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Comparing Mechanisms For Selling Correlated Goods

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הקדמה

נבחן בעית מקסום רווח של מוכר יחיד לקונה יחיד. בידי המוכר יש k מוצרים הטרוגנים, חסרי ערך עבורו, וברצונו למקסם רווח ממכירתם. שווי המוצרים בעיני הקונה אדיטיבי, ומתקבל מהתפלגות k ממדית ידועה (שווי המוצרים לא בהכרח בלתי תלוי). על המוכר להציע תפריט הכולל את האפשרויות השונות של הקונה לרכוש את המוצרים, כאשר כל אפשרות מכילה את המוצרים המוצעים ומחיר נדרש, אשר לא בהכרח אדיטיבי. הקונה יבחר באפשרות אחת מהתפריט לכל היותר, זו אשר תמקסם את ערך המוצרים בעיניו פחות העלות שלהם. ידוע זה מכבר שעבור שני מוצרים (או יותר), מכניזמים "פשוטים" עלולים להניב חלק זניח מהרווח האופטימלי. לכן, בעת הערכת מכניזמים נעזר בשתי דרכים פשוטות למכירת המוצרים כנקודת ייחוס, הראשונה היא מכירת כל המוצרים כחבילה והשניה היא מכירת כל אחד מהמוצרים בנפרד. מכניזם ייקרא דטרמיניסטי אם בתפריט מוצעים תתי קבוצות של k המוצרים (אין הקצאה חלקית של מוצרים). מכניזם ייקרא מונוטוני אם התשלום של הקונה כתלות בערך המוצרים בעיניו מהווה פונקציה מונוטונית. בתזה זו נראה שמכניזמים מונוטונים לא יכולים להניב רווח גדול פי יותר מ k מהרווח ממכירה בנפרד ונציג מספר תוצאות עבור מכניזמים דטרמיניסטים.

תודות

ברצוני להודות ראשית כל להוריי, על התמיכה והליווי בכל שאלך, ובפרט על עידוד דרכי האקדמית עוד מגיל התיכון. תודה לחבריי אשר היוו אוזן קשבת לרעיונות השונים המובאים בתזה זו. תודה מיוחדת אני חב למנחה שלי, פרופסור סרג'יו הרט, על הסבלנות, הבהירות והחדות אשר סייעו רבות בהפיכת רעיונות בוסריים לטענות בשלות ולהוכחות אלגנטיות.

Abstract

Consider the problem of maximizing the revenue from selling a number of heterogeneous goods to a single buyer whose private values for the goods are drawn from a (possibly correlated) known distribution, and whose valuation for the goods is additive. It is already known that when there are two (or more) goods, simple mechanisms may yield only a negligible fraction of the optimal revenue. This thesis compares revenues from various classes of mechanisms to revenues from the two simplest mechanisms – selling the goods separately and selling them as a bundle – by using previously defined tools, namely, multiple of separated revenue (MoS) and multiple of bundled revenue (MoB). We show in particular that monotonic mechanisms cannot yield more than k times the separated revenue (where k is the number of goods), and obtain bounds on the revenue of deterministic mechanisms.

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Chapter 1

Introduction

The problem of maximizing the revenue from selling a number of goods to a single buyer is known for its difficulty. Myerson's classic result [Mye81] shows that when selling a single good, the optimal revenue can be obtained by a simple "take it or leave it" offer. While we have an explicit formula for the optimal revenue from selling one good, we have relatively little information on how to obtain the optimal revenue from selling multiple goods. There is a significant body of work on the multiple goods case (see [HN17] and [HN19] for results and literature surveys), where it is shown that when the buyer's values for the different goods are independent, simple mechanisms are approximately optimal: selling each good separately for its optimal price extracts a constant fraction of the optimal revenue. On the other hand, when the buyer's valuations of the goods are correlated, simple mechanisms may yield only a negligible fraction of the optimal revenue. The last result is the motivation to research the correlated goods case.

In this thesis, we consider a setting in which a single seller, sells k goods and a single buyer has a valuation for the goods that is given by a random variable $X = (X_1, X_2, \ldots, X_k)$ with values in $\mathbb{R}_+^k$. The buyer's private values for the goods are drawn from an arbitrary – possibly correlated – but known prior distribution, and the value for bundles is additive. Without loss of generality, we assume that the seller offers a fixed menu and the buyer chooses a menu entry. Each menu entry specifies the probability q_i that a good i is allocated to the buyer and the payment s to the seller. In the case where the allocations for all goods $(q_1, q_2, \ldots, q_k)$ are in $\{0,1\}^k$, the mechanism is called *deterministic*. A pricing function $p\colon \mathcal{P}(K) \to \mathbb{R}_+$ is defined for deterministic mechanisms $p(A) = s(1_A)$, where 1_A is the indicator vector of a subset $A \subset K$ ($K = \{1, \ldots, k\}$). A *menu size* of a mechanism is the number of non-trivial possible outcomes, i.e., $|\left\{(q(x), s(x)) : x \in \mathbb{R}_+^k\right\} \backslash \{(0, \ldots, 0)\}|$.

An important result of Hart and Nisan ([HN19], Theorem A) states that a mechanism with bounded menu size cannot guarantee any positive fraction of the optimal revenue; i.e., for any $\epsilon > 0$ there exists a distribution X for which for any finite-size mechanism μ the fraction $\frac{\mathrm{Rev}(\mu,X)}{\mathrm{Rev}(X)} < \epsilon$. For this reason, we compare mechanisms' revenue to $\mathrm{BRev}(X)$ and $\mathrm{SRev}(X)$, the revenue achievable by selling the goods as a bundle and separately, instead of comparing it to the optimal revenue.

Two useful tools for analyzing this ratio are the multiple of bundled revenue (MoB) and the multiple of separated revenue (MoS). Given a mechanism μ, $\mathrm{MoB}(\mu)$ measures how many times better the revenue from μ can be relative to the bundled revenue. It is defined as the maximum, over all valuations X, of the ratio $\frac{\mathrm{Rev}(\mu,X)}{\mathrm{BRev}(X)}$. Given a class of mechanisms $\mathcal{N}$ and a class of valuations $\mathbb{X}$, we define $\mathrm{MoB}(\mathcal{N}, \mathbb{X})$ as the maximal (sup) ratio of the optimal revenue that can be achieved by mechanisms μ in $\mathcal{N}$ to the bundled revenue, over all valuations $X \in \mathbb{X}$. Our first result (Theorem 3.1) is an explicit expression for MoB over all deterministic k-good mechanisms:

$$\mathrm{MoB}(\text{Deterministic};\ k \text{ goods}) = \sum_{\ell=1}^{k} \frac{1}{\ell} \binom{k}{\ell}. \tag{1.1}$$

Next, we try to get a similar result for MoS. Unfortunately, the value of $\mathrm{MoS}(\text{Deterministic};\ k \text{ goods})$ remains an open question, but we improve the known upper bounds for three subclasses of deterministic mechanisms. The first class contains mechanisms with supermodular prices, i.e., increasing marginal cost, for which we prove that

$$\mathrm{MoS}\,(\text{Deterministic; supermodular; } k \text{ goods}) = \frac{2^k - 1}{k}. \tag{1.2}$$

The second class contains mechanisms with submodular prices, i.e., decreasing marginal cost, for which we show that

$$\mathrm{MoS}\,(\text{Deterministic; submodular; } k \text{ goods}) \leq k. \tag{1.3}$$

A surprising result presented by Hart and Reny [HR15], called *revenue non-monotonicity*, is that the seller's maximal revenue (from a mechanism or a class of mechanisms) may decrease when the buyer's valuation for the goods increases. They describe two settings in which the seller's maximal revenue increases as the buyer's distribution increases. The first is when the class of mechanisms is restricted to deterministic and symmetric mechanisms and the second is when the class is restricted to mechanisms with a submodular pricing function. In both cases, the payment function is monotonic. A mechanism $\mu = (q, s)$ is called monotonic if its payment function s is monotonic. Our most important result is an upper bound for MoS of monotonic mechanisms:

$$\mathrm{MoS}\,(\text{Monotonic; } k \text{ goods}) \leq k. \tag{1.4}$$

It turns out that monotonic mechanisms are quite limited in what they can achieve relative to selling the goods separately, and the interesting case remains MoS(Deterministic; k goods).

Chapter 2

Preliminaries

2.1 The Model

Consider the problem[1]of a seller who seeks to maximize revenue from selling a number $k \geq 1$ of goods to one buyer. The goods are worth nothing to the seller and their value to the buyer is nonnegative and additive. The valuation of the goods is given by a vector $(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k_+$, where x_i is the buyer's valuation for the good i. Therefore, the buyer's value for a subset $I \subseteq \{1, 2, \ldots, k\}$ is given by $\sum\limits_{i \in I} x_i$. The valuation vector of the buyer $x = (x_1, \ldots, x_k)$ is private and the seller knows only the probability distribution $\mathcal{F}$ on $\mathbb{R}^k_+$ from which the valuations are drawn.

A mechanism for selling k goods is given by an allocation function $q\colon \mathbb{R}^k_+ \to [0,1]^k$ and a payment function $s\colon \mathbb{R}^k_+ \to \mathbb{R}_+$. Therefore, for a valuation vector x, the buyer's payoff is $b(x) = q(x) \cdot x - s(x)$ and the seller's payoff is $s(x)$. The range of the function $q \times s\colon \mathbb{R}^k_+ \to [0,1]^k \times \mathbb{R}_+$ is called the *menu* of the mechanism and its cardinality, excluding the trivial outcome $((0, \ldots, 0), 0)$, is called its *menu size.*

A mechanism $\mu = (q, s)$ satisfies *individual rationality* (IR) if $b(x) \geq 0$ for all $x \in \mathbb{R}^k_+$. It satisfies *incentive compatibility* (IC) if $b(x) \geq q(y) \cdot x - s(y)$ for all $x, y \in \mathbb{R}^k_+$. The revenue of a mechanism $\mu = (q, s)$ from a buyer with a random valuation X is the expected payment of the buyer, $R(\mu, X) \coloneqq \mathbb{E}[s(X)]$. Given a class of mechanisms $\mathcal{N}$ and a valuation $X \in \mathbb{X}$,

[1]All concepts, results, and proofs are taken from [HN17], [HN19], and [HR15]. The new results are in Section 3.

we define $\mathcal{N}$-Rev $= \sup_{\mu \in \mathcal{N}} R(\mu; X)$, i.e., the maximal revenue that can be obtained by any mechanism from the class $\mathcal{N}$. The optimal revenue of a valuation X, denoted by $\text{Rev}(X)$, equals $\mathcal{N}$-$\text{Rev}(X)$ when $\mathcal{N}$ is the class of all IC and IR mechanisms. As seen in Hart and Nisan ([HN17], Proposition 6), we can restrict attention without loss of generality to IR and IC mechanisms that satisfy the *no positive transfer* (NPT) property, namely, $s(x) \geq 0$ for every $x \in \mathbb{R}^k_+$. From now on, it is assumed that all mechanisms satisfy IR, IC, and NPT.

We present three important subclasses. In the subclass of *separated mechanisms*, the seller offers each good separately, with optimal revenue $\text{SRev}(X) := \text{Rev}(X_1) + \cdots + \text{Rev}(X_k)$. In the subclass of *bundled mechanisms*, the seller offers all goods as a bundle, with optimal revenue $\text{BRev}(X) := \text{Rev}(X_1 + \cdots + X_k)$. In the subclass of *deterministic mechanisms*, the seller offers all subsets of the k goods, and each good is either fully allocated or not at all. The revenue of the last subclass is denoted by $\text{DRev}(X)$. Calculating the revenue of the last subclass is a multidimensional and hence much harder problem relative to the other two subclasses, where finding the revenue is a one-dimensional maximization problem.

2.2 Menu and Menu Size

Given a mechanism $\mu = (q, s)$, its *menu* is its range excluding the trivial outcome, $M_\mu = \{(q(x), s(x)) \colon x \in \mathbb{R}^k_+\} \setminus \{((0, 0, \ldots, 0), 0)\}$. Each element in the menu in called a *menu entry*. A construction for a mechanism from a given set (a given menu) $M \subset [0,1]^k \times \mathbb{R}_+$ is possible as well. Take $(q(x), s(x)) = (g, t)$, where $(g, t) \in M$ maximizes the "buyer's payoff" $g \cdot x - t$ and the mechanism's menu is a subset of M. We define the *menu size* as the cardinality of the menu, $|M_\mu|$. The class of mechanisms whose menu size is at most m turns out to be interesting: it implies the other two mentioned (finite) subclasses and its revenue is denoted by $\text{Rev}_{[m]}(X)$. Two basic results are presented by Hart and Nisan ([HN19], Proposition 3.1):

Proposition 2.1. *Given a k-good random valuation X,*

$$\text{Rev}_{[1]}(X) = \text{BRev}(X), \quad \text{and} \tag{2.1}$$

$$\frac{1}{m} \text{Rev}_{[m]}(X) \textit{ is weakly decreasing.} \tag{2.2}$$

Since the menu size of deterministic mechanisms is bounded by $2^k - 1$, we deduce that

$$\mathrm{DRev}(X) \leq \mathrm{Rev}_{[2^k-1]}(X) \leq (2^k - 1) \cdot \mathrm{Rev}_{[1]}(X) = (2^k - 1) \cdot \mathrm{BRev}(X).$$

A tighter bound for $\frac{\mathrm{DRev}(X)}{\mathrm{BRev}(X)}$ is shown in Section 3.1.

2.3 Monotonicity and Non-monotonicity

A mechanism $\mu = (q, s)$ is called monotone if its payment function $s(x)$ is monotone. An example of the non-monotonicity of a mechanism is given in Hart and Reny ([HR15], Figure 1). The following are a few essential definitions:

- A mechanism is called *seller-favorable* if when the buyer has different menu entries with the same payoff and a different cost, he chooses the more expensive one. For all $x \in \mathbb{R}_+^k$, $q(x) \cdot x - s(x) = q(y) \cdot x - s(y) \to s(y) \leq s(x)$.

- A deterministic mechanism is called *symmetric* if its pricing function $p\colon \mathcal{P}(K) \to \mathbb{R}_+$ depends only on the size of the element in $\mathcal{P}(K)$, i.e., $A, B \subset \{1, \ldots, k\}$ and $|A| = |B|$ implies $p(A) = p(B)$.

- A deterministic mechanism is called *submodular* if its pricing function $p\colon \mathcal{P}(K) \to \mathbb{R}_+$ is submodular, i.e., for each $I, J \subset \{1, \ldots, k\}$, $p(I) + p(J) \geq p(I \cup J) + p(I \cap J)$. In addition, it is called *supermodular* if $p(I) + p(J) \leq p(I \cup J) + p(I \cap J)$.

Two different results for monotonicity of mechanisms are presented by Hart and Reny ([HR15], Theorems 4 and 7). Let $\mu = (q, s)$ be a deterministic, symmetric, seller-favorable IC mechanism on $\mathbb{R}_+^k$; then its payment function s is nondecreasing. Alternatively, if it is a submodular seller-favorable IC mechanism, its payment function s is nondecreasing.

2.4 Revenue Comparisons

How can we evaluate mechanisms? We seek to find the optimal mechanism for maximizing the seller's revenue. A common approach is to compare the revenue of a class of mechanisms with the optimal revenue. As mentioned earlier, in our setup (of not necessarily independent $k \geq 2$ goods), no class of finite-size mechanisms can guarantee a positive fraction of the optimal revenue. Therefore, it does not help us to evaluate mechanisms. Instead, we compare the maximal revenue $\mathcal{N}$-Rev (achieved by the class of mechanisms $\mathcal{N}$) with some basic mechanisms.

2.4.1 MoB

The multiple of bundled revenue takes BRev $=$ Rev$_{[1]}$ as a benchmark. It is the sup of the multiple of $\mathcal{N}$-Rev(X) with bundled revenue over all random valuations in $\mathbb{X}$:

$$\text{MoB}(\mathcal{N};\mathbb{X}) = \sup_{X\in\mathbb{X}} \frac{\mathcal{N}\text{-Rev}(X)}{\text{BRev}(X)}. \tag{2.3}$$

The immediate results derived from Proposition 2.1 are

$$\text{MoB}\,(\text{menu size} \leq m;\ k \text{ goods}) \leq m, \quad \text{and} \tag{2.4}$$

$$\text{MoB}(\text{Deterministic};\ k \text{ goods}) \leq 2^k - 1. \tag{2.5}$$

A precise tool for measuring the MoB of a specific k-good mechanism $\mu = (q, s)$ is provided by Hart and Nisan ([HN19], Theorem 5.1):

Theorem 2.2. *Let $\mu = (q, s)$ be a $k-$good mechanism. Then,*

$$\text{MoB}(\mu) = \int_0^\infty \frac{1}{v(t)}dt \quad \text{when} \quad v(t) = \inf\{\|x\|_1 : s(x) \geq t\}. \tag{2.6}$$

Remark 2.3. From IR we deduce that $q(x) \cdot x - s(x) \geq 0$. Given a valuation x such that $s(x) \geq t$, $||x||_1 \geq q(x) \cdot x \geq s(x) \geq t$, we deduce that $v(t) \geq t$. In addition, for deterministic mechanisms, by the definition of $v(t)$, $v(\mathbb{R}_+)$ is uniquely determined by its values on the different prices of the $2^k - 1$ sub-bundles.

A lower bound for MoB(Deterministic) is achieved using this formula for a specific k-good deterministic mechanism and a k-good random valuation.

The improved result is

$$\sum_{\ell=1}^{k} \frac{1}{\ell}\binom{k}{\ell} \leq \mathrm{MoB}(\text{Deterministic};\ k \text{ goods}) \leq 2^k - 1.$$

In contrast to the lower bound, the upper bound is not tight, not even for 2 goods as shown by Hart and Nisan ([HN19], Proposition A.1):

$$\mathrm{MoB}\left(\text{Deterministic};\ 2 \text{ goods}\right) = \frac{5}{2}. \tag{2.7}$$

2.4.2 MoS

The multiple of separated revenue takes SRev as a benchmark:

$$\mathrm{MoS}(\mathcal{N};\mathbb{X}) = \sup_{X\in\mathbb{X}} \frac{\mathcal{N}\text{-Rev}(X)}{\mathrm{SRev}(X)}. \tag{2.8}$$

A similar analysis presented by Hart and Nisan ([HN19], Theorem A.7) states:

Theorem 2.4. *Let $\mu = (q, s)$ be a k-good mechanism. Then,*

$$\frac{1}{k}\int_0^\infty \frac{1}{w(t)}dt \leq \mathrm{MoS}(\mu) \leq \int_0^\infty \frac{1}{w(t)}dt \quad \text{when} \quad w(t) = \inf\{\|x\|_\infty : s(x) \geq t\}. \tag{2.9}$$

Notice that we do not have an accurate expression for the MoS of mechanism but rather a range. Later, we will apply the MoS value to a few classes of mechanisms.

A general construction presented by Hart and Nisan ([HN19], Proposition 7.3) yields a k-good mechanism μ that turns out to be useful for evaluating a lower bound for both the MoB and MoS of deterministic mechanisms. A similar (self-contained) proof is as follows:

Proposition 2.5.

$$\mathrm{MoB}(\text{Deterministic};\ k \text{ goods}) \geq \sum_{\ell=1}^{k} \frac{1}{\ell}\binom{k}{\ell}, \quad \text{and} \tag{2.10}$$

$$\mathrm{MoS}(\text{Deterministic};\ k \text{ goods}) \geq \frac{1}{k}\sum_{\ell=1}^{k} \binom{k}{\ell}. \tag{2.11}$$

Proof. Let $\epsilon > 0$ and let $I_0, I_1, \ldots, I_{2^k-1}$ be the 2^k subsets of $\{1, \ldots, k\}$ ordered by weakly increasing size, and let g_n be their indicator vector, i.e., $g_n[i] = \begin{cases} 1 & i \in I_n \\ 0 & i \notin I_n \end{cases}$. The prices are set to be a positive sequence $\{t_n\}_{n=0}^{2^k-1}$ that increases fast enough so that $\frac{t_{n+1}}{t_n} \geq \frac{1}{\epsilon}$ for all $n \geq 1$. Notice that for a valuation $x_n = t_n \cdot g_n$ and $0 \leq j < n$,

$$g_n \cdot x_n - g_j \cdot x_n = t_n \cdot g_n \cdot (g_n - g_j) \geq t_n \geq t_n - t_j,$$

and so $g_n \cdot x_n - t_n \geq g_j \cdot x_n - t_j$. The payment for x_n would be at least t_n and so $s(x_n) \geq t_n$. Therefore $v(t_n) \leq \|x_n\|_1 = t_n \|g_n\|_1$. Now,

$$\mathrm{MoB}(\mu) = \sum_{n=1}^{2^k-1} \frac{t_n - t_{n-1}}{v(t_n)} \geq \sum_{n=1}^{2^k-1} \frac{t_n - t_{n-1}}{t_n \cdot \|g_n\|_1} \geq \sum_{n=1}^{2^k-1} (1-\epsilon) \cdot \frac{1}{\|g_n\|_1} = (1-\epsilon) \sum_{\ell=1}^{k} \frac{1}{\ell} \binom{k}{\ell}$$

A similar computation is made for MoS: we use $w(t)$ with $\| \cdot \|_\infty$ instead of $v(t)$, and so $w(t_n) \leq \|x_n\|_\infty = t_n$:

$$\mathrm{MoS}(\mu) \geq \frac{1}{k} \sum_{n=1}^{2^k-1} \frac{t_n - t_{n-1}}{w(t_n)} \geq \frac{1}{k} \sum_{n=1}^{2^k-1} \frac{t_n - t_{n-1}}{t_n} \geq \frac{1}{k} \sum_{n=1}^{2^k-1} (1-\epsilon) = \frac{1}{k} (1-\epsilon) \sum_{\ell=1}^{k} \binom{k}{\ell}.$$

Since we took an arbitrary $\epsilon > 0$, we get the desired result. $\square$

Remark 2.6. The prices $\{t_n\}_{n=0}^{2^k-1}$ described in the previous proposition satisfy supermodularity for $0 < \epsilon < \frac{1}{2}$. Let $A \subset B$ and let $i \in A$ and $t_i = p_B$, then,

$$(p_B - p_{B \setminus \{i\}}) - (p_A - p_{A \setminus \{i\}}) \geq t_i - p_{B \setminus \{i\}} - p_A \geq t_i - t_{i-1} - t_{i-1} = t_i - 2t_{i-1} \geq 0.$$

Chapter 3

New results

3.1 The Multiple of Bundled Revenue

Theorem 3.1.

$$\mathrm{MoB}(\text{Deterministic};\ k \text{ goods}) = \sum_{\ell=1}^{k} \frac{1}{\ell}\binom{k}{\ell}.$$

Proof. It is left to prove that $\mathrm{MoB}(\text{Deterministic};\ k\,\text{goods}) \leq \sum_{\ell=1}^{k} \frac{1}{\ell}\binom{k}{\ell}$ since the other inequality is proved in Proposition 2.5.

Let μ be a k-good deterministic mechanism with a finite menu $\{(1_A, p_A) : A \in \mathcal{A}\}$, where $\mathcal{A} \subset \mathcal{P}(K)$. 1_A and p_A are the indicator vector and the payment for all goods with index in A, respectively. If $A \subset B \in \mathcal{A}$ and $p_A > p_B$, then for each $x \in \mathbb{R}^k_+$ we have

$$1_A \cdot x - p_A = \sum_{a \in A} x_a - p_A < \sum_{b \in B} x_b - p_B = 1_B \cdot x - p_B.$$

The option $\sum\limits_{a \in A} x_a - p_A$ is never chosen and so the menu entry $(1_A, p_A)$ does not affect the mechanism. Hence, $p_A \leq p_B$ is assumed for every $A \subset B \in \mathcal{A}$.

We extend the payment function's range $\mathcal{A}$ to $\mathcal{P}(K)$ by $p_A \coloneqq \min\{p_B : A \subset B \in \mathcal{A}\}$, and so $p : \mathcal{P}(K) \to \mathbb{R}_+$ is a nondecreasing function. Let $P \coloneqq \{p_A : A \subset K\}$ be the set of distinct prices used. For every $p \in P$ define $\tilde{v}(p)$ as

$$\tilde{v}(p) \coloneqq \inf_{x \in \mathbb{R}^k_+} \{\|x\|_1 \colon s(x) = p\};$$

then $\tilde{v}(p) \geq v(p)$, and

$$v(u) = \min\{\tilde{v}(p) \colon p \in P, p \geq u\}$$

for every $u \geq 0$.

Let $T \coloneqq \{p \in P \colon v(p) = \tilde{v}(p)\}$; then

$$v(u) = \min\{v(t) : t \in T, t \geq u\}$$

for every $u \geq 0$. Indeed, let $p \geq u$ in P be such that $v(u) = \tilde{v}(p)$; then $v(p) \geq v(u) = \tilde{v}(p)$. Therefore, $v(p) = \tilde{v}(p)$ and so $p \in T$, as needed. Let $T = \{t_i\}_{i=0}^{r}$ with $0 = t_0 < t_1 < \cdots < t_r$; then we have $v(u) = v(t_i)$ for every $u \in (t_{i-1}, t_i]$ and so

$$\mathrm{MoB}(\mu) = \int_0^\infty \frac{1}{v(u)} du = \sum_{i=1}^{r} \frac{t_i - t_{i-1}}{v(t_i)}.$$

Let x be such that $s(x) = t_i$, and let A be such that $q(x) = 1_A$ (thus $p_A = t_i$).

Lemma 3.2. *For every $c \in A$, if $p_{A\setminus\{c\}} > t_{i-1}$, then $v(p_{A\setminus\{c\}}) = v(p_A)$.*

Proof. Necessarily $p_{A\setminus\{c\}} \leq p_A$ and so $p_{A\setminus\{c\}} \in (t_{i-1}, t_i]$.
Therefore, $v(p_{A\setminus\{c\}}) = v(t_i) = v(p_A)$. □

Define

$$n_A \coloneqq |\{c \in A \colon v(p_A) = v(p_{A\setminus\{c\}})\}|.$$

Lemma 3.3. $\|x\|_1 \geq (|A| - n_A) \cdot (t_i - t_{i-1})$.

Proof. For every c in A, $(1_A, p_A)$ is preferred to $(1_{A\setminus\{c\}}, p_{A\setminus\{c\}})$ and so $x_c \geq p_A - p_{A\setminus\{c\}}$. By Lemma 3.2, if $v(p_{A\setminus\{c\}}) \neq v(p_A)$, then $p_{A\setminus\{c\}} \leq t_{i-1}$, and so $p_A - p_{A\setminus\{c\}} \geq t_i - t_{i-1}$. It follows that $\|x\|_1 \geq (|A| - n_A) \cdot (t_i - t_{i-1})$. □

Let

$$\mathcal{A}_i \coloneqq \{A \subset K \colon v(p_A) = v(t_i)\}$$

and let $\tilde{A} \in \mathcal{A}_i$ minimize $|A| - n_A$ over $\mathcal{A}_i$.

Corollary 3.4.

$$v(t_i) \geq (|\tilde{A}| - n_{\tilde{A}}) \cdot (t_i - t_{i-1}).$$

Proof. Recall that $v(t_i) = \tilde{v}(t_i) = \inf_{x \in \mathbb{R}_+^k} \{\|x\|_1 \colon s(x) = t_i\}$. For every x such that $s(x) = t_i$, there exists a chosen menu entry $(1_B, p_B)$ where $B \in \mathcal{A}_i$. By Lemma 3.3, we have $\|x\|_1 \geq (|B| - n_B) \cdot (t_i - t_{i-1}) \geq (|\tilde{A}| - n_{\tilde{A}}) \cdot (t_i - t_{i-1})$. Therefore,

$$v(t_i) \geq (|\tilde{A}| - n_{\tilde{A}}) \cdot (t_i - t_{i-1}).$$

□

Lemma 3.5.

$$\frac{t_i - t_{i-1}}{v(t_i)} \leq \sum_{B \in \mathcal{A}_i} \frac{1}{|B|}.$$

Proof. $v(u) \geq u$ for every $u > 0$ and so $\frac{t_i - t_{i-1}}{v(t_i)} \leq \frac{t_i}{v(t_i)} \leq 1$. By Corollary 3.4, we have

$$\begin{cases} \frac{t_i - t_{i-1}}{v(t_i)} \leq 1 = \frac{1}{|\tilde{A}|} & |\tilde{A}| = 1 \\ \frac{t_i - t_{i-1}}{v(t_i)} \leq 1 \leq n_{\tilde{A}} \cdot \frac{1}{|\tilde{A}| - 1} & |\tilde{A}| > 1, n_{\tilde{A}} = |\tilde{A}| \\ \frac{t_i - t_{i-1}}{v(t_i)} \leq \frac{(t_i - t_{i-1})}{(|\tilde{A}| - n_{\tilde{A}}) \cdot (t_i - t_{i-1})} \leq^{1} \frac{1}{|\tilde{A}|} + n_{\tilde{A}} \cdot \frac{1}{|\tilde{A}| - 1} & |\tilde{A}| > 1, n_{\tilde{A}} < |\tilde{A}|. \end{cases}$$

Either way, $\frac{t_i - t_{i-1}}{v(t_i)}$ is bounded by summing $\frac{1}{|B|}$ over sets B that satisfy $v(p_B) = v(t_i)$. Hence, $\frac{t_i - t_{i-1}}{v(t_i)} \leq \sum\limits_{B \in \mathcal{A}_i} \frac{1}{|B|}$. □

By the definition of $\mathcal{A}_i$, we have $\bigcup\limits_{i=1}^{r} \mathcal{A}_i \subset K$ and $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for $i \neq j$.

Summing over i completes the proof:

$$\mathrm{MoB}(\mu) = \sum_{i=1}^{r} \frac{t_i - t_{i-1}}{v(t_i)} \leq \sum_{i=1}^{r} \sum_{B \in \mathcal{A}_i} \frac{1}{|B|} \leq \sum_{\emptyset \neq B \subset K} \frac{1}{|B|} = \sum_{\ell=1}^{k} \binom{k}{\ell} \frac{1}{\ell}.$$

□

[1] $\forall m, n \in \mathbb{N}, n < m : \frac{1}{m-n} - \frac{1}{m} = \frac{n}{m \cdot (m-n)} \leq \frac{n}{m-1} \implies \frac{1}{m-n} \leq \frac{1}{m} + \frac{n}{m-1}$.

3.2 The Multiple of Separated Revenue

This case turns out to be more complicated than MoB. As shown in the proof of example 2.7, even for two goods, the value $w(p_{1,2})$ depends on the connection between p_{12} and p_1+p_2. In the next results, we distinguish between the submodular case and the supermodular case.

Theorem 3.6.

$$\mathrm{MoS}(\text{Deterministic; supermodular; } k \text{ goods}) = \frac{2^k-1}{k}.$$

Proof. Notice that one inequality is already proved (Proposition 2.5, Remark 2.6).

For the other inequality, let μ be a k-good deterministic and supermodular mechanism with a finite menu $\{(1_A, p_A) : A \in \mathcal{A}\}$, where $\mathcal{A} \subset \mathcal{P}(K)$. As in Theorem 3.1, $p_A \leq p_B$ is assumed for every $A \subset B \in \mathcal{A}$. We denote by α_A the probability that the menu entry $(1_A, p_A)$ is chosen. By definition, $\mathrm{Rev}(\mu; X) = \sum_{\emptyset \neq A \subset K} \alpha_A p_A$.

Let

$$Z := \sum_{A \neq \emptyset} \frac{1}{|A|} \sum_{i \in A} (p_A - p_{A \setminus \{i\}}) \sum_{B \supset A} \alpha_B.$$

By Lemma 3.8 we have

$$Z \leq \frac{2^k-1}{k} \mathrm{SRev}(X).$$

By Lemma 3.9 we have

$$Z = \mathrm{Rev}(\mu; X) + \sum_{B \neq \emptyset} \alpha_B \left(\sum_{A \subsetneq B} p_A \cdot \frac{2|A| + 1 - |B|}{|A|+1} \right).$$

By Lemma 3.10 we have that the described sum is positive, which yields the result. □

Lemma 3.7. *For every $A \subset K$ and every $i \in A$, we have*

$$(p_A - p_{A \setminus \{i\}}) \cdot \sum_{B \supset A} \alpha_B \leq \mathrm{Rev}(X_i).$$

Proof. Assume that the buyer chose the menu entry $(1_A, p_A)$; then, all $i \in A$, we have

$$\sum_{j\in A} x_j - p_A \geq \sum_{j\in A\setminus\{i\}} x_j - p_{A\setminus\{i\}} \text{ and so } x_i \geq p_A - p_{A\setminus\{i\}}.$$

From the supermodularity of the mechanism, it follows that for all $A \subset B$ and $i \in A$, we have $p_B - p_{B\setminus\{i\}} \geq p_A - p_{A\setminus\{i\}}$. Hence, $\mathbb{P}(X_i \geq p_A - p_{A\setminus\{i\}}) \geq \sum_{B\supset A} \alpha_B$. Recall that for a single good X_i:

$$\mathrm{Rev}(X_i) = \sup_{t\geq 0} t \cdot \mathbb{P}(X_i \geq t).$$

It follows that

$$\mathrm{Rev}(X_i) \geq (p_A - p_{A\setminus\{i\}}) \cdot \mathbb{P}(X_i \geq p_A - p_{A\setminus\{i\}}) \geq (p_A - p_{A\setminus\{i\}}) \cdot \sum_{B\supset A} \alpha_B.$$

□

Lemma 3.8.

$$Z \leq \frac{2^k - 1}{k} \mathrm{SRev}(X).$$

Proof. By Lemma 3.7,

$$Z \leq \sum_{A\neq\emptyset} \frac{1}{|A|} \sum_{i\in A} \mathrm{Rev}(X_i) = \sum_{i=1}^{k} \mathrm{Rev}(X_i) \left(\sum_{A\ni i} \frac{1}{|A|} \right).$$

Therefore,

$$\sum_{A\ni i} \frac{1}{|A|} = \sum_{\ell=1}^{k} \binom{k-1}{\ell-1} \frac{1}{l} = \sum_{\ell=1}^{k} \binom{k}{\ell} \frac{1}{k} = \frac{2^k - 1}{k}.$$

□

Lemma 3.9.

$$Z = \mathrm{Rev}(\mu; X) + \sum_{B\neq\emptyset} \alpha_B \left(\sum_{A\subsetneq B} p_A \cdot \frac{2|A| + 1 - |B|}{|A| + 1} \right).$$

Proof. Fix $B \subset K$ and consider all terms that include α_B. For each $A \subset B$ the term $\frac{1}{|A|} p_A \alpha_B$ appears $|A|$ times (once for each $i \in A$), yielding $p_A \alpha_B$ in total. For each $A \subsetneq B$, the term $\frac{1}{|A|+1}(-p_A \alpha_B)$ appears $|B| - |A|$ times (once for each $A' \subset B$ such that $A = A'\setminus\{i\}$ when

$i \in B \backslash A$), yielding $\frac{|B|-|A|}{|A|+1}(-p_A\alpha_B)$ in total. Thus,

$$Z = \sum_{B\neq\emptyset}\alpha_B\left(\sum_{A\subset B}p_A\cdot\left(1-\frac{|B|-|A|}{|A|+1}\right)\right) = \sum_{B\neq\emptyset}\alpha_B p_B + \sum_{B\neq\emptyset}\alpha_B\left(\sum_{A\subsetneq B}p_A\cdot\left(1-\frac{|B|-|A|}{|A|+1}\right)\right).$$

$\square$

Showing the following lemma for every $B \subset K$ completes the proof:

Lemma 3.10.

$$\sum_{A\subsetneq B}p_A\cdot[\frac{2|A|+1-|B|}{|A|+1}] \geq 0.$$

Proof. Let $n := |B|$; for each $m = 1, \ldots, n-1$, we define

$$\pi_m = \binom{n}{m}^{-1}\sum_{A\subset B:|A|=m}p_A, \qquad \lambda_m = \frac{2m+1-n}{m+1}\binom{n}{m}$$

and so we need to show that

$$\sum_{m=1}^{n-1}\lambda_m\pi_m \geq 0.$$

We will show that each term whose coefficient is negative (which happens when $m < \frac{n-1}{2}$) is covered by the corresponding term $\lambda_{n-m-1}\pi_{n-m-1}$, i.e.,

$$\lambda_m\pi_m + \lambda_{n-m-1}\pi_{n-m-1} \geq 0.$$

Indeed, for each $m < \frac{n-1}{2}$ we have

$$|\lambda_m| = \frac{-(2m+1-n)}{m+1}\binom{n}{m} = \frac{n-2m-1}{n-m}\binom{n}{n-m-1} = |\lambda_{n-m-1}|.$$

Below, we show that π_m is a nondecreasing sequence; then it follows that $\lambda_m\pi_m + \lambda_{n-m-1}\pi_{n-m-1} \geq 0$. Summing over all $m < \frac{n-1}{2}$ completes the proof. $\square$

Lemma 3.11. *Let N be a set of size n, and for each $m = 0, 1, \ldots, n$, let π_m be the average price of subsets of N of size m, i.e.,*

$$\pi_m = \binom{n}{m}^{-1} \sum_{C \subset N : |C| = m} p_C;$$

then $\{\pi_m\}_{m=0}^{n}$ is a nondecreasing sequence.

Proof. Let $m < n$; we will show that $\pi_m \leq \pi_{m+1}$. The function p is nondecreasing and so

$$\sum_C \sum_D p_C \leq \sum_C \sum_D p_D,$$

where both sums range over all pairs (C, D) such that $C \subset D$ and $|C| = m$, $|D| = m + 1$. Each p_C appears in the left-hand sum $n - m$ times (once for each $i \in N \backslash C$), and each p_D appears in the right-hand sum $m + 1$ times (once for each $i \in D$). Therefore,

$$(n - m)\binom{n}{m}\pi_m \leq (m + 1)\binom{n}{m + 1}\pi_{m+1}.$$

Since $(n - m)\binom{n}{m} = (m + 1)\binom{n}{m+1}$, we get $\pi_m \leq \pi_{m+1}$. $\square$

As mentioned in Section 2.3, the class of submodular, seller-favorable, IC mechanisms is a subclass of monotonic mechanisms. Therefore, the next result holds for submodular mechanisms as well.

Theorem 3.12.

$$\mathrm{MoS}(\textit{monotonic};\ k\ \textit{goods}) \leq k.$$

Proof. Let $\mu = (q, s)$ be a k-good monotonic mechanism. Recall the definition of $w(p)$:

$$w_{(q,s)}(p) \coloneqq \inf\{\|x\|_\infty : x \in \mathbb{R}^k_+,\ s(x) \geq p\}.$$

First, we show that it is sufficient to consider vectors on the diagonal of $\mathbb{R}^k_+$ when defining $w(p)$.

Lemma 3.13.

$$w(p) = \inf\{t : t \in \mathbb{R}_+,\ s(t \cdot \bar{1}) \geq p\}.$$

Proof. One inequality is immediate:

$$\{t \cdot \bar{1} \colon t \in \mathbb{R}_+,\ s(t \cdot \bar{1}) \geq p\} \subset \{x \colon x \in \mathbb{R}^k_+,\ s(x) \geq p\}.$$

$$\Downarrow$$

$$\inf\{||t \cdot \bar{1}||_\infty \colon t \in \mathbb{R}_+,\ s(t \cdot \bar{1}) \geq p\} \geq \inf\{||x||_\infty \colon x \in \mathbb{R}^k_+,\ s(x) \geq p\}.$$

For the other inequality, let $x \in \mathbb{R}^k_+$ such that $s(x) \geq p$ and let $t \coloneqq \|x\|_\infty$. Then $t \cdot \bar{1} \geq x$ and by the monotonicity of μ we get $s(t \cdot \bar{1}) \geq s(x) \geq p$, which have same $|| \cdot ||_\infty$ value. Therefore,

$$\inf\{t \colon t \in \mathbb{R}_+,\ s(t \cdot \bar{1}) \geq p\} \leq \inf\{||x||_\infty \colon x \in \mathbb{R}^k_+,\ s(x) \geq p\} = w(p).$$

□

Let $\mu_1 = (q_1, s_1)$ be a one-dimensional mechanism obtained from the mechanism $\mu = (q, s)$

on the diagonal, i.e.,

$$
\begin{gathered}
(q_1, s_1) \colon \mathbb{R}_+ \to [0,1] \times \mathbb{R}_+ \\
q_1(t) \coloneqq \frac{q(t \cdot \bar{1}) \cdot \bar{1}}{k}, \qquad s_1(t) \coloneqq \frac{s(t \cdot \bar{1})}{k}
\end{gathered}
$$

(we divide by k so that $q_1(t) \in [0,1]$). We get

$$
w_{(q,s)}(p) = \inf\{t \colon s(t \cdot \bar{1}) \geq p\} = \inf\{t \colon k \cdot s_1(t) \geq p\} = \inf\{t \colon s_1(t) \geq \frac{p}{k}\} = w_{(q_1,s_1)}(\frac{p}{k}),
$$

which implies that

$$
\int_0^\infty \frac{1}{w_{(q,s)}(p)} dp \underset{t=\frac{p}{k}}{=} \int_0^\infty \frac{k}{w_{(q_1,s_1)}(t)} dt.
$$

By Equation 2.9, we have

$$
\frac{1}{k} \int \frac{1}{w_{(q,s)}(p)} dp \leq \mathrm{MoS}(\mu) \leq \int \frac{1}{w_{(q,s)}(p)} dp.
$$

Since μ_1 is a one-good mechanism ($k = 1$), it follows that $\mathrm{MoS}(\mu_1) = \int \frac{1}{w_{(q_1,s_1)}(p)} dp$. For one good, SRev is the optimal revenue and so $\mathrm{MoS}(\mu_1)$ cannot be greater than 1.

Therefore,

$$
\mathrm{MoS}(\mu) \leq \int \frac{1}{w_{(q,s)}(p)} dp = k \cdot \int \frac{1}{w_{(q_1,s_1)}(t)} dt = k \cdot \mathrm{MoS}(\mu_1) \leq k.
$$

$\square$

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