Thresholds and expectation thresholds

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Monotone properties and thresholds

We use 2^X for the collection of subsets of a finite set X. Given X of size n and $p \in [0,1]$, μ_p is the product measure on 2^X given by $\mu_p(S) = p^{|S|}(1-p)^{n-|S|}$. Recall that $\mathcal{F} \subseteq 2^X$ is often called a *property* of subsets of X, and is said to be *monotone (increasing)* if $B \supseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$. For a monotone $\mathcal{F} \subseteq 2^X$, denote by $p_c(\mathcal{F})$ that $p \in [0,1]$ for which $\mu_p(\mathcal{F})$ $(= \sum\{\mu_p(S) : S \in \mathcal{F}\}) = 1/2$. We will assume throughout this note that $\mathcal{F}$ is a monotone property, and will always exclude the trivial cases $\mathcal{F} = 2^X$ and $\mathcal{F} = \emptyset$, so in particular $p_c(\mathcal{F})$ exists and is unique.

Recall that for monotone $\mathcal{F}_n \subseteq 2^{X_n}$ $(n = 1, 2, \ldots)$, a function $p_0(n)$ is a *threshold* for the sequence $\{\mathcal{F}_n\}$ if

$$\mu_{p(n)}(\mathcal{F}_n) \to \begin{cases} 0 & \text{if } p(n)/p_0(n) \to 0 \\ 1 & \text{if } p(n)/p_0(n) \to \infty. \end{cases} \tag{1}$$

(See *e.g.* [2], [6] or [18]. Of course $p_0(n)$ is not quite unique, but following common practice we will often say "the" threshold when we should really say "a.") It follows from [8] that every $\{\mathcal{F}_n\}$ has a threshold, and that in fact (see *e.g.* [18, Proposition 1.23 and Theorem 1.24]) $p_0(n) := p_c(\mathcal{F}_n)$ is a threshold for $\{\mathcal{F}_n\}$. So the quantity p_c conveniently captures threshold behavior, in particular allowing us to dispense with sequences.

Conjecture 1 *For any (nontrivial monotone) $\mathcal{F} \subseteq 2^X$ there exist $\mathcal{G} \subseteq 2^X$ and $q \in [0,1]$ such that*

(a) *each $B \in \mathcal{F}$ contains some member of $\mathcal{G}$,*

(b) $\sum_{A \in \mathcal{G}} q^{|A|} < 1/2$, *and*

(c) $q > p_c(\mathcal{F})/(K \log n)$,

where K is a universal constant and $n = |X|$.

Note that, for a given q, existence of a $\mathcal{G}$ for which (a) and (b) hold trivially gives $\mu_q(\mathcal{F}) < 1/2$; so the conjecture says that for any $\mathcal{F}$ as above there is a trivial lower bound on $p_c(\mathcal{F})$ which is within a factor $O(\log n)$ of the truth. To put this another way, define $q(\mathcal{F})$ to be the supremum of those q's for which there exists $\mathcal{G}$ satisfying (a) and (b). Then $p_c(\mathcal{F}) \geq q(\mathcal{F})$ is trivial, and Conjecture 1 would say $p_c(\mathcal{F}) < Kq(\mathcal{F}) \log n$. It is easy to see (familiar examples will be recalled below) that the factor $\log n$ cannot be improved.

(Condition (b) is also considered by Talagrand in [30] and [31]. In particular a hope of [30] (which is also primarily a problem paper) is to say that in certain geometrically-motivated situations the threshold essentially coincides with its trivial lower bound (more precisely, the gap is $O(1)$). This is different from what we are asking, but it may be that the underlying difficulties are similar.)

In the rest of this note we will briefly mention a few consequences of, and questions related to, Conjecture 1. As some of these indicate, the conjecture is extremely strong. It would probably be more sensible to conjecture that it is *not* true; but it does not seem easy to disprove, and we think a counterexample would also be quite interesting.

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Graph properties

A basic result for random graphs states that a threshold for the (usual) random graph $G_{n,p}$ to contain a given (fixed) subgraph H is $n^{-1/m(H)}$, where $m(H) = \max\{|E(H')|/|V(H')| : H' \subseteq H\}$. (This was proved for "balanced" graphs in [12], and for general graphs in [4].) Equivalently, one may take as a threshold the least $p = p(n)$ such that for each $H' \subseteq H$ the expected number of copies of H' in $G_{n,p}$ is at least 1.

This last $p(n)$—call it the *"expectation threshold"*—still makes sense if, instead of a fixed H, we consider a *sequence* $\{H_n\}$, where, allowing isolated vertices, we assume $|V(H_n)| = n$. Formally, for an arbitrary H, define $p_{\mathsf{E}}(H)$ to be the least p such that, for every spanning $H' \subseteq H$, $(|V(H')|!/|Aut(H')|)p^{|E(H')|} \geq 1$. Then the $p(n)$ of the preceding paragraph is the same as $p_{\mathsf{E}}(H_n)$, where H_n is gotten from H by adding $n - |V(H)|$ isolated vertices.

Of course in this more general situation, p_{E} may no longer capture the true threshold behavior. Well-known examples are perfect matchings and Hamiltonian cycles in $G_{n,p}$, for each of which the expectation threshold is (easily seen to be) on the order of $1/n$, while the actual threshold is $\log n/n$. (This was proved for matchings in [12], and for Hamiltonian cycles in [27] and [24]. It's easy to see that $\log n/n$ is a lower bound, since existence of a perfect matching and Hamiltonicity require minimum degree at least 1 and 2 respectively. The beautiful "stopping time" versions of these results ([12] for matchings, [23] and [5] for Hamiltonian cycles) say, in a precise way, that in each case failure to satisfy the degree requirements is the main source of failure to have the desired property.)

Now, for an n-vertex H, write $p_c(H)$ and $q(H)$ for $p_c(\mathcal{F})$ and $q(\mathcal{F})$, where $\mathcal{F} \subseteq 2^{\binom{[n]}{2}}$ is the collection of graphs on $[n]$ that contain copies of H. So again, $p_c(H_n)$ is a threshold function for "$G_{n,p} \supseteq H_n$." Moreover, for any H, $q(H) \geq p_{\mathsf{E}}(H)/2$ (the 1/2 should of course be ignored). To see this, notice that we have an alternate definition of $p_{\mathsf{E}}(H)$ analogous to that of $q(H)$: it is the supremum of those p's for which there exists $H' \subseteq H$ so that, with $\mathcal{G}$ the collection of copies of H', $\sum\{p^{|A|} : A \in \mathcal{G}\} < 1$. So the following conjecture, which was actually our starting point, contains Conjecture 1 for $\mathcal{F}$'s of this (subgraph containment) type.

Conjecture 2 *For any H, $p_c(H) < Kp_{\mathsf{E}}(H) \log |V(H)|$, where K is a universal constant.*

On the other hand, it could be that Conjecture 1 implies Conjecture 2, as would follow from a positive answer to

Question 3 *Is it true that, for some fixed K, $q(H) < Kp_{\mathsf{E}}(H)$ for every H?*

In other words, is it true that in the present situation the value of q does not change significantly if we require $\mathcal{G}$ (in the definition of q) to consist of all copies of a single graph? Pushing our luck, we could extend this to a general graph property $\mathcal{F}$ (meaning, as usual, that membership in $\mathcal{F}$ depends only on isomorphism type): can we in this case require that $\mathcal{G}$ also be a graph property (without significantly affecting q)? In fact, we don't even know that the answer to the following is negative.

Question 4 *For $\mathcal{F} \subseteq 2^X$ let $q^*(\mathcal{F})$ be the supremum of those q's for which there is some $Aut(\mathcal{F})$-invariant $\mathcal{G}$ satisfying (a) and (b) of Conjecture 1. Is it true that $q(\mathcal{F}) \leq Kq^*(\mathcal{F})$ for some universal K?*

Returning briefly to subgraph containment, it is not hard to see that for an n-vertex tree T with maximum degree $\Delta(T) = \Delta$, both $p_{\mathsf{E}}(T)$ and $q(T)$ are always between $\Omega(1/n)$ and $O(\Delta/n)$, and are $\Theta(\Delta/n)$ if $\Delta = \Omega(\log n)$. In particular, either of the above conjectures would imply that for any fixed Δ we have $p_c(T) < O(\log n/n)$ for any T of maximum degree at most Δ. For unbounded degrees, the conjectures are not as precise, but as an aside we may mention the natural guess that there are constants $K_1 > 0$ and K_2 such that for any tree T (with n vertices)

$$K_1 \max\{\log n/n, \Delta(T)/n\} < p_c(T) < K_2 \max\{\log n/n, \Delta(T)/n\}. \tag{2}$$

Finally, we don't even know a counterexample to

Conjecture 5 *Given $\varepsilon > 0$ there is a K such that any H with $\varepsilon < p_c(H) < 1 - \varepsilon$ satisfies $p_c(H) < Kp_{\mathsf{E}}(H)$.*

Hypergraph matching

Write $\mathcal{H}_k(n,p)$ for the random k-uniform hypergraph on vertex set $[n]$ in which each k-set is an edge with probability p, independent of other choices. A celebrated question first raised and studied by Schmidt and Shamir [29, 11] asks: for fixed k and n ranging over multiples of k, what is the threshold for $\mathcal{H}_k(n,p)$ to contain a perfect matching (meaning, of course, a collection of edges partitioning the vertex set)? The best published progress to date, building in part on [16] and some antecedents, is due to J.H. Kim [22], who shows that the threshold is at most $O(n^{-\alpha_k})$ with $\alpha_k = k - 1 - 1/(5 + 2(k-1))$; and very recently A. Johansson [19] announced that he can improve this to $O(n^{-k+1+o(1)})$. But it is natural to expect that, as for $G_{n,p}$, one usually has a perfect matching as soon as there are no isolated vertices, which would say in particular that (this was perhaps first proposed explicitly in [10]) the true threshold is $n^{-k+1}\log n$. In fact this would follow from Conjecture 1, since it's easy to see that for $\mathcal{F}$ the collection of k-uniform hypergraphs on $[n]$ containing perfect matchings (a property of subsets of $X = \binom{[n]}{k}$), one has $q(\mathcal{F}) = \Theta(n^{-k+1})$.

A related problem asks for the threshold for $G_{n,p}$ to contain a "triangle factor" (collection of triangles partitioning the vertex set). Here again one expects that isolated vertices (i.e. vertices not contained in triangles) are the main obstruction, which would make the threshold $n^{-2/3}\log^{1/3} n$. In this case Conjecture 1 is a little off, saying only that the threshold is at most $n^{-2/3}\log n$; but this is still much better than what's presently known. (Here again the best published bound is from [22] ($n^{-2/3+1/18}$, improving on [25]), and $n^{-2/3+o(1)}$ has been announced in [19].)

Isoperimetry

We would like to suggest a possible, if speculative, connection between the preceding questions and isoperimetric behavior. The most obvious difficulty posed by Conjecture 1 is that of identifying the requisite $\mathcal{G}$. While such an identification based purely on combinatorial information seems difficult, a beautiful result of Friedgut [13] does achieve something of the sort using the Fourier transform (see the comments following Conjecture 6). This somewhat motivates the approach proposed here.

Given a monotone $\mathcal{F} \subset 2^X$, let $m(p) = \mu_p(\mathcal{F})$. The derivative $m'(p)$ is, according to a fundamental observation of Russo ([28] or e.g. [17]), equal to the "total influence" of $\mathcal{F}$ w.r.t. μ_p, namely

$$I = I_p(\mathcal{F}) = \sum_{i=1}^{n} \mu_p(|\mathcal{F} \cap \{S, S\triangle\{i\}\}| = 1).$$

This measure of the edge boundary of $\mathcal{F}$ has recently been of importance in a number of contexts; see e.g. [20],[13],[21]. (The combination of Russo's lemma and lower bounds on influence has been a powerful tool for proving sharp threshold behavior. A few recent results have also used such sharp threshold information as a tool in estimating *location* of thresholds; see e.g. [1],[7].)

In the present context the edge isoperimetric inequality takes the form

$$p \cdot I_p(\mathcal{F}) \geq m(p) \log_p m(p). \tag{3}$$

Equality holds whenever $\mathcal{F}$ is a subcube, $\{S \subset X : S \supseteq R\}$ for some R. Though one would think (3) would be known, we could not find it in the literature, so indicate a proof at the end of this note. A derivation, based on log-Sobolev inequalities, of something containing a slightly weaker version of (3) (in which the right hand side is multiplied by a constant C_p) is given in [26, p.111, (5.28)].

For $\mathcal{G} \subseteq 2^X$, write $\langle \mathcal{G} \rangle$ for the monotone property generated by $\mathcal{G}$; that is, $\langle \mathcal{G} \rangle = \{B \subseteq X : \exists A \in \mathcal{G}, A \subseteq B\}$. (So, for example, (a) of Conjecture 1 is "$\mathcal{F} \subseteq \langle \mathcal{G} \rangle$.") We would like to say, very roughly, that if (3) is fairly tight for a pair $(\mathcal{F}, p)$, then $\mathcal{F}$ is close (in μ_p) to some property generated by small sets. We will say that $\mathcal{F}$ is (C,p)-*optimal* if (3) is tight to within the factor C; that is,

$$p \cdot m'(p) \leq C m(p) \log_p m(p).$$

Thus subcubes are $(1,p)$-optimal for every p.

Conjecture 6 *Given C, there are $K, \delta > 0$ such that for any $(C \log(1/p), p)$-optimal $\mathcal{F}$ (and $m(p) = \mu_p(\mathcal{F})$):*

(a) *There is an $R \subset X$ of size at most $K \log(1/m(p))$, such that*

$$\mu_p(S \in \mathcal{F} | S \supseteq R) \geq (1+\delta) m(p).$$

(b) *There is a family $\mathcal{G} \subset \{A \subseteq X : |A| \leq K \log(1/m(p))\}$ such that $\mu_p(\mathcal{F} \triangle \langle \mathcal{G} \rangle) < o(m(p))$.*

Furthermore, there is some fixed $C > 0$ such that for any $(C \log(1/p), p)$-optimal $\mathcal{F}$,

(c) *there is $\mathcal{G} \subset 2^X$ such that $\mathcal{F} \subseteq \langle \mathcal{G} \rangle$ and $\sum \{p^{|S|} : S \in \mathcal{G}\} \leq 1/2$.*

For $\mu_p(\mathcal{F})$ bounded away from 0 and 1, (b) is a conjecture of Friedgut [13], which he proved in case $\mathcal{F}$ is a graph property, and (a) is similar to a theorem of Bourgain [9]. The extra $\log(1/p)$ in Conjecture 6 seems a little unnatural, but makes sense in light of whatever examples we've looked at (see in particular the prototypical "dual tribes" example below). There is also some vague feeling that in "cases of interest" one can multiply the right hand side of (3) by $\Omega(\log(1/p))$. (Of course Conjecture 6 may be regarded as a statement to this effect. Also, for instance, the extra log is not needed if $\mu_p(\mathcal{F})$ is bounded away from 0 and 1, since (3) applied to the *dual* family, $\mathcal{F}^* = \{A \subseteq X : X \setminus A \notin \mathcal{F}\}$, gives (in general)

$$(1-p) I \geq (1 - \mu_p(\mathcal{F})) \log_{1-p} (1 - \mu_p(\mathcal{F}))$$

(note $\mu_{1-p}(\mathcal{F}^*) = 1 - \mu_p(\mathcal{F})$ and $I_{1-p}(\mathcal{F}^*) = I_p(\mathcal{F})$).)

It is not hard to see (a proof is included below) that for every $\epsilon > 0$ there is a C such that for every monotone $\mathcal{F}$ (on a set of size n)

$$\exists\ p \in [n^{-\varepsilon} p_c(\mathcal{F}), p_c(\mathcal{F})] \text{ for which } \mathcal{F} \text{ is } (C \log(1/p), p)\text{-optimal.} \tag{4}$$

We don't know whether $n^{-\varepsilon}$ can be improved to $\Omega(1/\log n)$; in fact, as far as we know even the following is possible.

Conjecture 7 *For each $C > 0$ there is an $\varepsilon > 0$ such that for any $\mathcal{F}$*

$$\exists\ p \in [\varepsilon p_c(\mathcal{F})/\log n, p_c(\mathcal{F})] \text{ for which } \mathcal{F} \text{ is } (C\log(1/p), p)\text{-optimal.}$$

Of course this combined with Conjecture 6(c) would give Conjecture 1. (Considering that $\mu_p(\mathcal{F})$ depends only on the sequence $(a_0, \ldots, a_n)$, where $a_k = a_k(\mathcal{F}) := |\{A \in \mathcal{F} : |A| = k\}|$, and that possibilities for this sequence are completely determined by the Kruskal-Katona Theorem, it's a little odd that Conjecture 7 should not be easy to settle one way or the other.)

Duality.

We would like to say—but don't know how—that even the $\log n$ "gap" of Conjecture 1 is not typical and, more particularly, implies some kind of simplicity in the dual. Could it be, for instance, that if $p = p_c(\mathcal{F})$ is bounded away from 0 and 1 (and the $\log n$ gap is tight), then there is some $R \subseteq [n]$ of size $O(\log n)$ for which $\mu_{1-p}(S \in \mathcal{F}^*|S \supseteq R) > 1/2 + \Omega(1)$ (equivalently, $\mu_p(S \in \mathcal{F}^*|S \cap R = \emptyset) < 1/2 - \Omega(1)$)? And might this even extend to more general situations if we replace $O(\log n)$ by $O(p^{-1}\log n)$? A similar conclusion—existence of R of size $O(\log n)$ for which either $\mu_p(\mathcal{F}|S \supseteq R) \approx 1$ or $\mu_p(\mathcal{F}|S \cap R = \emptyset) \approx 0$—is conjectured in [14, Conjecture 2.6], in case both $p_c(\mathcal{F})$ and $\mu_{p_c}(\mathcal{F})$ are bounded away from 0 and 1, and the isoperimetric inequality of [20] is tight to within a constant factor.

Slightly related thoughts about the interplay of $\mathcal{F}$ and $\mathcal{F}^*$ may be found in [15]; in particular, a variant of Conjecture 4.1 of that paper would imply that for any *graph property* $\mathcal{F}$ a weaker version of our Conjecture 1—in which one only asks that $\langle\mathcal{G}\rangle$ contain most of $\mathcal{F}$—holds for either $\mathcal{F}$ or $\mathcal{F}^*$. In fact [15]—which deals with randomized decision tree complexity, especially of graph properties—partly motivated our Conjecture 2; but it is not clear how far this connection goes. (For one thing, the aforementioned Conjecture 4.1, and the complexity conjecture that motivated it, are not correct for general properties.)

A little example

The following example (the dual of the "tribes" example of Ben-Or and Linial [3]) may help to put some of the preceding discussion in perspective. Several of the above conjectures are based (to the extent they're based on anything) on the idea that this simple construction is about as bad as things get.

Suppose $k|n$, and let X be a set of size n, and $X_1 \cup \cdots \cup X_m$ a partition of X, with $m = n/k$ and $|X_i| = k$ for each i. Let $\mathcal{F}$ consist of all sets meeting each X_i. Then $m(p)$ $(= \mu_p(\mathcal{F}))$ $= (1-(1-p)^k)^{n/k}$ and $m'(p) = n(1-(1-p)^k)^{n/k-1}(1-p)^{k-1}$.

If we take $k = \log n - \log\log n$, then $p_c(\mathcal{F})$ is bounded away from 0 and 1, while the conclusion of Conjecture 1 requires $q < O(1/\log n)$ (and we may then take $\mathcal{G} = \mathcal{F}$). Here we achieve $(O(\log(1/p)), p)$-optimality when $p = \theta(1/\log n)$, while $(O(1), p)$- and $(1 - o(1), p)$-optimality require, respectively, $p = \log^{-(1+\Omega(1))} n$, and $p = \log^{-\omega(1)} n$.

Little proofs

Proof of (3). We actually show this for general (i.e. not necessarily monotone) $\mathcal{F}$. Note I makes sense in this generality, though its interpretation as a derivative is only valid when $\mathcal{F}$ is monotone.

We proceed by induction on n. Write I', μ' $(= \mu'_p)$ for influence and measure in $\{0,1\}^{n-1}$. Let $\mathcal{A} = \{x \in \mathcal{F} : x_n = 0\}$, $\mathcal{B} = \{x \in \mathcal{F} : x_n = 1\}$. By induction,

$$pI = p(1-p)I'(\mathcal{A}) + p^2 I'(\mathcal{B}) + p\mu'(\mathcal{A}\triangle\mathcal{B}) \geq (1-p)\mu'(\mathcal{A})\log_p \mu'(\mathcal{A}) + p\mu'(\mathcal{B})\log_p \mu'(\mathcal{B}) + p\mu'(\mathcal{A}\triangle\mathcal{B}),$$

and we would like to say that the r.h.s. is at least

$$((1-p)\mu'(\mathcal{A})+p\mu'(\mathcal{B}))\log_p((1-p)\mu'(\mathcal{A})+p\mu'(\mathcal{B})).$$

Set $\alpha=\mu'(\mathcal{A})$, $\beta=\mu'(\mathcal{B})$. Then assuming (w.l.o.g.) that $\beta\geq\alpha$, it's enough to verify the numerical statement

$$(1-p)\alpha\log_p\alpha+p\beta\log_p\beta+p(\beta-\alpha)-((1-p)\alpha+p\beta)\log_p((1-p)\alpha+p\beta)\geq 0.$$

But this holds with equality when $\beta=\alpha$, and an easy calculation shows that the derivative of the l.h.s. w.r.t. β is nonnegative (for *all* β, though we only need $\beta\geq\alpha$).

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Proof of (4). We prove this with $C\approx 1/(\varepsilon\ln 2)$ (with the log in (4) interpreted as $\log_2$). Set $p_c(\mathcal{F})=p_c$. Suppose $\mathcal{F}$ is not $(C\log(1/p),p)$-optimal for any $p\in[n^{-\varepsilon}p_c,p_c]$; that is,

$$pm'(p)>Cm(p)\log(1/m(p))$$

for each such p. We show this implies that $m(n^{-\varepsilon}p_c)\leq 2^{-n}$, a contradiction since $pI=\sum\{\mu_p(x)|\{y\sim x:y\notin\mathcal{F}\}|:x\in\mathcal{F}\}\leq nm(p)$ gives $(\log(1/p),p)$-optimality whenever $m(p)\leq 2^{-n}$.

Notice that if $m(p)\leq 2^{-r}$ and $tm'(t)>Cm(t)\log(1/m(t))$ for $t\in[(1-1/(rC))p,p]$, then $m(p(1-1/(rC)))\leq 2^{-(r+1)}$. Thus our assumption gives $m(p)\leq 2^{-n}$ for

$$p=p_c(1-1/C)(1-1/(2C))\cdots(1-1/((n-1)C))\approx n^{-\varepsilon}p_c.$$

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