Choice Games*

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Consider the following two-person game GAME1:[1]

- Player 1 chooses a countably infinite sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of real numbers, and puts them in boxes labeled $1, 2, ...$
- Player 2 opens all the boxes except one, in some order, and reads the numbers there; then he writes down a real number ξ.
- The unopened box, say box number i, is opened; if $x_i = \xi$ then Player 2 wins, and if $x_i \neq \xi$ then Player 1 wins.

Theorem 1 *For every $\varepsilon > 0$ Player 2 has a mixed strategy in* GAME1 *guaranteeing him a win with probability at least $1 - \varepsilon$.*

Remark. The proof uses the Axiom of Choice.

Proof. Fix an integer K. We will construct K pure strategies of Player 2 such that against every sequence $\mathbf{x}$ of Player 1 at least $K-1$ of these strategies yield a win for Player 2. The mixed strategy that puts probability $1/K$ on each one of these pure strategies thus guarantees a probability of at least $1 - 1/K$ of winning.

Let $X = \mathbb{R}^{\mathbb{N}}$ be the set of countable infinite sequences of real numbers. Consider the equivalence relation on X where $\mathbf{x} \sim \mathbf{x}'$ if and only if there is N such that $x_n = x'_n$ for all $n \geq N$ (i.e., $\mathbf{x}$ and $\mathbf{x}'$ coincide except for finitely many coordinates). Apply the Axiom of Choice to choose an element in each equivalence class; let $F(\mathbf{x})$ denote the chosen element in the equivalence class of $\mathbf{x}$ (thus $F : X \to X$ satisfies $\mathbf{x} \sim \mathbf{x}'$ iff $F(\mathbf{x}) = F(\mathbf{x}')$).

For every sequence $\mathbf{x} \in X$ and $k = 1, ..., K$, let $\mathbf{y}^k$ denote the subsequence of $\mathbf{x}$ consisting of all coordinates x_n with indices $n \equiv k$ (thus $y^k_m = x_{k+(m-1)K}$), and let $\mathbf{z}^k := F(\mathbf{y}^k)$. Since $\mathbf{y}^k \sim \mathbf{z}^k$, let R^k be the minimal index r such that $y^k_m = z^k_m$ for all $m \geq r$ (thus the last coordinate where $\mathbf{y}^k$ and $\mathbf{z}^k$ differ is coordinate $R^k - 1$), and let $R^{-j} := \max_{k\neq j} R^k$.

For each $j = 1, 2, ..., K$ we define a pure strategy σ_j of Player 2 as follows:

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[1] Source unknown. I heard it from Benjy Weiss, who heard it from ..., who heard it from ... . For a related problem, see http://xorshammer.com/2008/08/23/set-theory-and-weather-prediction/

- Open all boxes belonging to the sequences $\mathbf{y}^k$ for all $k \neq j$.
- Determine $\mathbf{z}^k = F(\mathbf{y}^k)$, and thus R^k for each $k \neq j$.
- Compute $R^{-j} = \max_{k \neq j} R^k$.
- Open all boxes belonging to the sequence $\mathbf{y}^j$ except for the R^{-j}-th box.
- Determine $\mathbf{z}^j = F(\mathbf{y}^j)$.
- Guess that the number in the unopened box, $y^j_{R^{-j}}$, equals $z^j_{R^{-j}}$.

The strategy σ_j wins against the sequence $\mathbf{x}$ that has $y^j_{R^{-j}} = z^j_{R^{-j}}$, which is implied by $R^j \leq R^{-j}$. Thus, if σ_j loses against $\mathbf{x}$ then necessarily $R^j > R^{-j}$, i.e., $R^j > R^k$ for all $k \neq j$, which means that R^j is the unique maximizer among all the R^k. Therefore, against any $\mathbf{x}$, at most one σ_j can lose. ■

A similar result, but now *without using the Axiom of Choice.*[2] Consider the following two-person game GAME2:

- Player 1 chooses a rational number in the interval $[0, 1]$ and writes down its infinite decimal expansion[3] $0.x_1x_2...x_n...$, with all $x_n \in \{0, 1, ..., 9\}$.
- Player 2 asks (in some order) what are the digits x_n except one, say x_i; then he writes down a digit $\xi \in \{0, 1, ..., 9\}$.
- If $x_i = \xi$ then Player 2 wins, and if $x_i \neq \xi$ then Player 1 wins.

By choosing i arbitrarily and ξ uniformly in $\{0, 1, ..., 9\}$, Player 2 can guarantee a win with probability $1/10$. However, we have:

Theorem 2 *For every $\varepsilon > 0$ Player 2 has a mixed strategy in* GAME2 *guaranteeing him a win with probability at least $1 - \varepsilon$.*

Proof. The proof is the same as for Theorem 1, except that here we do not use the Axiom of Choice. Because there are only countably many sequences $\mathbf{x} \in \{0, ..., 9\}^{\mathbb{N}}$ that Player 1 may choose (namely, those $\mathbf{x}$ that become eventually periodic), we can order them—say $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, ..., \mathbf{x}^{(m)}, ...$—and then choose in each equivalence class the element with minimal index (thus $F(\mathbf{x}) = \mathbf{x}^{(m)}$ iff m is the minimal natural number such that[4] $\mathbf{x} \sim \mathbf{x}^{(m)}$). ■

Remark. When the number of boxes is *finite* Player 1 can guarantee a win with probability 1 in GAME1, and with probability $9/10$ in GAME2, by choosing the x_i independently and uniformly on $[0, 1]$ and $\{0, 1, ..., 9\}$, respectively.

[2] Due to Phil Reny.

[3] When there is more than one expansion, e.g., $0.1000000... = 0.0999999...$, Player 1 chooses which expansion to use.

[4] Explicit strategies σ^j may also be constructed, based on R^j being the index where the sequence $\mathbf{y}^j$ becomes periodic.