(b-a)^2\,(b/a)\operatorname{diam} G}{1 + \big|\log\big((a/b)\,Y/\operatorname{diam} G\big)\big|}\,,$$

where C is a universal constant.

One can also prove similar estimates for the least ω-length of a closed path in the $m \times n$ torus $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ whose projection on the first coordinate has degree 1. The details are left to the reader.

2 Proof of Theorem 2

Let $f(\omega) := \operatorname{diam}_\omega(G)$. Let $u_1, u_2 \in V(G)$ be a (random) pair of vertices where $\operatorname{dist}_\omega(u_1, u_2) = \operatorname{diam}_\omega(G)$, and let β be a path from u_1 to u_2 such that $\sum_{e\in\beta} \omega_e = \operatorname{diam}_\omega(G)$. We use some arbitrary but fixed method for choosing between all possible choices for the triple (u_1, u_2, β). Clearly, $\mathrm{diam}_\omega(G) \le b \cdot \mathrm{diam}(G)$. Therefore,

$$|\beta| \le (b/a)\,\mathrm{diam}(G)\,, \tag{3}$$

where $|\beta|$ denotes the number of edges in β.

Let $e \in E(G)$. Note that if $\rho_e\, f(\omega) < 0$, then we must have $e \in \beta$. By (3), this gives

$$\sum_{e \in E(G)} \mathbf{P}[\,\rho_e\, f(\omega) < 0\,] \le \mathbf{E}\big[\,|\beta|\,\big] \le (b/a)\,\mathrm{diam}(G)\,. \tag{4}$$

Fix $e \in E(G)$. By symmetry, $\mathbf{P}[\,\rho_e f < 0\,] = \mathbf{P}[\,\rho_{e'} f < 0\,]$ for every $e' \in \Gamma e$. Consequently,

$$|\Gamma e|\,\mathbf{P}[\,\rho_e f < 0\,] \le \sum_{e' \in E(G)} \mathbf{P}[\,\rho_{e'} f < 0\,] \le (b/a)\,\mathrm{diam}(G)\,. \tag{5}$$

Now clearly, $\mathbf{P}[\,\rho_e f \ne 0\,] = 2\,\mathbf{P}[\,\rho_e f < 0\,]$ and $\|\rho_e f\|_\infty \le (b-a)/2$. Therefore,

$$\|\rho_e f\|_2^2 \le (1/2)\,(b-a)^2\,\mathbf{P}[\,\rho_e f < 0\,]\,. \tag{6}$$

By Cauchy-Schwarz,

$$\|\rho_e f\|_1 \le \sqrt{\mathbf{P}[\,\rho_e f \ne 0\,]}\,\|\rho_e f\|_2\,. \tag{7}$$

By (2), we have

$$\mathrm{var}(f) \le C\,\frac{\sum_{e \in E(G)} \|\rho_e f\|_2^2}{\min_{e \in E(G)} \log\big(\|\rho_e\, f\|_2 / \|\rho_e\, f\|_1\big)}\,.$$

To estimate the numerator, we use (6) and (4), and for the denominator (5) and (7). The theorem easily follows. $\square$

3 Proof of Theorem 1

If $v, u \in \mathbb{Z}^d$, and α is a path from v to u, then α will be called an ω-**geodesic** if it minimizes ω-length; that is, $\mathrm{dist}_\omega = \sum_{e \in \alpha} \omega_e$. Given ω, let γ be an ω-geodesic from 0 to v. Although there may be more than one such geodesic, we require that γ depends only on ω. (For example, we may use an arbitrary deterministic choice among any possible collection of ω-geodesics.)

The general strategy for the proof of Theorem 1 is as for Theorem 2. However, the difficulty is that there is not enough symmetry to get a good bound on $\mathbf{P}\big[\,\rho_e\,\mathrm{dist}_\omega(0,v)<0\,\big]$. It would have been enough to show that $\mathbf{P}\big[\,e\in\gamma\,\big] < C\,|v|^{-1/C}$ holds with the exception of at most $C\,|v|/\log|v|$ edges, for some constant $C>0$. But we could not prove this. Therefore, we will need an averaging argument, for which the following lemma will be useful.

Lemma 3. *There is a constant $c>0$ such that for every $m\in\mathbb{N}$ there is a function*

$$g = g_m : \{a,b\}^{m^2} \to \{0,1,2,\dots,m\}$$

satisfying

$$\|\sigma_j\, g - g\|_\infty \le 1$$

for every $j=1,2,\dots,m^2$ and

$$\max_y \mathbf{P}[\,g(x)=y\,] \le c/m\,,$$

where x is random-uniform in $\{a,b\}^{m^2}$.

The easy proof of the lemma is left to the reader.

Fix some $v\in\mathbb{Z}^d$ with $|v|$ large, and set $f=f(\omega):=\mathrm{dist}_\omega(0,v)$. Clearly, $f(\omega)\le b\,|v|$, and therefore $|\gamma|\le(b/a)\,|v|$, where $|\gamma|$ denotes the number of edges in γ. In particular, f depends on only finitely many of the coordinates in ω. Also, $|\gamma|\le(b/a)|v|$ implies

$$\sum_{e\in E}\mathbf{P}[\,e\in\gamma\,] = \mathbf{E}\big[\,|\gamma|\,\big] \le (b/a)\,|v|\,. \tag{8}$$

Fix $m:=\lfloor |v|^{1/4}\rfloor$, and let $S:=\{1,\dots,d\}\times\{1,\dots,m^2\}$. Let $c>0$ and $g=g_m$ be as in Lemma 3. Given $x=\big(x_{i,j}:\{i,j\}\in S\big)\in\{0,1\}^S$ let

$$z = z(x) := \sum_{i=1}^{d} g(x_{i,1},\dots,x_{i,m^2})\,\mathbf{e}_i\,,$$

where $\{\mathbf{e}_1,\dots,\mathbf{e}_d\}$ is the standard basis for $\mathbb{R}^d$. Define

$$\tilde f(x,\omega) := \mathrm{dist}_\omega(z, v+z)\,.$$

We think of $\tilde{f}$ as a function on the space $\tilde{\Omega} := \{a, b\}^{S\cup E}$. Since $|z| \le md$, it follows that $|\tilde{f} - f| \le 2mdb$. In particular,

$$\mathrm{var}(f) \le \mathrm{var}(\tilde{f}) + 4mdb\sqrt{\mathrm{var}(\tilde{f})} + 4m^2d^2b^2\,. \tag{9}$$

It therefore suffices to find a good estimate for $\mathrm{var}(\tilde{f})$.

Let $e \in E$ be some edge. We want to estimate its influence:

$$I_e(\tilde{f}) := \mathbf{P}\big[\,\sigma_e\tilde{f}(x,\omega) \ne \tilde{f}(x,\omega)\,\big] = 2\,\mathbf{P}\big[\,\sigma_e\tilde{f}(x,\omega) > \tilde{f}(x,\omega)\,\big]\,.$$

Note that if the pair $(x,\omega) \in \tilde{\Omega}$ satisfies $\sigma_e\tilde{f}(x,\omega) > \tilde{f}(x,\omega)$, then e must be on every ω-geodesic from z to $v + z$. Consequently, conditioning on z and translating ω and e by $-z$ gives

$$I_e(\tilde{f}) = 2\,\mathbf{P}\big[\,\sigma_e\tilde{f}(x,\omega) > \tilde{f}(x,\omega)\,\big] \le 2\,\mathbf{P}\big[\,e - z \in \gamma\,\big]\,. \tag{10}$$

Let Q be the set of edges $e' \in E(\mathbb{Z}^d)$ such that $\mathbf{P}[\,e - z = e'\,] > 0$. The L^1 diameter of Q is $O(m)$. (We allow the constants in the $O(\cdot)$ notation to depend on d, a and b, but not on v.) Hence, the diameter of Q in the dist_ω metric is also $O(m)$, and therefore $|\gamma \cap Q| \le O(m)$. But the lemma gives

$$\max_{z_0} \mathbf{P}\big[\,z = z_0\,\big] \le (c/m)^d\,.$$

By conditioning on γ and summing over the edges in $\gamma \cap Q$, we therefore get

$$\mathbf{P}\big[\,e \in \gamma + z \bigm| \gamma\,\big] \le O(1)\,m^{1-d}\,.$$

Consequently, (10) and the choice of m give

$$I_e(\tilde{f}) \le O(1)\,|v|^{-1/4}\,. \tag{11}$$

Also, (8) implies

$$\sum_{e\in E} \mathbf{P}\big[\,e - z \in \gamma \bigm| z\,\big] \le (b/a)\,|v|\,.$$

Combining this with (10) therefore gives

$$\sum_{e\in E} I_e(\tilde{f}) \le 2\,(b/a)\,|v|\,.$$

Applying (11) yields

$$\sum_{e\in E} \frac{I_e(\tilde f)}{1+\left|\log I_e(\tilde f)\right|} \le O(1)\,|v|/\log|v|\,. \tag{12}$$

On the other hand, $I_s(\tilde f) \le b-a$ for $s \in S$. As $|S| = O(1)\,|v|^{1/2}$ and $\|\rho_q f\|_\infty = O(1)$ for $q \in S\cup E$, we get from (12) and (2)

$$\operatorname{var}(\tilde f) \le O(1) \sum_{q\in E\cup S} \frac{I_q(\tilde f)}{1+\left|\log I_q(\tilde f)\right|} \le O(1)\,|v|/\log|v|\,.$$

Therefore, Theorem 1 now follows from (9). $\square$

4 A proof of Talagrand's inequality (2)

To prove (2), it clearly suffices to take $a=0, b=1$. For $f:\{0,1\}^J \to \mathbb{R}$, consider the Fourier-Walsh expansion of f,

$$f = \sum_{S\subset J} \widehat f(S)\, u_S\,,$$

where $u_S(\omega) = (-1)^{S\cdot\omega}$ and $S\cdot\omega$ is shorthand for $\sum_{s\in S}\omega_s$. For each $p\in\mathbb{R}$ define the operator

$$T_p(f) := \sum_{S\subset J} p^{|S|}\widehat f(S) u_S\,,$$

which is of central importance in harmonic analysis. The Bonami-Beckner [6, 2] inequality asserts that

$$\|T_p f\|_2 \le \|f\|_{1+p^2}\,. \tag{13}$$

Set $f_j := \rho_j\, f$. Because $\rho_j\, u_S = u_S$ if $j\in S$ and $\rho_j\, u_S = 0$ if $j\notin S$, we have

$$\widehat{f_j}(S) = \begin{cases} \widehat f(S) & j\in S\,,\\ 0 & j\notin S\,. \end{cases}$$

Since $\|f\|_2^2 = \sum_{S\subset J}\widehat f(S)^2$, it follows that

$$\operatorname{var}(f) = \sum_{\emptyset\ne S\subset J}\widehat f(S)^2 = \sum_{S\subset J}\sum_{j\in J}\widehat{f_j}(S)^2/|S| = 2\sum_{j\in J}\int_0^1 \|T_p\, f_j\|_2^2\, dp\,.$$

Therefore, (13) gives

$$\operatorname{var}(f) \le 2 \sum_{j \in J} \int_0^1 \|f_j\|_{1+p^2}^2 \, dp \,. \tag{14}$$

An instance of the Hölder inequality

$$\mathbf{E}[\,|f_j|^{1+p^2}\,] \le \mathbf{E}[\,f_j^2\,]^{p^2}\, \mathbf{E}[\,|f_j|\,]^{1-p^2}$$

implies

$$\begin{aligned}
\int_0^1 \|f_j\|_{1+p^2}^2 \, dp &\le \int_0^1 \Big(\mathbf{E}\big[\,f_j^2\,\big]^{p^2}\, \mathbf{E}\big[\,|f_j|\,\big]^{1-p^2} \Big)^{2/(1+p^2)} dp \\
&= \|f_j\|_2^2 \int_0^1 \big(\|f_j\|_1 / \|f_j\|_2 \big)^{2(1-p^2)/(1+p^2)} \, dp \\
&\le 2\, \|f_j\|_2^2 \int_{1/2}^1 \big(\|f_j\|_1 / \|f_j\|_2 \big)^{2(1-p^2)/(1+p^2)} \, dp \,.
\end{aligned}$$

Let $s(p) := 2(1-p^2)/(1+p^2)$. Since $s'(p) \le s'(1) = -2$ when $p \in [1/2, 1]$, the above gives

$$\begin{aligned}
\int_0^1 \|f_j\|_{1+p^2}^2 \, dp &\le 2\, \|f_j\|_2^2 \int_{s(1/2)}^{s(1)} \big(\|f_j\|_1 / \|f_j\|_2 \big)^s \, \frac{ds}{s'(p)} \\
&\le \|f_j\|_2^2 \int_0^{6/5} \big(\|f_j\|_1 / \|f_j\|_2 \big)^s \, ds \\
&= \|f_j\|_2^2 \, \frac{1 - \big(\|f_j\|_1 / \|f_j\|_2 \big)^{6/5}}{\log\big(\|f_j\|_2 / \|f_j\|_1 \big)} \,.
\end{aligned}$$

Now (14) implies

$$\operatorname{var}(f) \le 2 \sum_{j \in J} \|f_j\|_2^2 \, \frac{1 - \big(\|f_j\|_1 / \|f_j\|_2 \big)^{6/5}}{\log\big(\|f_j\|_2 / \|f_j\|_1 \big)} \tag{15}$$

from which (2) follows. $\square$

Remark: It is worth noting that relation (1) can also be derived from relation (2). Indeed, let $f = dist_\omega(0, v)$ and for a real number s define

$g_s = \max(f(x), s)$ and choose s so that $\mathbf{P}\Big[(f(x) \geq s)\Big] = u$. Then it follows from (2) that

$$var(g) \leq C|v|_1 u/\big(1 + \min_j(\log\big(\|\rho_j g\|_2/\|\rho_j g\|_1\big))\big).$$

It follows that $var(g) \leq C \cdot |v|_1 u/\log(1/u)$. Therefore,

$$\mathbf{P}\Big[(f(x) > s + C'\sqrt{|v|_1/\log(1/u)}\Big] \leq u/2.$$

This implies relation (1). (This argument is fairly general and applies to general events in the product space $\{0,1\}^J$. For our case we get an even slight improvement for the tail estimates in certain ranges. We omit the details.)

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